which reduce in the particular case (29) to

$$\left.\begin{array}{ll} \nabla_\gamma x^\alpha = 2\,\mathcal{A}^\alpha_\gamma\,, & \nabla_\gamma x_\beta = 2\,\epsilon_{\gamma\beta}\,, \\ \nabla_\gamma q^\alpha = 0 \quad , & \nabla_\gamma q_\beta = 0 \quad , \\ \nabla_\gamma X^\alpha = -\mathcal{A}^\alpha_\gamma, & \nabla_\gamma X_\beta = -\,\epsilon_{\gamma\beta}, \\ \nabla_\gamma P^\alpha = +\mathcal{A}^\alpha_\gamma, & \nabla_\gamma P_\beta = +\,\epsilon_{\gamma\beta}. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

In conclusion the author wishes to thank Dr. J. HAANTJES and Professor J. A. SCHOUTEN whose criticisms on this note have led to several improvements

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Mathematics. — *Conformal representations of an n-dimensional euclidean space with a non-definite fundamental form on itself.* By J. HAANTJES. (Communicated by Prof. W. v. D. WOUDE).

(Communicated at the meeting of September 25, 1937.)

Introduction.

It is wellknown that every real conformal point transformation in an n-dimensional ($n > 2$) euclidean space (R_n) with a *definite* fundamental quadratic form can be brought about by a motion and an inversion or a dilatation [1]). This theorem, which for $n = 3$ is due to LIOUVILLE and is called LIOUVILLE's *theorem*, does not hold in a euclidean space with a fundamental form which is not definite.

The problem with which we are here concerned is to find the extension of the above theorem to a euclidean manifold, the fundamental form of which is not definite. This leads to a new class of conformal transformations (formula (26)). If $\mathfrak{M}$ denotes this class, then, as we shall see, the extension of LIOUVILLE's theorem may be formulated as follows. Every real conformal point transformation in an R_n ($n > 2$) is composed of a motion and a transformation T, where T is either a dilatation or an inversion or a transformation belonging to the class $\mathfrak{M}$.

It will appear that every transformation belonging to $\mathfrak{M}$ is the product of two inversions. Thus the following theorem holds in any euclidean space. The inversions and motions in an R_n define together the conformal group of point transformations.

§ 1. *Conformal transformations of the fundamental tensor.*

Let $a_{\lambda\varkappa}$ be the fundamental tensor in an n-dimensional RIEMANNian space V_n. A transformation of the form

$$'a_{\lambda\varkappa} = \sigma\, a_{\lambda\varkappa}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

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[1]) S. LIE, Ueber Komplexe, insbesondere Linien- und Kugelkomplexe, mit Anwendung auf die Theorie partieller Differentialgleichungen, Math. Ann., 5 (1872) p. 184.

where σ is a function of the coordinates $x^\varkappa$, which leads to another fundamental tensor $'a_{\lambda\varkappa}$, is called *a conformal transformation of the fundamental tensor*. It is clear from the angle definition that the transformation (1) preserves angles.

We shall first show that these transformations are closely connected with the conformal point transformations. Let

$$y^\varkappa = f^\varkappa (x^\lambda) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

be a real conformal point transformation. Thus we have

$$a_{\lambda\varkappa}(y) \frac{\partial y^\varkappa}{\partial x^\mu} \frac{\partial y^\lambda}{\partial x^\nu} = \sigma\, a_{\mu\nu}(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Such a transformation defines a 1—1 point correspondence. Hence we can solve the equations (2) for $x^\varkappa$ in terms of y^λ

$$x^\varkappa = F^\varkappa (y). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

We now pass to another coordinate system $(\varkappa')$ by the transformation

$$x^{\varkappa'} = F^{\varkappa'} (x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where the functions $F^{\varkappa'}$ are identical with the functions $F^\varkappa$. Then the coordinates of the point $y^\varkappa$ with respect to the system $(\varkappa')$ are

$$y^{\varkappa'} = F^{\varkappa'} (y^\lambda) = F^{\varkappa'} (f^\lambda (x)) \overset{*}{=} x^\varkappa \text{ [1]}). \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and the components of the fundamental tensor at the point $y^\varkappa$ with respect to $(\varkappa')$ are, as follows from (3),

$$a_{\lambda'\varkappa'}(y) \overset{*}{=} \sigma\, a_{\lambda\varkappa}(x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Thus, given a conformal point transformation (2), there exists always a coordinate system $(\varkappa')$ so that with respect to $(\varkappa')$ the point y has the same coordinates as the corresponding point x with respect to the system $(\varkappa)$, whereas the components of the fundamental tensor at the new point y with respect to $(\varkappa')$ are obtained from the components with respect to $(\varkappa)$ at the point x by multiplying with a factor σ. This means, however, that *every conformal point transformation corresponds to a conformal transformation of the fundamental tensor.*

If the V_n is a euclidean space, R_n, the curvature affinor defined by

$$K_{\nu\mu\lambda}^{\cdot\cdot\cdot\varkappa} = 2\,\partial_{[\nu} \left\{ {}_{\mu]\lambda}^{\;\varkappa} \right\} + 2 \left\{ {}_{[\nu|\varrho|}^{\;\;\varkappa} \right\} \left\{ {}_{\mu]\lambda}^{\;\varrho} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

vanishes at every point. It is zero at the point y as well as at the point x, from which it follows in consequence of (7), that the curvature affinor belonging to the tensor $\sigma\, a_{\lambda\varkappa}$ also vanishes. A conformal transformation in R_n corresponds, therefore, to a conformal transformation of the fundamental tensor

$$'a_{\lambda\varkappa} = \sigma\, a_{\lambda\varkappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

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[1]) The sign $\overset{*}{=}$ means that the equation holds with respect to the coordinate system or systems used in the equation itself; it needs not to hold with respect to other systems.

so that the curvature tensor $'K_{\nu\mu\lambda}^{\cdot\cdot\cdot\varkappa}$ belonging to the fundamental tensor $'a_{\lambda\varkappa}$ vanishes. In the next section these conformal transformations of the fundamental tensor will be investigated.

§ 2. *Conformal transformations of the fundamental tensor in R_n, which lead again to euclidean spaces.*

Let us consider the transformation (9). The CHRISTOFFEL symbols computed from the tensor $'a_{\lambda\varkappa}$ are

$$'\left\{ {\varkappa \atop \mu\lambda} \right\} = \left\{ {\varkappa \atop \mu\lambda} \right\} + \frac{1}{2} A_\mu^\varkappa s_\lambda + \frac{1}{2} A_\lambda^\varkappa s_\mu - \frac{1}{2} a_{\mu\lambda} s^\varkappa \quad . \quad . \quad . \quad (10)$$

where $A_\lambda^\varkappa$ is the unit affinor and

$$s_\lambda = \partial_\lambda \log \sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

From this it follows that the curvature affinor $'K_{\nu\mu\lambda}^{\cdot\cdot\cdot\varkappa}$ belonging to the tensor $'a_{\lambda\varkappa}$ is related with $K_{\nu\mu\lambda}^{\cdot\cdot\cdot\varkappa}$ by the following equation

$$'K_{\nu\mu\lambda}^{\cdot\cdot\cdot\varkappa} = K_{\nu\mu\lambda}^{\cdot\cdot\cdot\varkappa} + a_{[\nu[\lambda}\, s_{\mu]\tau]}\, a^{\tau\varkappa}\ ^{1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where

$$s_{\mu\lambda} = 2 \nabla_\mu s_\lambda - s_\mu s_\lambda + \tfrac{1}{2} a_{\mu\lambda} s_\varrho s^\varrho \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

We now suppose the space to be euclidean with respect to the fundamental tensor $a_{\lambda\varkappa}$ as well as with respect to the tensor $'a_{\lambda\varkappa}$. Then both $K_{\nu\mu\lambda}^{\cdot\cdot\cdot\varkappa}$ and $'K_{\nu\mu\lambda}^{\cdot\cdot\cdot\varkappa}$ vanish and, when $n > 2$, it follows from (12) that

$$s_{\mu\lambda} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

which equation is equivalent to

$$2 \nabla_\mu s_\lambda = s_\mu s_\lambda - \tfrac{1}{2} a_{\mu\lambda} s_\varrho s^\varrho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Every solution s_λ of this differential equation is a gradient, as is easily shown by alternating both sides of (15), and gives a conformal transformation of the fundamental tensor, which leads again to a euclidean space.

In this paper we consider the case that $a_{\lambda\varkappa}\, dx^\lambda\, dx^\varkappa$ is a *non-definite* quadratic form. Then the real solutions s_λ may be divided into three groups:

1. The solution $s_\lambda = 0$
2. The real solutions s_λ for which $s_\varrho s^\varrho \neq 0$
3. The real solutions s_λ, different from zero, for which $a_{\lambda\varkappa} s^\lambda s^\varkappa = 0$.

The solution $s_\lambda = 0$ gives $\sigma =$ constant.

If s_λ is a solution of the second kind, then a cartesian coordinate system can be chosen in such a way that

$$\sigma = C^2 (x^\varkappa x_\varkappa)^{-2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

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1) Cf. J. A. SCHOUTEN und D. J. STRUIK. Einführung in die neueren Methoden der Differentialgeometrie I, Groningen, p. 129.

C being an arbitrarily chosen constant. For the proof we refer to the literature [1]).

In the present paper we shall examine the solutions belonging to the third group. Such a solution satisfies the differential equation

$$2\nabla_\mu s_\lambda = s_\mu s_\lambda. \quad \ldots \ldots \quad (17)$$

Hence we have according to (11)

$$2\nabla_\mu \sigma^{-\frac{1}{2}} s_\lambda = 2\sigma^{-\frac{1}{2}}\nabla_\mu s_\lambda - \sigma^{-\frac{1}{2}} s_\lambda \nabla_\mu \log \sigma = \sigma^{-\frac{1}{2}} s_\mu s_\lambda - \sigma^{-\frac{1}{2}} s_\mu s_\lambda = 0, \quad (18)$$

which means that the vector

$$b_\lambda = \tfrac{1}{2}\sigma^{-\frac{1}{2}} s_\lambda = -\partial_\lambda \sigma^{-\frac{1}{2}} \quad \ldots \ldots \quad (19)$$

is covariant constant, hence it is constant with respect to a cartesian coordinate system. In the following we suppose the coordinate system to be an orthogonal cartesian one. From (19) we obtain by integration

$$\sigma^{-\frac{1}{2}} = c - b_\lambda x^\lambda;\ c = \text{constant} \quad \ldots \ldots \quad (20)$$

So we have

$$s_\lambda = \frac{2\,b_\lambda}{c - b_\varkappa x^\varkappa}, \quad \ldots \ldots \quad (21)$$

where b_λ is a constant null-vector ($b_\lambda b^\lambda = 0$). By the orthogonal coordinate transformation $x^{\varkappa'} = \delta_\varkappa^{\varkappa'} x^\varkappa + a^{\varkappa'}$, where the $\delta_\varkappa^{\varkappa'}$ denote the generalized KRONECKER symbols and the $a^{\varkappa'}$ are constants, c alters. It is, of course, always possible to find a coordinate system for which $c = 1$.

§ 3. *The corresponding conformal point transformations.*

As we have seen, to every conformal point transformation corresponds a conformal transformation of the fundamental tensor.

A conformal transformation corresponding to the transformation $'a_{\lambda\varkappa} = \sigma\, a_{\lambda\varkappa}$ with σ = constant is the *dilatation*

$$y^\varkappa = \sqrt{\sigma}\, x^\varkappa, \quad \ldots \ldots \quad (22)$$

as is easily seen by substituting this expression in (3).

Furthermore the *inversion*

$$y^\varkappa = C\,\frac{x^\varkappa}{x^\lambda x_\lambda} \quad (C = \text{constant}) \quad \ldots \quad (23)$$

corresponds to a conformal transformation of the fundamental tensor which belongs to the second group. Indeed from (23) follows

$$dy^\varkappa\, dy_\varkappa = \sigma\, dx^\varkappa\, dx_\varkappa, \quad \ldots \ldots \quad (24)$$

[1]) J. A. SCHOUTEN, Der Ricci-Kalkül, p. 173.

where σ stands for the expression (16). Every other conformal transformation, which corresponds to the same transformation of the fundamental tensor, is the product of the inversion (23) and a motion.

Let us now consider the conformal transformation of the fundamental tensor (9) with (comp. (20))

$$\sigma = (1 - b_\lambda x^\lambda)^{-2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

hence a transformation which belongs to the third group. We shall now show that one of the corresponding conformal point transformations is given by

$$y^\varkappa = \frac{x^\varkappa - \frac{1}{2} x^\lambda x_\lambda b^\varkappa}{1 - b^\lambda x_\lambda}; \; b^\lambda b_\lambda = 0. \quad . \quad . \quad . \quad . \quad . \quad (26)$$

From (26) we have by differentiation

$$dy^\varkappa = \frac{dx^\varkappa - (x^\lambda dx_\lambda) b^\varkappa}{1 - b^\lambda x_\lambda} + \frac{(b^\mu dx_\mu)(x^\varkappa - \frac{1}{2} x^\lambda x_\lambda b^\varkappa)}{(1 - b^\lambda x_\lambda)^2}, \quad . \quad . \quad . \quad (27)$$

from which it follows after some calculation

$$a_{\lambda\varkappa} dy^\lambda dy^\varkappa = \frac{a_{\lambda\varkappa} dx^\lambda dx^\varkappa}{(1 - b^\varrho x_\varrho)^2} = \sigma a_{\lambda\varkappa} dx^\lambda dx^\varkappa, \quad . \quad . \quad . \quad . \quad (28)$$

where σ stands for the expression (25). Consequently the conformal representation (26) of the space upon itself corresponds indeed to the transformation

$$'a_{\lambda\varkappa} = (1 - b_\lambda x^\lambda)^{-2} a_{\lambda\varkappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and every other conformal point transformation which corresponds to the transformation (29) is the product of the transformation (26) and a motion. In consequence of these results we have the following extension of LIOUVILLE's theorem:

Any real conformal representation of an n-dimensional ($n > 2$) *euclidean space with a non-definite fundamental form upon itself can be brought about by the product of one of the transformations* (22) *(dilatation),* (23) *(inversion) or* (26) *with a motion.*

In a euclidean space with a definite fundamental form a real solution s_λ, for which $s_\lambda s^\lambda = 0$, does not exist. In this case, therefore, we do not find real conformal representations of the form (26).

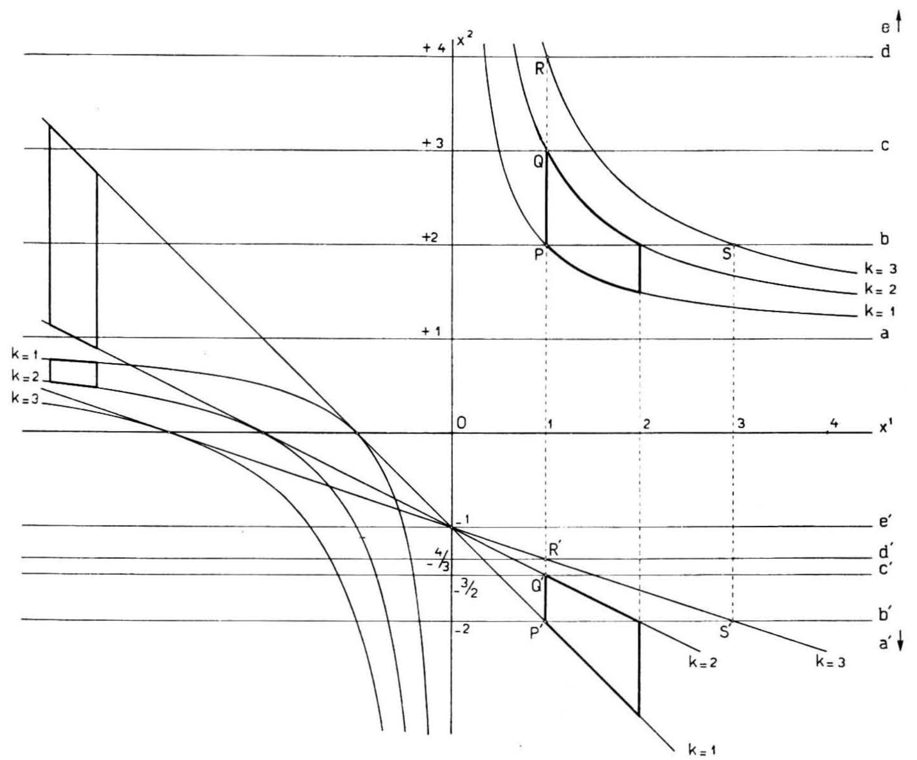
In an R_2 with a non-definite fundamental form the representations mentioned in the above theorem are the conformal representations with the property that "circles" remain "circles". As an example let us write in full one of these transformations. The coordinate system may be chosen in such a way that

$$ds^2 = 2\,dx^1 dx^2, \quad a_{11} = 0, \quad a_{12} = 1, \quad a_{22} = 0 \quad . \quad . \quad . \quad (30)$$

Taking $b^1 = 1$, $b^2 = 0$, we obtain the following representation

$$\left.\begin{aligned} y^1 &= \frac{x^1 - x^1 x^2}{1 - x^2} = x^1 \\ y^2 &= \frac{x^2}{1 - x^2}. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (31)$$

The system of "circles" $x^1 (x^2 - 1) = k$ transforms into the system $y^1 + k (y^2 + 1) = 0$, as is represented in the figure.



It is interesting to note that we may look upon the transformation (26) as the product of two inversions, one with the centre in $x^{\varkappa} = 0$ and one with the centre $x^{\varkappa} = \frac{1}{2} b^{\varkappa}$:

$$\left.\begin{aligned} &a) \quad z^{\varkappa} = \frac{x^{\varkappa}}{x^{\lambda} x_{\lambda}} \\ &b) \quad y^{\varkappa} = \frac{z^{\varkappa} - \frac{1}{2} b^{\varkappa}}{(z^{\lambda} - \frac{1}{2} b^{\lambda})(z_{\lambda} - \frac{1}{2} b_{\lambda})}, \quad b_{\lambda} b^{\lambda} = 0 \end{aligned}\right\} \quad \ldots\ldots \quad (32)$$

So we have the following theorem:

The inversions and motions in an R_n together define the conformal group.

This theorem holds both in an R_n with a non-definite fundamental form and in an R_n with a definite form.