Physics. — *The decay of the penetrating cosmic rays.* By E. M. BRUINS.
(Communicated by Prof. J. CLAY.)

(Communicated at the meeting of December 17, 1938.)

§ 1. Recently it has been suggested that the penetrating component found by examination of the cosmic rays is produced by the positive and negative electrons falling from the cosmic space at a certain height in the atmosphere. The penetrating component itself would consist of heavy electrons which decay with a lifetime proportional to the energy. When they decay, ordinary electrons are formed which are absorbed according to the cascade theory. H. EULER and W. HEISENBERG [1]) estimated the lifetime of this heavy electron from absorption measurements and from the ratio of the number of penetrating and soft rays at great depths. However, some peculiarities, such as the dependence of the ratio of the soft and hard rays on the angle to the vertical at sea-level, are not given by their final formula. Below we shall discuss the modification of the energy-spectrum of the heavy electrons in the atmosphere.

§ 2. Let $m(h')$ be the mass per cm^2 per cm at the height h'. In the atmosphere may then be applied

$$m(h') = m_0 \, e^{-\beta h'}.$$

Suppose a is the loss of energie of a ray per cm per gram per cm^2 in electron-volt. The energyloss of a ray on proceeding along a path h, reckoned from the layer at a height H, where the penetrative rays are formed, is then

$$\triangle E = \int_0^h m_0 \, e^{-\beta(H-h)} \, a \, dh = A\,(e^{\beta h} - 1).$$

The magnitude A occurring here

$$A = \frac{m_0 \, e^{-\beta H} \, a}{\beta}$$

has the dimension of an energy and *is characteristic of the condition in the atmosphere.* For $H = 20$ KM, $a = 2 \times 10^6$ eV cm per gram, is $A = 1.64 \times 10^8$ eV. The constant of decay per cm of the path length is

1) H. EULER und W. HEISENBERG, Ergebnisse der exakten Naturw., **17**. H. EULER, Z. f. Physik., **110**, 692 (1938).

inversely proportional to the energy. To the number of penetrating rays N, formed with an energy E at a height H, consequently applies:

$$\frac{dN}{N} = -\frac{\alpha}{E - \triangle E}\, dh$$

from which for the number of particles with an energy $E - \triangle E$ at a distance h follows:

$$N = C(E) \left(\frac{e^{\beta h}}{E + A - A\, e^{\beta h}} \right)^{-\frac{\alpha}{\beta(E+A)}}.$$

If the energyspectrum of the rays at their formation is

$$N(E) = N_0 \, E^{-s}$$

then follows from this

$$N = N_0 \, E^{-s} \left[\frac{E + A - A\, e^{\beta h}}{E\, e^{\beta h}} \right]^{\frac{\alpha}{\beta(E+A)}}.$$

On measuring the energy in A, which is characteristic of the atmosphere, we find that

$$\underline{E = Ax}$$

$$N(x) = \overline{N}_0 \, x^{-s} \left(\frac{x + 1 - e^{\beta h}}{x\, e^{\beta h}} \right)^{\frac{\alpha}{A\beta(x+1)}}.$$

Consequently, by absorption measurements only $\alpha/A\beta$ may be determined exclusively. The primary energy spectrum is multiplied by a factor which for high values of x approaches unity. For great energies, therefore, at every depth

$$N(x) \backsim \overline{N}_0 \, x^{-s}.$$

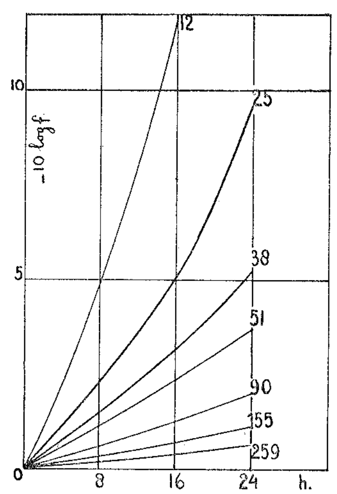

Fig. 1. Correction factor for the energyspectrum at $\tau = 10^{-6}$ sec. at different heights for different energies. Lines for constant energy expressed in A.

A different choice of H, a, m_0, α, β only causes a change of the exponent of the correction factor. The magnitude $\alpha = \mu c^2/\tau c$ measures the proportion between the "rest-energy" and the "lifetime". We suppose $\mu c^2 = 8 \,.\, 10^7$ eV. The correction may be obtained by means of fig. 1. *The energy spectrum shows a maximum which with increasing depth and diminishing lifetime is shifted towards a higher energy.* In the figures 2, 3 and 4 and the tables I, II, III and IV the energy spectrum is found for varying lifetime at depths of 8, 16 and 24 KM. The general behaviour of the energy spectrum might also be described as a change of the exponent s, which holds good for great depths and runs via the value zero, reached in the maximum of the energy spectrum, to negative values. This change must then be found as well in the integral spectrum.

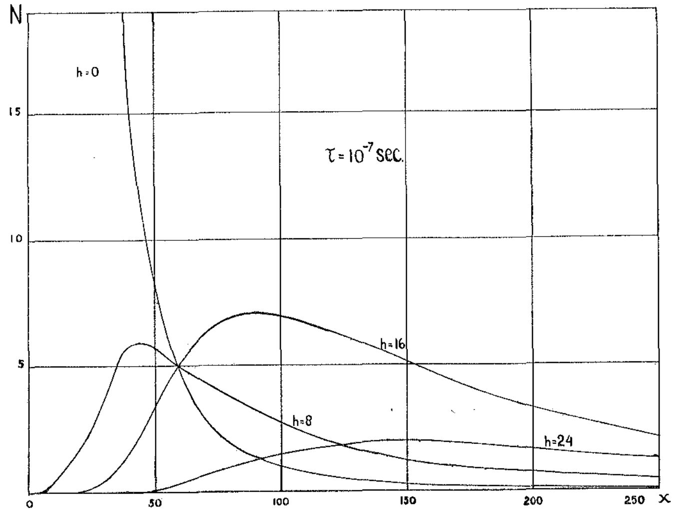

Fig. 2. Energyspectrum for $\tau = 10^{-7}$ sec. The given values have been multiplied for $h = 8$ KM by 10^7 and for $h = 16$ and 24 KM by 10^8.

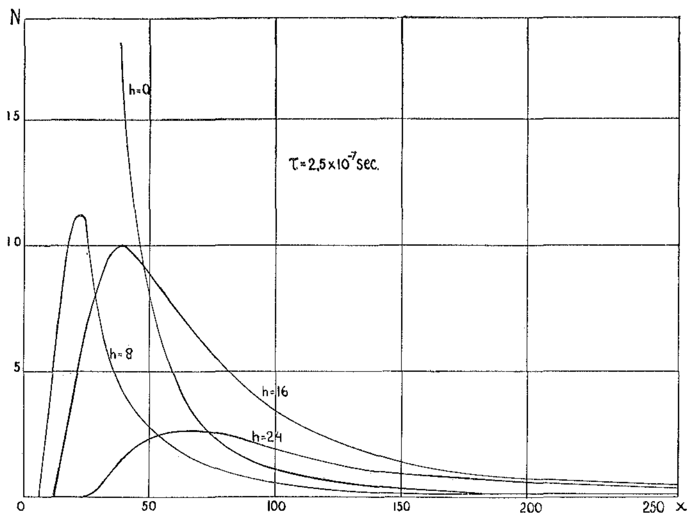

Fig. 3. Energyspectrum for $\tau = 2.5 \times 10^{-7}$ sec. The given values have been multiplied for $h = 8$ KM by 10^6 and for $h = 16$ and 24 KM by 10^7.

The total number of particles with an energy greater than qA is

$$N(q) = \overline{N}_0 \int_q^\infty x^{-s} \left(\frac{x + 1 - e^{\beta h}}{x\, e^{\beta h}} \right)^{\frac{\alpha}{A\beta(x+1)}} dx.$$

If we assume here that

$$xy = q \text{ and } s = 3$$

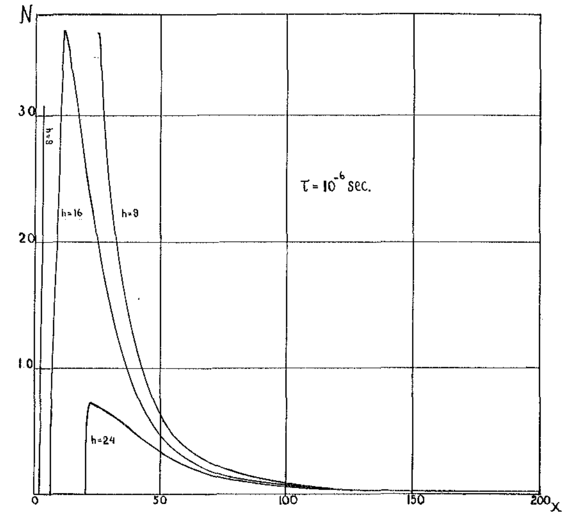

Fig. 4. Energyspectrum for $\tau = 10^{-6}$ sec. The given values have been multiplied by 10^6.

then is

$$N(q) = \frac{\overline{N}_0}{q^2} \int_0^1 y \left(\frac{q - (e^{\beta h} - 1)\, y}{q\, e^{\beta h}} \right)^{\frac{\alpha y}{A\beta(y+q)}} dy = \frac{\overline{N}_0\, C(q)}{2\, q^2}.$$

TABLE I. $\tau = 10^{-7}$ sec.

x	$h = 0$	$h = 8$	$h = 16$	$h = 24$
5.5	6.011—3	6.89 —15	—	—
12	5.787—4	6.90 — 8	—	—
25	6.400—5	2.426— 7	6.433—10	1.45 —14
38	1.823—5	5.575— 7	1.256—8	8.09 —11
51	7.521—6	5.668— 7	3.624—8	1.292— 9
64	3.815—6	4.692— 7	5.655—8	4.658— 9
77	2.190—6	3.975— 7	6.761—8	9.180— 9
90	1.372—6	3.200— 7	7.093—8	13.435— 9
142	3.492—7	1.361— 7	5.436—8	20.02 — 9
155	2.685—7	1.156— 7	4 896—8	19.76 — 9
168	2.109—7	9.696— 8	4.395—8	19.13 — 9
181	1.686—7	8.197— 8	3.923—8	18.27 — 9
194	1.370—7	6.992— 8	3.532—8	17.65 — 9
259	5.756—8	3.480— 8	2.091—8	12.36 — 9

$E = Ax$ $A = 1.64 \times 10^8$ eV $h =$ depth in KM

The whole numbers indicate the exponent of 10 of the factor by which all given values must be multiplied.

TABLE II. $\tau = 2.5 \times 10^{-7}$ sec.

x	$h=0$	$h=8$	$h=16$	$h=24$
5.5	6.011—3	1.00 —7	—	—
10	1.00 —3	—	6 —10	—
11	7.514—4	—	2.99 —11	—
12	5.787—4	6.208—6	6.15 — 9	—
25	6.400—5	1.075—5	6.531— 7	8.88 —9
38	1.823—5	4.518—6	9.910— 7	1.317 —7
51	7.521—6	2.674—6	8.910— 7	2.343 —7
64	3.815—6	1.650—6	7.081— 7	2.608 —7
77	2.190—6	1.108—6	5.448— 7	2.451 —7
90	1.372—6	7.664—7	4.195— 7	2.155 —7
129	4.658—7	3.106—7	2.051— 7	1.200 —7
142	3.492—7	2.396—7	1.659— 7	1.1131—7
155	2.685—7	1.917—7	1.359— 7	9.456 —8
164	2.109—7	1.546—7	1.126— 7	8.074 —8
181	1.686—7	1.246—7	9.409— 8	6.932 —8
194	1.370—7	1.047—7	7.965— 8	5.988 —8
259	5.756—8	4.706—8	3.839— 8	3.111 —8

TABLE III. $\tau = 5.0 \times 10^{-7}$.

x	$h=8$	$h=16$	$h=24$
5.5	2.459—5	—	—
10	—	7.968—7	—
11	—	14.99 —7	—
12	5.994—5	23.18 —6	—
25	2.098—5	6.465—6	7.538—7
38	9.075—6	4.251—6	15.50 —7
51	—	2.587—6	13.28 —7
64	3.525—6	1.644—6	12.55 —7
77	1.557—6	1.092—6	7.327—7
90	1.025—6	7.586—7	5.438—7
129	3.803—7	3.091—7	2.364—7
142	2.892—7	2.407—7	1.972—7
155	2.269—7	1.910—7	1.593—7
168	1.805—7	1.541—7	1.304—7
181	1.460—7	1.260—7	1.081—7
194	1.197—7	1.045—7	0.906—7
259	5.205—8	4.701—8	0.423—7

TABLE IV. $\tau = 10^{-6}$.

x	$h=8$	$h=16$	$h=24$
5.5	3.844—4	—	—
10	—	2.823—5	—
11	—	3.336—5	—
12	1.862—4	3.662—5	—
25	3.665—5	2.034—5	6.946—6
38	1.286—5	8.852—6	5.315—6
51	5.807—6	4.411—6	3.160—6
64	3.094—6	2.504—6	1.950—6
77	1.847—6	1.547—6	1.267—6
90	1.186—6	1.020—6	0.864—6
129	4.209—7	3.794—7	0.332—6
142	3.178—7	2.899—7	0.249—6
155	2.468—7	2.265—7	0.207—6
168	1.951—7	1.803—7	0.166—6
181	1.568—7	1.457—7	0.135—6
194	1.280—7	1.196—7	0.111—6
259	5.473—8	5.202—8	0.050—6

The value $\frac{1}{2}C(q)$ of the integral measures the deviation of the integral spectrum which would be found without the occurrence of the spontaneous decay

$$N_0(q) = \frac{\overline{N_0}}{2q^2}.$$

For high values of q the integrand approaches to y and consequently $C(q)$ to unity.

From measurements with absorbing material, where the influence of the decay may be neglected, the integral energyspectrum is to be found. By means of graphical integration we find the following values of C and τ for the total number of particles at $h = 24$ KM.

τ	10^{-5}	10^{-6}	5×10^{-7}	2.5×10^{-7}	10^{-7}
C	0.80	0.202	0.078	0.026	0.0076

Since further

$$\log N(q) = \log N_0(q) + \log C$$

we may obtain the lifetime τ from the size of the deviation from the straight line in the *log N—log d* diagram ($d =$ thickness of the absorbing layer of water), which was found for great thickness by J. CLAY, A. VAN GEMERT and P. H. CLAY [1]). The deviation from the straight line occurring between 0 and 50 meters of water yields

$$\tau = 1{,}5 \times 10^{-6} \text{ sec.}$$

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[1]) J. CLAY, A. VAN GEMERT and P. H. CLAY, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **41**, 694 (1938).

Mathematics. — *On the solutions of algebraic differential equations.* By K. MAHLER. (Communicated by Prof. J. G. VAN DER CORPUT).

(Communicated at the meeting of December 17, 1938).

Let

$$f(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \dots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

be a convergent or divergent power series with coefficients in a finite algebraic field K, which formally satisfies an algebraic differential equation; i.e. there is a polynomial $F(z, y_0, y_1, \dots, y_m) \not\equiv 0$ in K, such that identically in z [1])

$$F(z, f(z), f'(z), \dots, f^{(m)}(z)) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In his Groningen Thesis [2]), J. POPKEN proved the following

Theorem 1: *There is a positive number c independent of n, such that for all sufficiently large indices, either*

$$\alpha_n = 0, \quad or \quad |\alpha_n| \geqq exp\left(-cn\,(log\, n)^2\right).$$

The proof then given was rather complicated. In this note, I give a simpler proof, which depends on the following results of G. PÓLYA [3]):

Theorem 2: *There is an infinite sequence* $a_0, a_1, a_2, \dots$ *of positive integers, such that all numbers* $a_n \alpha_n$ $(n = 0, 1, 2, \dots)$ *are algebraic integers, and such that*

$$\frac{log\, a_n}{n\,(log\, n)^2} = O(1).$$

Theorem 3: *There is a positive number* c_1, *which does not depend on n, such that for all sufficiently large indices n,*

$$|\alpha_n| \leqq n!^{c_1}.$$

[1]) It suffices to suppose that the TAYLOR coefficients α_n are algebraic and that $f(z)$ satisfies an equation (2). For then, without loss of generality, the coefficients of the polynomial F may be assumed to be algebraic, and therefore the α's can be expressed as rational functions with rational coefficients in a finite number of the α's and in the coefficients of F.

[2]) Amsterdam 1935, N.V. Noord-Hollandsche Uitgeversmaatschappij, Satz 12.

[3]) C. R. **201** (1935), p. 444, first two theorems. I need these theorems only in the special case of rational coefficients α_n.