Mathematics. — *Ueber Differentialkovarianten erster Ordnung der binären kubischen Differentialform.* Von P. G. MOLENAAR. (Communicated by Prof. R. WEITZENBÖCK.)

(Communicated at the meeting of May 28, 1938.)

§ 1. Die binäre kubische Differentialform

$$f = a_{ikl}\, dx^i\, dx^k\, dx^l = a^3_{dx}$$

hat eine Differentialkovariante erster Ordnung und zweiter Stufe mit den Komponenten

$$n_{ik} = \frac{\partial a_{ik1}}{\partial x_2} - \frac{\partial a_{ik2}}{\partial x_1} + \tfrac{1}{2}\left(\frac{\partial \alpha^{pq}}{\partial x_i} Q_{kpq} + \frac{\partial \alpha^{pq}}{\partial x_k} Q_{ipq}\right) \qquad (R \neq 0). \qquad ^{1)}$$

Wegen der Relation

$$\alpha^{pq} Q_{pq\lambda} = 0 \quad (\lambda = 1, 2) \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

ist

$$n_{ik} = \frac{\partial a_{ik1}}{\partial x_2} - \frac{\partial a_{ik2}}{\partial x_1} - \tfrac{1}{2}\alpha^{pq}\left(\frac{\partial Q_{kpq}}{\partial x_i} + \frac{\partial Q_{ipq}}{\partial x_k}\right)$$

und da

$$\alpha^{pq} = \frac{2}{R}\,\alpha_{pq}$$

findet man

$$n_{ik} = \frac{\partial a_{ik1}}{\partial x_2} - \frac{\partial a_{ik2}}{\partial x_1} - \frac{\alpha_{pq}}{R}\left(\frac{\partial Q_{kpq}}{\partial x_i} + \frac{\partial Q_{ipq}}{\partial x_k}\right) \quad . \; . \; . \; . \quad (2)$$

Nun ist

$$\triangle = \alpha^2_{dx} = (f, f)^{(2)} \quad Q = (f, \triangle)^{(1)} \quad \text{und} \quad R = (\triangle, \triangle)^{(2)}. \; . \; . \quad (3)$$

Durch (2) und (3) wird also eine Operation

$$n = N(f)$$

definiert, welche aus der Grundform f eine Differentialkovariante n ableitet.

Diese Operation kann man auch ausüben auf die absolute Differentialkovariante

$$Q^*_{ikl} = \frac{Q_{ikl}}{\sqrt{\dfrac{-R}{2}}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

[1]) Vgl. P. G. MOLENAAR, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **41**, 278—288 (1938).

Hierzu muss man erst

$$\triangle_{Q^*}, \quad R_{Q^*} \quad \text{und} \quad Q_{Q^*}$$

berechnen.

Das Büschel

$$f_{\varkappa\lambda} = \varkappa f + \lambda Q$$

hat die Komitanten [2])

$$\left.\begin{aligned} \triangle_{\varkappa\lambda} &= \left(\varkappa^2 + \frac{R}{2}\lambda^2\right)\triangle \\ R_{\varkappa\lambda} &= \left(\varkappa^2 + \frac{R}{2}\lambda^2\right)^2 R \\ Q_{\varkappa\lambda} &= \left(\varkappa^2 + \frac{R}{2}\lambda^2\right)\left(\varkappa Q - \frac{R}{2}\lambda f\right) \end{aligned}\right\} \quad . \; . \; . \; . \; . \quad (5)$$

Setzt man $\varkappa = 0$ und $\lambda = 1$, so findet man

$$\triangle_Q = \tfrac{1}{2} R \triangle$$
$$R_Q = \tfrac{1}{4} R^3$$
$$Q_Q = -\tfrac{1}{4} R^2 f$$

woraus folgt

$$\triangle_{Q^*} = \frac{\tfrac{1}{2} R \triangle}{\sqrt{\dfrac{-R^2}{2}}} = -\triangle$$

$$R_{Q^*} = \frac{\tfrac{1}{4} R^3}{\sqrt{\dfrac{-R^4}{2}}} = R$$

$$Q_{Q^*} = \frac{-\tfrac{1}{4} R^2 f}{\sqrt{\dfrac{-R^3}{2}}} = -\sqrt{\frac{-R}{2}}\, f.$$

Daher ist

$$u_{ik} = N(Q^*)_{ik} = \frac{\partial Q^*_{ik1}}{\partial x_2} - \frac{\partial Q^*_{ik2}}{\partial x_1} + \frac{\alpha_{pq}}{R}\left\{\frac{\partial\left(-\sqrt{\frac{-R}{2}}\, a_{kpq}\right)}{\partial x_i} + \frac{\partial\left(-\sqrt{\frac{-R}{2}}\, a_{ipq}\right)}{\partial x_k}\right\} =$$

$$= \frac{\partial Q^*_{ik1}}{\partial x_2} - \frac{\partial Q^*_{ik2}}{\partial x_1} + \frac{\alpha_{pq}}{R}\left(-\sqrt{\frac{-R}{2}}\right)\left(\frac{\partial a_{kpq}}{\partial x_i} + \frac{\partial a_{ipq}}{\partial x_k}\right) -$$

$$- \frac{\alpha_{pq}}{R}\left(a_{kpq}\frac{\partial\sqrt{\frac{-R}{2}}}{\partial x_i} + a_{ipq}\frac{\partial\sqrt{\frac{-R}{2}}}{\partial x_k}\right)$$

[2]) Vgl. CLEBSCH—LINDEMANN, Vorlesungen über Geometrie I. S. 227.

also wegen

$$a_{pq}\, a_{pq\lambda} = 0 \quad (\lambda = 1, 2) \quad \ldots \ldots \quad (6)$$

$$u_{ik} = \frac{\partial Q^*_{ik1}}{\partial x_2} - \frac{\partial Q^*_{ik2}}{\partial x_1} + \frac{\frac{1}{2}\,\alpha_{pq}}{\sqrt{-\frac{R}{2}}}\left(\frac{\partial a_{kpq}}{\partial x_i} + \frac{\partial a_{ipq}}{\partial x_k}\right).$$

Setzt man

$$\frac{\alpha_{pq}}{\sqrt{\frac{-R}{2}}} = \alpha^*_{pq} \quad \ldots \ldots \quad (7)$$

so wird

$$u_{ik} = \frac{\partial Q^*_{ik1}}{\partial x_2} - \frac{\partial Q^*_{ik2}}{\partial x_1} + \tfrac{1}{2}\,\alpha^*_{pq}\left(\frac{\partial a_{kpq}}{\partial x_i} + \frac{\partial a_{ipq}}{\partial x_k}\right) \quad \ldots \quad (8)$$

Ferner folgt aus (2) nach (4)

$$n_{ik} = \frac{\partial a_{ik1}}{\partial x_2} - \frac{\partial a_{ik2}}{\partial x_1} - \frac{\alpha_{pq}}{R}\left\{\frac{\partial\left(\sqrt{\frac{-R}{2}}\, Q^*_{kpq}\right)}{\partial x_i} + \frac{\partial\left(\sqrt{\frac{-R}{2}}\, Q^*_{ipq}\right)}{\partial x_k}\right\}$$

oder wegen (1) und (7)

$$n_{ik} = \frac{\partial a_{ik1}}{\partial x_2} - \frac{\partial a_{ik2}}{\partial x_1} + \tfrac{1}{2}\,\alpha^*_{pq}\left(\frac{\partial Q^*_{kpq}}{\partial x_i} + \frac{\partial Q^*_{ipq}}{\partial x_k}\right) \quad \ldots \quad (9)$$

Man kann zeigen, dass die Differentialkovariante u_{ik} durch Ueberschiebung von n_{ik} über α^*_{ik} entsteht.

Es gibt eine lineare Transformation [3]), welche

$$f = a_{ikl}\, dx^i\, dx^k\, dx^l$$

überführt in

$$f = a_{111}\, dx^{1\,3} + a_{222}\, dx^{2\,3} \quad \ldots \ldots \quad (10$$

Durch diese Transformation wird

$$Q = a^2_{111}\, a_{222}\, dx^{1\,3} - a^2_{222}\, a_{111}\, dx^{2\,3}$$
$$\triangle = 2\, a_{111}\, a_{222}\, dx^1\, dx^2$$
$$R = -2\, a^2_{111}\, a^2_{222}$$
$$Q^* = a_{111}\, dx^{1\,3} - a_{222}\, dx^{2\,3}$$
$$\triangle^* = 2\, dx^1\, dx^2$$

$$n_{11} = \frac{\partial a_{111}}{\partial x_2} \qquad n_{12} = 0 \qquad n_{22} = -\frac{\partial a_{222}}{\partial x_1}$$

$$u_{11} = \frac{\partial a_{111}}{\partial x_2} \qquad u_{12} = 0 \qquad u_{22} = \frac{\partial a_{222}}{\partial x_1}$$

[3]) Vgl. GORDAN—KERSCHENSTEINER, Invariantentheorie II, S. 177.

Die Kovarianten n_{ik} und u_{ik} sind also verschieden.

Durch Ueberschiebung von n_{ik} über α^*_{ik} bekommt man

$$(n, \alpha^*)^{(1)} = (n_{11}\alpha^*_{12} - n_{12}\alpha^*_{11})\, dx^{1\,2} + (n_{11}\alpha^*_{22} - n_{22}\alpha^*_{11})\, dx^1\, dx^2 + (n_{12}\alpha^*_{22} - n_{22}\alpha^*_{12})\, dx^{2\,2}.$$

Durch die obige Transformation entsteht hieraus

$$(n, \alpha^*)^{(1)} = n_{11}\, dx^{1\,2} - n_{22}\, dx^{2\,2} = u_{11}\, dx^{1\,2} + u_{22}\, dx^{2\,2}.$$

u_{ik} ist also die erste Ueberschiebung von n_{ik} über α^*_{ik}.

Da

$$Q^* = (f, \alpha^*)^{(1)}$$

hat man die Relation

$$N\left((f, \alpha^*)^{(1)}\right) = (N(f), \alpha^*)^{(1)}$$

§ 2. Setzt man in dem Büschel

$$f_{\varkappa\lambda} = \varkappa f + \lambda Q$$

$$\lambda = \frac{1}{\sqrt{\frac{-R}{2}}} \quad \text{also} \quad \lambda Q = Q^*$$

so ist

$$f_\varkappa = \varkappa f + Q^*$$

eine absolute Differentialkovariante ($\varkappa$ ist unabhängig von x_i).

Aus (5) folgt

$$\triangle_\varkappa = (\varkappa^2 - 1)\, \triangle$$
$$R_\varkappa = (\varkappa^2 - 1)^2\, R$$
$$Q_\varkappa = (\varkappa^2 - 1)\left(\varkappa Q + \sqrt{\frac{-R}{2}}\, f\right) = \sqrt{\frac{-R}{2}}\,(\varkappa^2 - 1)(\varkappa Q^* + f).$$

Durch Anwendung der Operation N auf $f_\varkappa$ entsteht

$$S^{(\varkappa)}_{ik} = \frac{\partial(\varkappa a_{ik1} + Q^*_{ik1})}{\partial x_2} - \frac{\partial(\varkappa a_{ik2} + Q^*_{ik2})}{\partial x_1} -$$

$$- \frac{(\varkappa^2 - 1)\,\alpha_{pq}}{(\varkappa^2 - 1)^2\, R}\left\{\frac{\partial \sqrt{\frac{-R}{2}}\,(\varkappa^2 - 1)(\varkappa Q^*_{kpq} + a_{kpq})}{\partial x_i} + \frac{\partial \sqrt{\frac{-R}{2}}\,(\varkappa^2 - 1)(\varkappa Q^*_{ipq} + a_{ipq})}{\partial x_k}\right\} =$$

$$= \varkappa\left(\frac{\partial a_{ik1}}{\partial x_2} - \frac{\partial a_{ik2}}{\partial x_1}\right) + \left(\frac{\partial Q^*_{ik1}}{\partial x_2} - \frac{\partial Q^*_{ik2}}{\partial x_1}\right) + \frac{\frac{1}{2}\,\alpha_{pq}}{\sqrt{\frac{-R}{2}}}\left\{\frac{\partial(\varkappa Q^*_{kpq} + a_{kpq})}{\partial x_i} + \frac{\partial(\varkappa Q^*_{ipq} + a_{ipq})}{\partial x_k}\right\} -$$

$$- \frac{\alpha_{pq}}{R}\left\{(\varkappa Q^*_{kpq} + a_{kpq})\frac{\partial \sqrt{\frac{-R}{2}}}{\partial x_i} + (\varkappa Q^*_{ipq} + a_{ipq})\frac{\partial \sqrt{\frac{-R}{2}}}{\partial x_k}\right\}$$

also wegen (1) und (6)

$$S_{ik}^{(\varkappa)} = \varkappa \left\{ \frac{\partial a_{ik1}}{\partial x_2} - \frac{\partial a_{ik2}}{\partial x_1} + \tfrac{1}{2} \overset{*}{\alpha}_{\underline{pq}} \left(\frac{\partial Q^*_{kpq}}{\partial x_i} + \frac{\partial Q^*_{ipq}}{\partial x_k} \right) \right\} +$$
$$+ \left\{ \frac{\partial Q^*_{ik1}}{\partial x_2} - \frac{\partial Q^*_{ik2}}{\partial x_1} + \tfrac{1}{2} \overset{*}{\alpha}_{\underline{pq}} \left(\frac{\partial a_{kpq}}{\partial x_i} + \frac{\partial a_{ipq}}{\partial x_k} \right) \right\}$$

oder

$$S_{ik}^{(\varkappa)} = \varkappa\, n_{ik} + u_{ik} \qquad (\varkappa^2 \neq 1) \quad \ldots\ldots \quad (11)$$

Für $\varkappa^2 = 1$ wird die Operation N sinnlos, da dann $\triangle_\varkappa$, $R_\varkappa$ und $Q_\varkappa$ gleich Null werden. Man findet jedoch durch Addition und Subtraktion von (8) und (9) zwei nicht verschwindende Kovarianten:

$$S_{ik}^{(1)} = \frac{\partial (a_{ik1} + Q^*_{ik1})}{\partial x_2} - \frac{\partial (a_{ik2} + Q^*_{ik2})}{\partial x_1} +$$
$$+ \tfrac{1}{2} \overset{*}{\alpha}_{\underline{pq}} \left\{ \frac{\partial (a_{kpq} + Q^*_{kpq})}{\partial x_i} + \frac{\partial (a_{ipq} + Q^*_{ipq})}{\partial x_k} \right\}$$

$$S_{ik}^{(-1)} = \frac{\partial (-a_{ik1} + Q^*_{ik1})}{\delta x_2} - \frac{\partial (-a_{ik2} + Q^*_{ik2})}{\partial x_1} -$$
$$- \tfrac{1}{2} \overset{*}{\alpha}_{\underline{pq}} \left\{ \frac{\partial (-a_{kpq} + Q^*_{kpq})}{\partial x_i} + \frac{\partial (-a_{ipq} + Q^*_{ipq})}{\partial x_k} \right\}.$$

Setzt man noch

$$f + Q^* = v_x^3 \qquad\qquad -f + Q^* = w_x^3$$

worin v_x^3 und w_x^3 reine Kuben sind [3]), so bekommen diese Kovarianten die neue Gestalt:

$$S_{ik}^{(1)} = \frac{\partial v_{ik1}}{\partial x_2} - \frac{\partial v_{ik2}}{\partial x_1} + \tfrac{1}{2} \overset{*}{\alpha}_{\underline{pq}} \left(\frac{\partial v_{kpq}}{\partial x_i} + \frac{\partial v_{ipq}}{\partial x_k} \right)$$

$$S_{ik}^{(-1)} = \frac{\partial w_{ik1}}{\partial x_2} - \frac{\partial w_{ik2}}{\partial x_1} - \tfrac{1}{2} \overset{*}{\alpha}_{\underline{pq}} \left(\frac{\partial w_{kpq}}{\partial x_i} + \frac{\partial w_{ipq}}{\partial x_k} \right)$$

Unterwirft man diese Kovarianten der Transformation, welche f überführt in die kanonische Gestalt (10), so wird

$$S^{(1)} = 2 \frac{\partial a_{111}}{\partial x_2} dx'^2 \quad \text{und} \quad S^{(-1)} = 2 \frac{\partial a_{222}}{\partial x_1} dx'^2.$$

Ihre Diskriminanten sind Null.

$S_{ik}^{(1)}$ und $S_{ik}^{(-1)}$ sind also die entarteten Exemplare des Büschels $S_{ik}^{(\varkappa)} = \varkappa\, n_{ik} + u_{ik}$. Die Differentialkovarianten erster Ordnung der binären kubischen Differentialform führen zurück auf n_{ik} und ihre Ueberschiebung mit $\overset{*}{\alpha}_{ik}$.

Biochemistry. — *Behaviour of Microscopic Bodies consisting of Biocolloid Systems and suspended in an Aequeous Medium. I. Pulsating Vacuoles in Coacervate Drops.* By H. G. BUNGENBERG DE JONG. (Communicated by Prof. J. VAN DER HOEVE.)

(Communicated at the meeting of May 28, 1938.)

1. *General introduction.*

Experimental cytology studies the behaviour of microscopically small systems from biocolloids and it is faced by a remarkable difficulty in the interpretation of the morphological changes brought about under the influence of external or internal factors. It is tried to find some connection with the results of colloid-chemical researches, e.g. on sols and gels. However, the dimensions of the latter objects of research are usually such that their properties are practically determined only by those of the three-dimensional content of these systems and the influence of the bordering surfaces is not expressed in it.

With the objects of cytology, however, the proportion of bordering surface and content is totally different and the morphological changes observed by it consist of changes of the biocolloid systems, in which both, surface and content, are simultaneously contained and influence each other mutually.

For this reason we are inclined to think that, if colloid chemistry wishes to be of use to cytology, it will have to occupy itself with the study of microscopically small colloid bodies; by bodies is meant here: colloid systems surrounded by an external surface.

In the study of the coacervation phenomena the present writer already applied this method, in so far as not only the properties of the three-dimensional coacervates were examined but also those of microscopical coacervate drops. The significance of these systems in biology, which consequently may be regarded as fluid biocolloid bodies, has been stated elsewhere [1]).

In so far as the study of these fluid and other kinds of biocolloid bodies

---

[1]) Summarizing articles on coacervation:
H. G. BUNGENBERG DE JONG, Die Koazervation und ihre Bedeutung für die Biologie, Protoplasma, **15**, 110 (1932).
H. G. BUNGENBERG DE JONG, La Coacervation et son importance en Biologie. Tome I et II. Hermann et Cie; Paris 1936.
H. G. BUNGENBERG DE JONG, Koazervation, Kolloid Z., **79**, 223, 334 (1937), **80**, 221, 350 (1937).