Nun ist

$$R = (ab)^2\,(cd)^2\,(ad)\,(bc) = (a\,Q)^3$$

also

$$\frac{\partial R}{\partial x_i} = 4 \sum_a \frac{\partial R}{\partial a}\,\frac{\partial a}{\partial x_i} = 4\,(\varphi Q)^3\,\psi_i$$

Da aber jetzt $R = 0$ und $Q = k\,\alpha^3_{dx}$ ist, folgt

$$(\varphi\alpha)^3\,\psi_i = 0.$$

Hieraus folgt also, dass *in diesem Fall I identisch verschwindet.*

Mathematics. — *On the singular series in* WARING's *problem and in the problem of the representation of integers as a sum of powers of primes.* By H. D. KLOOSTERMAN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of January 28, 1939.)

The object of the present note is to give a simple proof of Theorem 2 in G. H. HARDY and J. E. LITTLEWOOD, Some problems of "Partitio Numerorum": IV. The singular series in WARING's problem and the value of the number $G(k)$, Math. Zeitschr. **12** (1922), 166—188. The original proof of HARDY and LITTLEWOOD (as it is also reproduced in LANDAU's Vorlesungen über Zahlentheorie, Band 1, p. 280—285) contained rather complicated arguments about the number of solutions of certain congruences. It is shown here, that these arguments can be replaced by some much simpler lemmas about the generalized GAUSSian sums. The same method of proof is also applicable to the singular series in the problem of the representation of integers as a sum of powers of primes.

Notations. k, s, n, q, a are integers, $k \geqq 3$, $s \geqq 1$, $n \geqq 1$, $q \geqq 1$. p is always a prime. A summation over h, in which h is running through a complete system of residues mod q or a complete system of residues prime to q is denoted by

$$\sum_{h \bmod q} \quad \text{and} \quad \sum_{h \bmod q}{}'$$

respectively. Further we write

$$e_q(x) = e^{\frac{2\pi i x}{q}}$$

and

$$S_q(a) = \sum_{h \bmod q} e_q(a\,h^k), \quad C_q(a) = \sum_{h \bmod q}{}' e_q(a\,h^k), \quad c_q(a) = \sum_{h \bmod q}{}' e_q(a\,h),$$

$$A_q(n) = \sum_{a \bmod q}{}' \left(\frac{S_q(a)}{q}\right)^s e_q(-a\,n)$$

($c_q(n)$ is RAMANUJAN's sum). If p is a prime, then p^θ and p^μ (where θ and μ are integers $\geqq 0$) denote the highest powers of p, which divide k and n respectively and we write

$$\mu = \beta k + \nu$$

(where β and ν are integers $\geqq 0$ and $0 \leqq \nu < k$). The integer γ is defined by

$$\gamma = \theta + 1 \text{ if } p > 2; \qquad \gamma = \theta + 2 \text{ if } p = 2.$$

Then with the exception of the case $k=4$, $p=2$, $\theta=2$ (in which $k=\gamma$) we have always $k > \gamma$ (if $k \geqq 3$). Therefore in any case we have $k \geqq \gamma$. Further let

$$\chi_p = 1 + \sum_{\lambda=1}^{\infty} A_{p^\lambda}(n).$$

Lemma 1. If λ is an integer $> \gamma$, $q = p^\lambda$ and $(a, q) = 1$, then

$$C_q(a) = 0.$$

Proof. Let

$$h = h_1 + h_2 p^{\lambda-\theta-1},$$

where h_1 runs through a complete system of residues prime to $p^{\lambda-\theta-1}$ and h_2 through a complete system of residues mod $p^{\theta+1}$, so that h runs through a complete system of residues prime to p^λ. Then

$$h^k \equiv h_1^k + k h_1^{k-1} h_2 p^{\lambda-\theta-1} \,(mod\, p^\lambda),$$

(LANDAU, Vorlesungen über Zahlentheorie, Band 1, Satz 290) and therefore

$$C_q(a) = \sum_{h_1 \bmod p^{\lambda-\theta-1}}{}' e_q(a h_1^k) \sum_{h_2 \bmod p^{\theta+1}} e_{p^{\theta+1}}(k h_1^{k-1} h_2).$$

The sum over h_2 is zero for every h_1, which proves the lemma.

Lemma 2. Let λ be an integer $\geqq 1$ and $\lambda - 1 = tk + r$, where t and r are integers $\geqq 0$ and $0 \leqq r < k$. Then if $q = p^\lambda$ and $(a, q) = 1$, we have

$$\frac{S_q(a)}{q} = \frac{1}{p^t}\,\frac{S_{p^{r+1}}(a)}{p^{r+1}} = \frac{1}{p^t}\,\frac{C_{p^{r+1}}(a) + p^r}{p^{r+1}}.$$

Proof. Those terms $e_q(a h^k)$ of $S_q(a)$, for which $p^\varkappa$ $(0 \leqq \varkappa \leqq \lambda)$ is the highest power of p, which divides h, are together (if we put $h = h_1 p^\varkappa$):

$$\sum_{h_1 \bmod p^{\lambda-\varkappa}}{}' e_q(a p^{k\varkappa} h_1^k)$$

and therefore

$$p^{\varkappa(k-1)} C_{p^{\lambda-k\varkappa}}(a) \quad \text{or} \quad \varphi(p^{\lambda-\varkappa})$$

according as $0 \leqq \varkappa \leqq t$ or $t < \varkappa \leqq \lambda$. But if $0 \leqq \varkappa < t$, then

$$\lambda - k\varkappa \geqq \lambda - k(t-1) = r + k + 1 \geqq k + 1 > \gamma$$

and therefore it follows from lemma 1, that

$$\frac{S_q(a)}{q} = p^{-\lambda}\,[p^{t(k-1)} C_{p^{\lambda-kt}}(a) + \sum_{\varkappa=t+1}^{\lambda} \varphi(p^{\lambda-\varkappa})] = p^{-t-r-1} C_{p^{r+1}}(a) + p^{-t-1}$$
$$= p^{-t-r-1}\,[C_{p^{r+1}}(a) + p^r] = p^{-t-r-1} S_{p^{r+1}}(a)$$

(since those terms of $S_{p^{r+1}}(a)$ for which h is divisable by p are all equal to 1).

Lemma 3. With the same notations as in lemma 2 we have

$$\begin{aligned} A_q(n) &= p^{t(k-s)} A_{p^{r+1}}(0) && \text{if} \quad 0 \leqq t < \beta; \\ &= p^{\beta(k-s)} A_{p^{r+1}}\left(\frac{n}{p^{k\beta}}\right) && \text{if} \quad t = \beta \quad ; \\ &= 0 && \text{if} \quad t > \beta \quad ; \\ &= p^{t(k-s)-s} c_{p^{r+1}}\left(-\frac{n}{p^{kt}}\right) && \text{if} \quad 0 \leqq t \leqq \beta \quad \text{and} \quad \gamma \leqq r \leqq k. \end{aligned}$$

Proof. According to lemma 2 we have

$$A_q(n) = p^{-ts} \sum_{a \bmod q}{}' \left(\frac{S_{p^{r+1}}(a)}{p^{r+1}}\right)^s e_q(-a\,n).$$

We put

$$a = a_1 + a_2 p^{r+1},$$

where a_1 is running through a complete system of residues prime to p^{r+1} and a_2 through a complete system of residues mod p^{kt}. Then we get:

$$A_q(n) = p^{-ts} \sum_{a_1 \bmod p^{r+1}}{}' \left(\frac{S_{p^{r+1}}(a_1)}{p^{r+1}}\right)^s e_q(-a_1 n) \sum_{a_2 \bmod p^{kt}} e_{p^{kt}}(-a_2\,n).$$

The inner sum is zero unless $n \equiv 0 \,(\text{mod } p^{kt})$, that is if $t \leqq \beta$, and then it is p^{kt}. Therefore

$$A_q(n) = p^{t(k-s)} A_{p^{r+1}}\left(\frac{n}{p^{kt}}\right).$$

But if $0 \leqq t < \beta$, then

$$A_{p^{r+1}}\left(\frac{n}{p^{kt}}\right) = A_{p^{r+1}}(0)$$

and the first three statements of the lemma are proved. In order to prove the fourth statement of the lemma, we observe, that for $r \geqq \gamma$ we have in consequence of lemma 1:

$$C_{p^{r+1}}(a) = 0$$

and therefore lemma 2 gives

$$\frac{S_q(a)}{q} = \frac{1}{p^{t+1}}.$$

Hence

$$A_q(n) = \sum_{a \bmod q}' p^{-s(t+1)} e_q(-an) = p^{-s(t+1)} c_q(-n) = p^{t(k-s)-s} c_{p^r+1}\left(-\frac{n}{p^{kt}}\right).$$

Theorem. If $\lambda > Max(\beta k + \nu + 1,\ \beta k + \gamma)$, then $A_{p^\lambda}(n) = 0$. We have also

$$\chi_p = p^{(1-s)\gamma} \sum_{t=0}^{\beta-1} p^{t(k-s)} N(p^\gamma, 0) + p^{(1-s)\gamma+\beta(k-s)} N\left(p^\gamma, \frac{n}{p^{k\beta}}\right),$$

where (if m is an integer and α is an integer $\geqq 1$) $N(p^\alpha, m)$ is the number of solutions of the congruence

$$h_1^k + h_2^k + \ldots + h_s^k \equiv m \quad (mod\ p^\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for which not every $h_i\ (i = 1, 2, \ldots, s)$ is divisable by p. (If $\beta = 0$, then $\sum_{t=0}^{\beta-1}$ is to be replaced by 0).

Proof. If $\lambda \geqq 1 + (\beta + 1)k$, then the first statement is contained in lemma 3. Now let

$$\lambda = \beta k + r + 1 \quad (0 \leqq r < k).$$

Then it follows from lemma 3, that if $\lambda \geqq \beta k + \gamma + 1$, that is if $r \geqq \gamma$, we have

$$A_{p^\lambda}(n) = p^{\beta(k-s)-s} c_{p^r+1}\left(-\frac{n}{p^{k\beta}}\right).$$

Since the RAMANUJAN sum is 0 for $r > \nu$, the first part of the theorem is proved.

In order to prove the second part of the theorem, we obtain from lemma 3:

$$\left.\begin{aligned} \chi_p &= 1 + \sum_{t=0}^{\beta-1} \sum_{r=0}^{k-1} A_{p^{kt+r+1}}(n) + \sum_{r=0}^{k-1} A_{p^{\beta k+r+1}}(n) = \\ &= 1 + \sum_{t=0}^{\beta-1} p^{t(k-s)} \sum_{r=0}^{k-1} A_{p^{r+1}}(0) + p^{\beta(k-s)} \sum_{r=0}^{k-1} A_{p^{r+1}}\left(\frac{n}{p^{k\beta}}\right). \end{aligned}\right\} \quad . \quad (2)$$

Next we consider the sum

$$\sigma = \sum_{h_i \bmod p^\gamma} \sum_{a \bmod p^\gamma} e_{p^\gamma}((h_1^k + h_2^k + \ldots + h_s^k - m)a),$$

where m is an integer and the $h_i\ (i = 1, 2, \ldots s)$ and a each run through a complete system of residues mod p^γ. Those terms of σ, for which every h_i is divisable by p are together equal to

$$p^{(\gamma-1)s} \sum_{a \bmod p^\gamma} e_{p^\gamma}(-ma).$$

Since

$$\sum_{a \bmod p^\gamma} e_{p^\gamma}((h_1^k + h_2^k + \ldots + h_s^k - m)a)$$

is zero unless

$$h_1^k + h_2^k + \ldots + h_s^k \equiv m \qquad (mod\ p^\gamma),$$

in which case it is p^γ, we get

$$\sigma = p^\gamma N(p^\gamma, m) + p^{(\gamma-1)s} \sum_{a \bmod p^\gamma} e_{p^\gamma}(-ma).$$

Again, collecting those terms of σ, for which

$$(a, p^\gamma) = p^{\gamma-\lambda} \qquad (0 \leqq \lambda \leqq \gamma)$$

we have also

$$\sigma = p^{\gamma s}(1 + A_p(m) + A_{p^2}(m) + \ldots + A_{p^\gamma}(m)).$$

Equating the two expressions for σ we get

$$\sum_{r=0}^{\gamma-1} A_{p^{r+1}}(m) = -1 + p^{\gamma(1-s)} N(p^\gamma, m) + p^{-s} \sum_{a \bmod p^\gamma} e_{p^\gamma}(-am) \quad . \quad . \quad (3)$$

Further it follows from lemma 3, that

$$\sum_{r=\gamma}^{k-1} A_{p^{r+1}}(m) = p^{-s} \sum_{r=\gamma}^{k-1} c_{p^{r+1}}(-m) = p^{-s} \sum_{a \bmod p^k} e_{p^k}(-am) - p^{-s} \sum_{a \bmod p^\gamma} e_{p^\gamma}(-am) \ (4)$$

and the same lemma gives also

$$\sum_{r=0}^{k-1} A_{p^{r+1}}(m) = -1 + p^{\gamma(1-s)} N(p^\gamma, m) + p^{-s} \sum_{a \bmod p^k} e_{p^k}(-am) \quad . \quad (5)$$

Subtracting (4) from (5) and substituting the result in (3) we find:

$$\chi_p = 1 + \sum_{t=0}^{\beta-1} p^{t(k-s)}(p^{k-s} - 1) + p^{(1-s)\gamma} \sum_{t=0}^{\beta-1} p^{t(k-s)} N(p^\gamma, 0) +$$

$$+ p^{(1-s)\gamma+\beta(k-s)} N\left(p^\gamma, \frac{n}{p^{k\beta}}\right) + p^{\beta(k-s)}\left(-1 + p^{-s} \sum_{a \bmod p^k} e_{p^k}\left(-a\frac{n}{p^{k\beta}}\right)\right).$$

Since

$$\frac{n}{p^{k\beta}} \not\equiv 0 \quad (mod\ p^k) \quad \text{and} \quad \sum_{t=0}^{\beta-1} p^{t(k-s)}(p^{k-s} - 1) = p^{\beta(k-s)} - 1$$

the theorem now follows immediately.

The corresponding theorem for the problem of the representation of integers as a sum of powers of primes is rather more simple. Let [1])

$$A'_q(n) = \sum_{a \bmod q}' \left(\frac{C_q(a)}{\varphi(q)}\right)^s e_q(-an), \quad \chi'_p = 1 + \sum_{\lambda=1}^{\infty} A'_{p^\lambda}(n).$$

Then we have the

Theorem. If $N'(p^\alpha, m)$ is the number of solutions of the congruence (1) for which all $h_i\,(i = 1, 2, \ldots s)$ are prime to p, then

$$\chi'_p = p^\gamma (\varphi(p^\gamma))^{-s} N'(p^\gamma, m).$$

Proof. From lemma 1 we get

$$\chi'_p = 1 + \sum_{\lambda=1}^{\gamma} A'_{p^\lambda}(n).$$

We now consider the sum

$$\sigma' = \sum_{h_i \bmod p^\gamma}' \quad \sum_{a \bmod p^\gamma} e_{p^\gamma}\left((h_1^k + h_2^k + \ldots + h_s^k - n)\, a\right).$$

Then in the same way as shown above for σ, we now find for σ' the two expressions

$$\sigma' = (\varphi(p^\gamma))^s \chi'_p = p^\gamma N'(p^\gamma, m)$$

and the theorem follows immediately.

[1]) I. VINOGRADOFF. Einige allgemeine Primzahlsätze. Travaux de l'Institut math. de Tbilissi 3, 1—67 (1938) (in Russian and in German).

Chemistry. — *On dissymmetrical synthesis in the case of complex metallic salts.* III. By I. LIFSCHITZ [1]). (Communicated by Prof. F. M. JAEGER.)

(Communicated at the meeting of January 28, 1939.)

§1. If a luteo-cobaltic salt be prepared from an optically inactive Co-salt and an optically active diamine — e.g. *l-propylene-diamine* (= *l-pn*), — a relatively-dissymmetrical reaction takes place and the obtained reaction-product is totally (or double) active. It might be expected to consist of a mixture of two optical isomers, viz. I. $[Co(l\text{-}pn)_3]_D X_3$ and II. $[Co(l\text{-}pn)_3]_L X_3$. However, in all cases so far examined only one single salt was obtained. Nevertheless, it may be easily demonstrated that a relatively dissymmetrical synthesis has, indeed, taken place. The obtained luteo-salt, namely, appears to possess a rotatory dispersion of a perfectly similar nature as $[Co(en)_3]_D X_3$, — which salt owes its activity exclusively to the axial symmetry of the complex ion. Moreover, already the occurrence of a COTTON-effect in the visible spectrum proves that the above-mentioned luteo-salt must be totally-active; for complexes, which are only partially-active with regard to the diamine, never manifest such a COTTON-effect [2]).

Totally different conditions are met with in the case of the complex cobaltic salts with *three* mol. of an optically-active α-amino-acid, e.g. $[Co(d\text{-}alanine)_3]$. A comparison with optically-active $[Co(glycine)_3]$ is impossible here, as the latter salt has *not* yet been obtained in the optically-active form; while, on the other hand, it is not known whether a complex-bound, optically-active α-amino-acid present in of themselves racemic complexes, causes a COTTON-effect to display itself or not. The occurrence of a relatively-dissymmetrical synthesis, consequently, could only be considered as rigorously proved in the case of these tri-α-amino-acid cobaltiates, if of the four theoretically possible reaction-products (α and β forms are geometrical isomers): III. α-$[Co(d\text{-}alan)_3]_D$; IV. α-$[Co(d\text{-}alan)_3]_L$; V. β-$[Co(d\text{-}alan)_3]_D$; VI. β-$[Co(d\text{-}alan)_3]_L$, either all four, or three, or at least III and IV or V and VI could be shown to be present in the reaction-product and could be isolated from it.

In the preceding communications the presence of a relatively-dissymmetrical synthesis could, indeed, in this way be proved in the case of the complex-formation with three mol. *d-alanine* or *d-glutaminic acid*. The results of these and not yet published researches [3]) now justify the

[1]) Cf. Proc. Kon. Akad. v. Wetensch., Amsterdam, 27, Nos. 9 and 10 (1924); ibid. 39, Nr. 10 (1936).

[2]) Cf. A. WERNER, Helv. Chim. Act., 1, 5 (1918).

[3]) From not yet completed experiments it was found, that also in the case of complexes with asparaginic acid a similar argumentation is feasible.