investigation, I am inclined to think that an eye vesicle under ectoderm can induce numerous lenses, if by the growth of neighbouring organs it is placed in different positions and, now in this place, then in another, comes into contact with the ectoderm. My previous investigation, namely, revealed that the eye vesicle during a long time possesses the capacity to induce lenses. The lenses, induced in this way, will develop differently, according as the eye sooner or later loses contact with the lens anlage.

If during that process the eye vesicle is turned, it may take the induced lens along with it. The latter, if replaced as well, may later give the incorrect impression that it has been induced out of other material than ectoderm. Thus the results of POPOFF (1937) might be explained. He thought, namely, that he had found lens formation out of different tissues if an abnormally orientated eye vesicle came into contact with them. Only his recent researches (grafting of different tissues into the cavity of the eye cup) can produce evidence in favour of his opinion.

Besides, the fact that a lens is found against the tapetum of the eye (POLITZER) may not lead to the conclusion that this layer also can induce a lens.

Finally I will mention that in *Triton taeniatus* I repeatedly noticed lens formation out of the margin of the iris and even directly out of the retina of the grafted eye vesicles. Nevertheless, these eyes had also induced lenses out of ectoderm. The presence of a regenerated lens consequently does not deprive the eye of its capacity to induce lenses.

The phenomenon that an eye vesicle, grafted at a very young stage without a lens rudiment, regenerates a lens out of its retina or iris may also account for some cases described in the literature of so-called lens induction out of other tissues than ectoderm. The observed lenses might have been regenerated lenses. In *Axolotl* and *Rana esculenta* I observed these regenerated lenses only rarely, but there is no doubt that this type of lens formation occurs. In *Triton taeniatus* the phenomenon may be frequently observed.

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Physics. — *Some considerations on the fields of stress connected with dislocations in a regular crystal lattice.* I. By J. M. BURGERS. (Mededeeling N⁰. 34 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hoogeschool te Delft.)

(Communicated at the meeting of January 28, 1939.)

1. In order to explain the mechanism of plastic deformation of a crystal in its most simple form, as it is presented by the shearing process due to slipping along planes of a definite crystallographic orientation, several authors have assumed that the basic phenomenon leading to slip is the migration through the lattice of a well defined type of deviation from the ideal structure, a so-called *dislocation* [1]).

It has been in particular TAYLOR who has investigated the characteristic properties of an elementary, two-dimensional type of dislocation, the possibilities for its displacement through the lattice, and the influence of the fields of stress connected with a system of such dislocations upon this displacement [2]). An account of some of the results of this work, together with suggestions for certain modifications which made it possible to construct a connection with views developed by BECKER and by OROWAN, has been given by W. G. BURGERS and the present author in the "First Report on Viscosity and Plasticity", pp. 199 and seq. The problem, however, presented itself whether the two-dimensional type of dislocation, which must extend in a straight line through the lattice from one boundary surface of the crystal to the opposite boundary, really leads to an appropriate description of what is to be found in an actual crystal; it would appear that dislocations characterized by disturbances of a more general, three-dimensional type, which for instance may be confined to a region of finite extent, might lead to a more adequate picture [3]). It is the object of the following pages to make a few contributions towards the development

[1]) Compare: "First Report on Viscosity and Plasticity" (Verhand. Kon. Nederl. Akad. v. Wetenschappen te Amsterdam, 1e sectie, XV, No. 3, 1935), p. 198 and the literature mentioned there; "Second Report on Viscosity and Plasticity" (ibidem, XVI, No. 4, 1938), p. 200.

See also papers by A. KOCHENDÖRFER, Zeitschr. f. Physik **108**, p. 244, 1938 and Zeitschr. f. Metallkunde **30**, p. 299, 1938.

[2]) G. I. TAYLOR, Proc. Roy. Soc. (London) **A 145**, p. 362, 1934.

[3]) "Second Report on Viscosity and Plasticity", p. 201. — KOCHENDÖRFER in his second paper (see footnote 1, above) alludes to the same problem; however, the few lines devoted by him to this matter (*l.c.* p. 300, second column) apparently are not based upon a sufficiently developed investigation of the geometric features of dislocations of three-dimensional type.

of such a picture, by investigating some of the geometrical relationships presented by dislocations of three-dimensional nature, and developing expressions for the fields of stress connected with them. It must be remarked that the treatment is of a preliminary character, and the reader will note several points where further work will be necessary.

2. *Introductory geometrical considerations on dislocations of various form.* — A schematical picture of a lattice with a two-dimensional "unit dislocation" of the type as considered by TAYLOR, is given in fig. 1. The

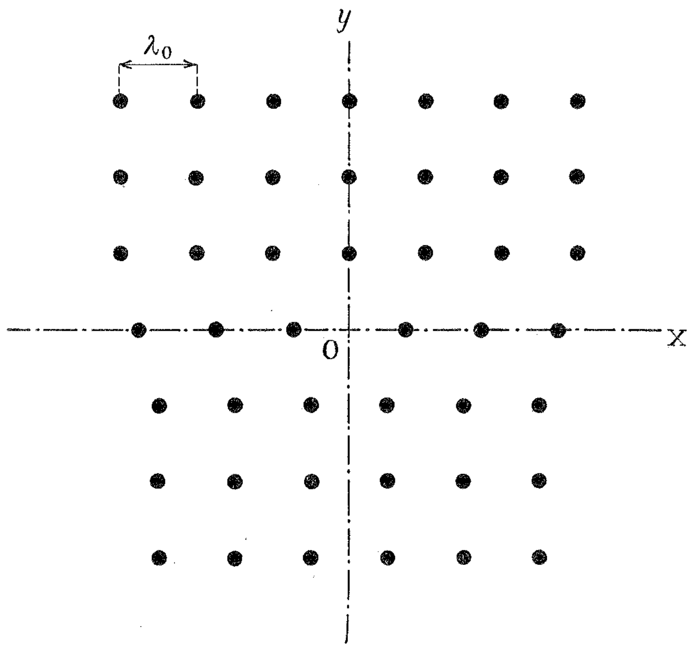

Fig. 1. Elementary type of a two-dimensional dislocation, having the z-axis (perpendicular to the plane of the paper) as its singular line.

disturbance presented by the lattice in this case can be described by stating that above a definite horizontal plane, say the x, z-plane, every row of atoms parallel to the x-axis contains one atom more than every row below this plane. The dislocation can be obtained by imagining the lattice to be cut along the upper half of the y, z-plane (*i.e.* the half plane $x = 0$, $y > 0$), and inserting an extra layer of atoms into this cut.

It will be evident that the deformations appearing in the lattice in consequence of this process (*i.e.* the deviations from their original cubical form, which are shown by the cells of the lattice) decrease indefinitely with increasing distances from the z-axis. Instead of the deformations of the cells we will consider the *displacements* of the atoms from their normal positions. When the components of the displacement are denoted by u, v, w, it will be seen that although these quantities in reality are defined only for the (enumerable) set of lattice points where atoms are to be found,

they can be considered as being determined by functions of the coordinates x, y, z, which in general are everywhere continuous and finite. It is only at the points of the z-axis and its immediate neighbourhood, that these functions lose their meaning; moreover, in the case of the function giving the values of u the following point is to be noted: When in the half plane $x = 0$, $y < 0$ we assume $u = 0$, then in the region where y is positive we shall find that u approaches to the value $+\frac{1}{2}\lambda_0$ for $x > 0$, and to the value $-\frac{1}{2}\lambda_0$ for $x < 0$; hence the corresponding function will be discontinuous at the points of the half plane $x = 0$, $y > 0$. The explicit introduction of this discontinuity into the function, however, leads to unnecessary complications in the analytical treatment of the problems before us: it would lead to an infinite value of the derivative $\partial u/\partial x$ at the points of the half plane, which is cumbersome as the actual deviation of the lattice cells from their normal form is finite here and in fact approaches to zero. It is more convenient therefore to consider the function defining u as a function of the coordinates which is continuous also at the half plane $x = 0$, $y > 0$, and which consequently is continuous through the whole of space, with the exception only of the z-axis; then there is no complication in the expression of deformations by means of the derivatives of this function. It is to be noted, however, that the function thus defined ceases to be a single-valued function: when we describe a closed circuit around the z-axis, considering u as a continuous function of the coordinates, then on coming back to our starting point we shall find that u will have either increased, or decreased by the amount λ_0.

The functions giving the values of v and w, on the other hand, although they likewise cease to have a meaning at the points of the z-axis, are single-valued throughout the whole of space.

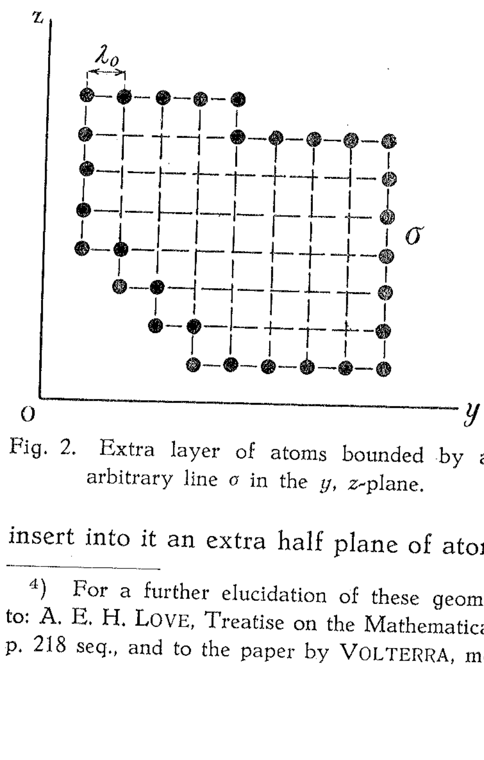

Fig. 2. Extra layer of atoms bounded by an arbitrary line σ in the y, z-plane.

The result arrived at can be expressed by saying that the function $u\,(x, y, z)$ possesses a cyclic constant for every closed line embracing the z-axis, which constant has the value λ_0.

The z-axis consequently is to be considered as a singular line for the dislocation, which transforms the space around it into a multiply-connected region [4]).

3. Instead of making a cut of half-infinite extent in order to insert into it an extra half plane of atoms, we may also imagine that a cut

[4]) For a further elucidation of these geometrical relationships the reader is referred to: A. E. H. LOVE, Treatise on the Mathematical Theory of Elasticity (Cambridge 1920), p. 218 seq., and to the paper by VOLTERRA, mentioned in footnote 8) below.

is made over a finite area Σ of the plane $x=0$ (or of a plane $x=$constant), lying wholly inside the crystal, and insert into it an extra layer of atoms. The boundary line σ of the cut, or rather of the extra layer of atoms introduced into the lattice, will consist of segments of rows of atoms, alternately parallel to the y-axis and to the z-axis (compare fig. 2); in the geometrically simplest case it may be a rectangle, but when observed on a scale large compared with the atomic distance λ_0, it may be of any form.

In this case again the components v and w can be represented by single-valued functions of the coordinates, whereas u can be described by a many-valued function, with the cyclic constant λ_0 for every line embracing the boundary line σ, which now is the singular line of the field.

4. It is possible to imagine a dislocation of another character, in which the many-valued function again represents the u-component of the displacement, but in which the singular line is the x-axis. In order to obtain such a case a discontinuity is introduced in the junction of the half-planes $x=const.$, $y<0$ with the half-planes $x=const.$, $y>0$, by making a shift of one atomic distance in passing from the region $z<0$ into the region $z>0$. Then, as indicated schematically in fig. 3, it will be found in moving along a line $x=const.$, $z=const.$, that the component u increases by the amount $\frac{1}{2}\lambda_0$ if $z>0$, whereas it decreases by the same amount if $z<0$.

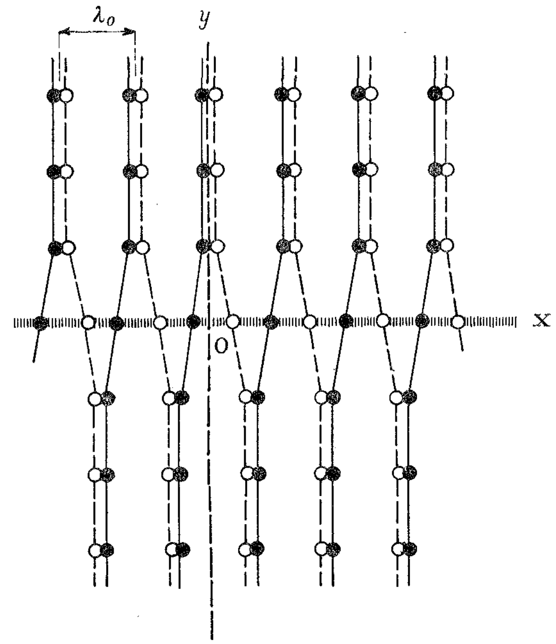

Fig. 3. Schematical picture of a dislocation having the x-axis as singular line. Continuous lines indicate rows of atoms above the x, y-plane; broken lines indicate rows of atoms below this plane.

It must be remarked that the vertical rows of atoms in general will not remain perfectly straight and parallel to the z-axis. From reasons of sym-

metry with respect to the x-axis we might expect that we should observe an increase of u by something like $\frac{1}{4}\lambda_0$ when we move in the direction of $+z$ along a row for which $y<0$; then in moving along a horizontal row in the direction of $+y$ there should be observed a further increase of u by $\frac{1}{4}\lambda_0$; next going downwards along a vertical row for which $y>0$ into the region $z<0$ there should again be an increase by a similar amount, etc. The exact calculation shows that u increases proportionally to the angle described around the x-axis, as will be seen from eq. (27) below.

5. In this way we see that it is possible for singular lines to run parallel to any one of the three coordinate axes. The case last considered may be combined with the other cases; an example is indicated schematically in fig. 4. Here on the right hand side of the plane $x=0$ we have the same

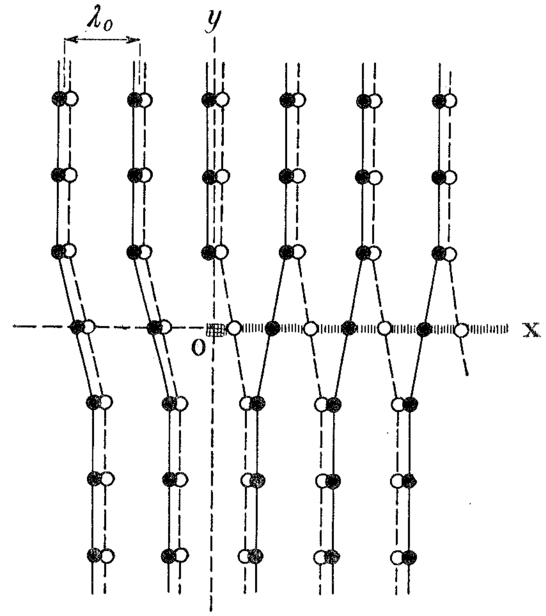

Fig. 4. Schematical picture of a dislocation with a singular line consisting of the positive x-axis together with the positive z-axis (directed upwards perpendicularly to the plane of the paper).

type of dislocation as sketched in fig. 3, with the positive x-axis as the singular line; an extra layer of atoms, however, has been introduced along the quarter plane $x=0$, $y>0$, $z>0$, in consequence of which there are no discontinuities in the region $x<0$, only deformations which will gradually decrease as we go further away in the direction of $-x$. The positive half of the z-axis now has become a singular line, being in fact the continuation of the segment which was formed by the positive x-axis.

Another case is indicated schematically in fig. 5a, which is obtained in the following way: In the x, y-plane a rectangle is imagined with sides $2a$, $2b$ respectively. This rectangle will intersect a number of layers of atoms which in the undisturbed state of the lattice were parallel to the plane $x = 0$. These layers are assumed to be cut along the lines of intersection (all cuts lying in the plane $z = 0$, and extending from $y = -b$ to $y = +b$); in joining them together again a shift of amount λ_0 has been introduced in the way as indicated in fig. 5*b* (representing a section according to the x, z-plane). The half plane $x = -a$, $z < 0$ then will possess a free upper border σ_1 along the segment extending from $y = -b$

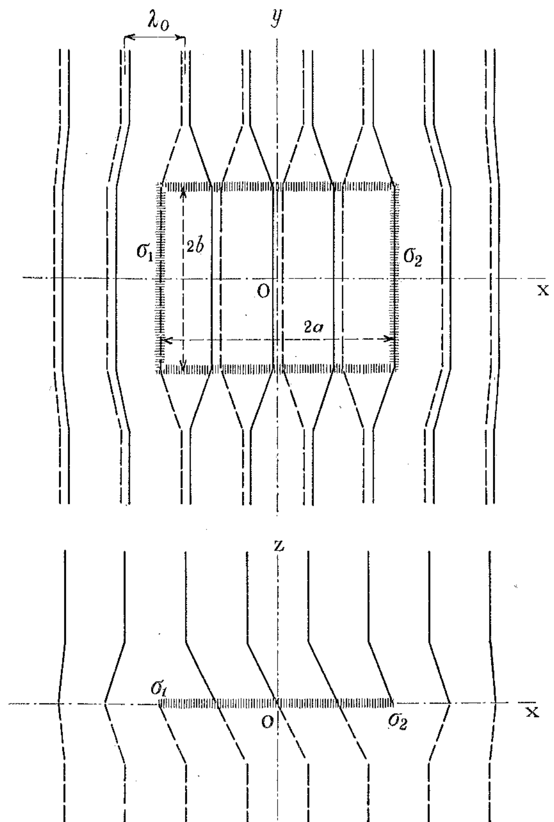

Fig. 5*a*.

Fig. 5*b*.

Fig. 5. Schematical picture of a dislocation with a singular line in the form of a rectangle in the x, y-plane. Fig. 5*a*: view in the direction of the negative z-axis; fig. 5*b*: section by the plane Oxz.

to $y = +b$; the half plane $x = +a$, $z > 0$ has a free lower border σ_2 along a segment of similar extent.

The singular line in this case is formed by the four sides of the rectangle, the segments σ_1 and σ_2 being two of these sides.

It must be stated, of course, that in actual cases the discontinuities possibly may not have the rather simple form assumed in the diagrams given here: there may be regions of irregular atomic arrangement, affecting several rows of atoms in the neighbourhood of what we have called the singular line. However, what is most important in every case is the mode

of connection between the planes or rows of atoms at larger distances from the disturbed region, and for the sake of simplicity in the mathematical formulation it is convenient to keep to the picture of singular lines as determining the geometry of the field.

We now may generalise to cases where the singular line consists of an arbitrary sequence of segments, each of which is parallel to one of the coordinate axes. Again viewing from a scale which is large compared with atomic distances, we may consider such a singular line as being of arbitrary form in space.

One important property of these singular lines, however, must be noticed at once: they can never end at an interior point of the lattice, and must be either closed in themselves, or extend from a point of the exterior surface to another point of this surface or to infinity, or from infinity to infinity.

6. *The field of stress accompanying a dislocation.*—It has been observed by TAYLOR [5]) that although it is not possible to calculate in a rigorous way the forces experienced by the atoms in the immediate neighbourhood of the singular line, at greater distances the mean stresses per unit area can be found with the aid of the equations of the theory of elasticity. In order to arrive at exact results it is necessary to make use of the equations valid for crystalline substances. Even in the case of substances of the regular class these equations are more complicated than those valid for isotropic bodies, the number of constants occurring in them being three, instead of two, while a still greater number occurs in the equations for crystalline substances of other classes [6]). The application of these equations consequently will lead to elaborate expressions, which are not easily handled. It will be useful, therefore, first to develop a provisional treatment, based upon the ordinary theory of elasticity for isotropic bodies, for which the mathematical technique has been built out much further. The results obtained in this way can give an insight into the principal features of the subject [7]), while the application of the exact equations for regular crystals will be considered afterwards in Part II.

The concept of dislocations (originally called "distortions"; the name dislocations is due to LOVE) was introduced into the theory of elasticity by VOLTERRA in order to describe the deformation that can be found in a body occupying a multiply-connected region, when the displacements of the points are given by many-valued functions of the coordinates, there being no exterior forces (neither volume-forces, nor surface-forces) acting on the body [8]). In our case this multiply-connected region is the space

5) G. I. TAYLOR, *l.c.* p. 375.

6) See A. E. H. LOVE, *l.c.* Chapter VI, and works on physical crystallography.

7) See G. I. TAYLOR, *l.c.* p. 377.

8) V. VOLTERRA, Sur l'équilibre des corps élastiques multiplement connexes, Ann. Ecole Norm. Supér. (3) **24**, p. 400, 1907; A. E. H. LOVE, *l.c.* p. 218. — A report on various types of structural stresses in elastic systems has been given by P. NEMÉNYI

which is obtained when thin cores are cut away from the body along the singular lines considered before.

In the following discussion we assume the crystal to be of infinite extent in all directions, and — unless especially stated — we restrict to singular lines of finite extent, and consequently closed in themselves.

The boundary conditions for the field of stress in this case require that all stress components shall vanish at infinity, in such a way that no resultant force nor resultant moment is transmitted through any plane which recedes to infinity. In virtue of the equations of elastic equilibrium these conditions at the same time ensure that there will be no resultant force or moment acting upon the matter in the immediate neighbourhood of the system of singular lines considered as a whole. It will be natural, however, to introduce the more stringent condition that the resultant force and moment must vanish for any one of the singular lines separately. And generally we must go still further: when the disturbed region along the singular line is of the nature of a relatively thin core, it is inconceivable that forces of considerable magnitude can be transmitted along it from one part to another. Hence, when we consider a small element of this core, bounded *e.g.* by a cylindrical surface having an element of the singular line as axis, the stresses acting on the cylindrical surface must balance each other already very nearly, so that there will remain only residues of an order of magnitude vanishing at the same time as the radius of the cylinder.

7. *General mathematical expressions for the components of the displacement in an elastic body, connected with a given dislocation.* — The general expressions for the displacement components have been deduced by VOLTERRA. However, before giving VOLTERRA's equations we will follow a more synthetic way, which will make clear the meaning of the various terms of these equations.

We start from a set of formulae giving the components of the displacement due to a force operating at a point in an indefinitely extended body [9]). The displacement may be considered as being the sum of two parts, one part having the same direction as the force and being equal in magnitude to the force divided by $4\pi\mu r$, while the second part can be written as the gradient of a certain function ψ: [10])

$$u_k = \frac{F_k}{4\pi\mu r} + \frac{\partial\psi}{\partial x_k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(Selbstspannungen elastischer Gebilde, Zeitschr. f. angew. Math. u. Mech. **11**, pp. 59—70, 1931).

[9]) A. E. H. LOVE, *l.c.* p. 183, eqs. (11). As stated by LOVE at p. 181, these equations originally are due to W. THOMSON.

[10]) For convenience we write x_1, x_2, x_3 for the coordinates; u_1, u_2, u_3 for the components of the displacement, etc. The force is acting at the point ξ_1, ξ_2, ξ_3 and

$$r^2 = (x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2.$$

The quantity μ is one of the two elastic constants (μ, λ) characteristic of an isotropic

the F_k representing the components of the force. For the present it is not necessary to give the "complementary function" ψ explicitly; it must be introduced in order to ensure that the equations of the theory of elasticity shall be fulfilled, but, as we shall see later, this can be done afterwards, so that we are entitled to leave it aside until further consideration. It may be remarked that whereas the first part (the "principal" part) of u_k satisfies the equation

$$\triangle\left(\frac{F_k}{4\pi\mu r}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the "complementary function" ψ is subjected to the equation:

$$\triangle\triangle\psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now it will have been seen from sections **2**—**5** that the condition expressing the multi-valuedness of the displacement component u in the cases considered is of the same kind as that of the potential φ associated with the velocity field determined by a *vortex line*, coinciding with the singular line σ. In the hydrodynamical case the cyclic constant of the potential function for every closed line embracing the vortex line is equal to the strength of the vortex line, which thus in our case should be numerically equal to λ_0. — In a more general case, where all three components u_1, u_2, u_3 may be multi-valued, we shall introduce three cyclic constants f_1, f_2, f_3.

We may, therefore, begin by tentatively writing down the following formula for the "principal" part of the components u_k:

$$u_k^* = f_k\,\varphi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where φ is the hydrodynamic potential for a vortex line of unit strength, coinciding with the singular line σ characteristic of the dislocation. The value of φ is equal to the solid angle which a surface Σ bounded by the line σ subtends at the point of the field considered, divided by 4π; it can also be represented by the integral [11]):

$$\varphi = \frac{1}{4\pi}\iint d\Sigma\,\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

ν being the normal to the element $d\Sigma$, drawn in the direction determined by that side of the surface Σ which is considered as the positive side.

8. Formula (5) induces us to interprete the components u_k^* considered in (4) as being due to a system of imaginary *doublets*, distributed over the surface Σ, the axis of the doublets everywhere being normal to $d\Sigma$, whereas the strength (the "moment") of the doublets has the components μf_k.

medium, as used by LOVE and other writers; μ is equal to the shear modulus G, while the ordinary modulus of elasticity (YOUNG's modulus) is given by $E = \mu\,(3\lambda + 2\mu)/(\lambda + \mu)$, the compression modulus being $H = \lambda + 2\mu/3$. POISSON's ratio $1/m$ of the lateral contraction to the longitudinal extension in an ordinary extension experiment is determined by: $m = 2(\lambda + \mu)/\mu$.

[11]) See H. LAMB, Hydrodynamics (Cambridge, 1932), p. 212.

As every doublet consists of two forces of equal and opposite magnitude, the resultant force due to the system is zero. There will be, however, a resultant moment, and it is not difficult to prove that the components of this moment are given by the expressions:

$$\left.\begin{aligned} &\mu\,(A_2 f_3 - A_3 f_2) \\ &\mu\,(A_3 f_1 - A_1 f_3) \\ &\mu\,(A_1 f_2 - A_2 f_1) \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (6)$$

where A_1, A_2, A_3 resp. represent the area's of the projections of Σ upon the three coordinate planes, taken with such signs that $A_k > 0$ when the normal ν to Σ is in the direction of the positive x_k-axis. Our force system consequently does not represent an equilibrium system.

In order to balance this moment, we introduce a system of imaginary forces $\mu\, g_k$ acting at the points of the boundary line, where:

$$\left.\begin{aligned} g_1 &= f_2 \frac{d\xi_3}{d\sigma} - f_3 \frac{d\xi_2}{d\sigma} \\ g_2 &= f_3 \frac{d\xi_1}{d\sigma} - f_1 \frac{d\xi_3}{d\sigma} \\ g_3 &= f_1 \frac{d\xi_2}{d\sigma} - f_2 \frac{d\xi_1}{d\sigma} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (7)$$

It is easily proved that this system yields a resultant force equal to zero, while it has a resultant moment which is the exact opposite of that given by eqs. (6). Consequently as a second contribution to the "principal" part of the components u_k we take the expressions:

$$u_k^{**} = \frac{1}{4\pi} \int d\sigma \, \frac{g_k}{r} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (8)$$

The whole system then will be balanced.

It is of importance to observe that formula (8) also can be written in the form of an integral over the surface Σ, as follows [12]:

$$u_k^{**} = \frac{1}{4\pi} \iint d\Sigma \left\{ (\nu k)\, f_l \frac{\partial}{\partial \xi_l}\left(\frac{1}{r}\right) - f_\nu \frac{\partial}{\partial \xi_k}\left(\frac{1}{r}\right) \right\} . \; . \; . \qquad (9)$$

9. We now turn to the determination of the "complementary function", to be denoted by Ψ. We put:

$$u_k = u_k^{*} + u_k^{**} + \frac{\partial \Psi}{\partial x_k} \quad . \; . \; . \; . \; . \; . \; . \; . \qquad (10)$$

[12]) In this equation and in the following ones it is assumed that when in a product or in a differential quotient an index, like l, occurs twice, summation is to be performed with respect to $l = 1, 2, 3$. — The quantities (νk) are the cosines of the angles between the normal ν to $d\Sigma$ and the coordinate axes, and $f_\nu = (\nu l) \cdot f_l$ (component of f_k normal to $d\Sigma$).

The dilatation θ then is given by:

$$\theta = \frac{\partial u_k}{\partial x_k} = \frac{\partial u_k^{*}}{\partial x_k} + \frac{\partial u_k^{**}}{\partial x_k} + \triangle\, \Psi \quad . \; . \; . \; . \; . \qquad (11)$$

The components u_k must satisfy the equations of the theory of elasticity:

$$\mu \triangle u_k + (\lambda + \mu) \frac{\partial \theta}{\partial x_k} = 0 \, . \; . \; . \; . \; . \; . \; . \; . \qquad (12)$$

As both $\triangle u_k^{*} = 0$ and $\triangle u_k^{**} = 0$, this equation will be satisfied, provided:

$$\mu \triangle \Psi + (\lambda + \mu)\, \theta = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \qquad (13)$$

from which it follows that Ψ must satisfy the equation:

$$\triangle\, \Psi = -\frac{\lambda + \mu}{\lambda + 2\mu} \left(\frac{\partial u_k^{*}}{\partial x_k} + \frac{\partial u_k^{**}}{\partial x_k} \right) \quad . \; . \; . \; . \; . \qquad (14)$$

Now from (4), combined with (5), and from (9) it is found that:

$$\frac{\partial u_k^{*}}{\partial x_k} = \frac{\partial u_k^{**}}{\partial x_k} = -\frac{1}{4\pi} \iint d\Sigma \, \frac{\partial}{\partial \nu} \left\{ \frac{f_l\,(x - \xi_l)}{r^3} \right\} \quad . \; . \; . \qquad (15)$$

The solution of (14) therefore can be given in the form:

$$\Psi = -\frac{\lambda + \mu}{4\pi(\lambda + 2\mu)} \iint d\Sigma \, f \, \frac{\partial^2 r}{\partial \nu \, \partial x_l} + \Psi' \, . \; . \; . \; . \qquad (16)$$

where Ψ' is a function which satisfies the equation $\triangle \Psi' = 0$. This function must be determined in such a way that the function Ψ shall not present a discontinuity at the surface Σ. It is found that this is obtained by taking:

$$\Psi' = -\frac{\lambda + \mu}{4\pi(\lambda + 2\mu)} \iint d\Sigma \, \frac{2 f_\nu}{r} \quad . \; . \; . \; . \; . \qquad (17)$$

so that after a slight reduction there results:

$$\Psi = -\frac{\lambda + \mu}{4\pi(\lambda + 2\mu)} \iint d\Sigma \left[\frac{f_l\,(\nu k)\,(x_k - \xi_k)\,(x_l - \xi_l)}{r^3} + \frac{f_\nu}{r} \right] . \qquad (18)$$

It is interesting to remark that Ψ also can be represented by a line integral taken along σ. We introduce a vector $\mu\, h_k$ with the components:

$$\left.\begin{aligned} h_1 &= f_2 \frac{x_3 - \xi_3}{r} - f_3 \frac{x_2 - \xi_2}{r} \\ h_2 &= f_3 \frac{x_1 - \xi_1}{r} - f_1 \frac{x_3 - \xi_3}{r} \\ h_3 &= f_1 \frac{x_2 - \xi_2}{r} - f_2 \frac{x_1 - \xi_1}{r} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \qquad (19)$$

Then it is found that:

$$\Psi = \frac{\lambda+\mu}{4\pi(\lambda+2\mu)} \int d\sigma \left(h_1 \frac{d\xi_1}{d\sigma} + h_2 \frac{d\xi_2}{d\sigma} + h_3 \frac{d\xi_3}{d\sigma} \right) \quad . \quad . \quad . \quad (20)$$

The fact that this transformation is possible proves that Ψ is independent of the particular form given to the surface Σ (provided it is bounded by the line σ), and that consequently Ψ and its derivatives will be continuous at the points of Σ.

Our final expression for u_k thus becomes:

$$u_k = f_k \varphi + \frac{1}{4\pi} \int d\sigma \frac{g_k}{r} + \frac{\partial \Psi}{\partial x_k} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

with φ given by (5) and Ψ given either by (18) or by (20). The first integral introduces the desired multi-valued character, while all three terms are independent of the particular form given to the surface Σ and exclusively depend upon the form of the boundary line σ.

10. The formulae deduced by VOLTERRA with the aid of a very elegant method, refer to a somewhat more general case than the one considered here [13]). When we restrict to the type of multi-valuedness considered above, these formulae can be given in the form:

$$u_k = \iint d\Sigma \, X_{kl} f_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where the X_{kl} ($l = 1, 2, 3$) represent the components of the stress acting on the element $d\Sigma$ at ξ_1, ξ_2, ξ_3, when a unit force in the direction of the x_k-axis is applied at the point x_1, x_2, x_3. When the calculations are worked out, it is found that [14]):

$$\left. \begin{aligned} X_{kl} = \frac{\delta_{kl}}{4\pi} \frac{\partial}{\partial \nu}\left(\frac{1}{r}\right) + \frac{1}{4\pi} \left\{ (\nu k) \frac{\partial}{\partial \xi_l}\left(\frac{1}{r}\right) - (\nu l) \frac{\partial}{\partial \xi_k}\left(\frac{1}{r}\right) \right\} - \\ - \frac{\lambda+\mu}{4\pi(\lambda+2\mu)} \left\{ \frac{\partial^3 r}{\partial x_k \, \partial x_l \, \partial \nu} - 2(\nu l) \frac{\partial}{\partial x_k}\left(\frac{1}{r}\right) \right\} \end{aligned} \right\} . \quad (23)$$

With these values of X_{kl} the expressions (22) are identical with (21).

11. *Application of equations (21) to some simple examples.* — We turn back for a moment to the cases indicated schematically in fig. 1 and 3, although they refer to fields where the singular line is of infinite extent, and will attempt to apply eqs. (21) to them. In these cases $f_2 = f_3 = 0$, while $f_1 = \lambda_0$.

A. In the case of fig. 1 the singular line is the z-axis. In order to find the value of φ by means of eq. (5) we may take an arbitrary half plane for the surface Σ, provided it has the z-axis as its boundary, as different positions of this plane lead to results differing only by an additive constant.

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13) V. VOLTERRA *l.c.* p. 425, eqs. (I), (II).

14) Here $\delta_{kl} = 1$ or 0, accordingly as $k = l$ or $k \neq l$.

It is necessary, however, to specify the positive direction of the normal ν, as this determines the side from which the solid angle must be viewed, and at the same time determines the direction in which the boundary line must be described in the integrals (8) and (20).

When for Σ we take the half plane $x = 0$, $y > 0$, and as the positive direction of the normal that of the positive x-axis, then the boundary line must be followed in the direction of $-z$; consequently along this line we shall have:

$$d\xi_1/d\sigma = 0; \qquad d\xi_2/d\sigma = 0; \qquad d\xi_3/d\sigma = d\zeta/d\sigma = -1.$$

We now obtain:

(*a*) $$\varphi = \frac{1}{2\pi} \operatorname{arctg} \frac{y}{x} + const. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24a)$$

(*b*) $$g_1 = 0, \qquad g_2 = +\lambda_0, \qquad g_3 = 0.$$

The integral $u_2^{**} = \frac{1}{4\pi} \int d\sigma \frac{g_2}{r}$ is divergent; the relevant part (*i.e.* the part dependent upon x and y), however, can be written:

$$u_2 = \frac{1}{4\pi} \int d\sigma \frac{g_2}{r} \simeq -\frac{\lambda_0}{2\pi} \ln \sqrt{x^2+y^2} + const. \quad . \quad . \quad (24b)$$

(*c*) $$h_1 = 0, \qquad h_2 = -\lambda_0 \frac{z-\zeta}{r}; \qquad h_3 = \lambda_0 \frac{y}{r}.$$

Equation (20) becomes:

$$\Psi = \frac{-(\lambda+\mu)\lambda_0}{4\pi(\lambda+2\mu)} \int d\sigma \frac{y}{r},$$

which, in the same sense as above, gives the result:

$$\Psi \simeq \frac{(\lambda+\mu)\lambda_0}{2\pi(\lambda+2\mu)} \, y \ln \sqrt{x^2+y^2} + const. \quad . \quad . \quad . \quad . \quad (24c)$$

Hence we obtain [15]):

$$\left. \begin{aligned} u_1 &= \frac{\lambda_0}{2\pi} \operatorname{arctg} \frac{y}{x} + \frac{(\lambda+\mu)\lambda_0}{2\pi(\lambda+2\mu)} \frac{xy}{x^2+y^2} \\ u_2 &= -\frac{\mu\lambda_0}{2\pi(\lambda+2\mu)} \ln \sqrt{x^2+y^2} + \frac{(\lambda+\mu)\lambda_0}{2\pi(\lambda+2\mu)} \frac{y^2}{x^2+y^2} \\ u_3 &= 0 \end{aligned} \right\} \quad . \quad (25)$$

B. In the case of fig. 3 the singular line is the x-axis. Along this axis we have $d\xi_2/d\sigma = 0$, $d\xi_3/d\sigma = 0$; hence the quantities g_k vanish; likewise

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15) It may be remarked that these expressions differ from those given by VOLTERRA,

the integral (20) vanishes. There remains only the function φ, which now is equal to:

$$\varphi = -\frac{1}{2\pi}\, arctg\, \frac{z}{y} + const. \quad \ldots \ldots \quad (26)$$

Hence [16]:

$$\left.\begin{aligned} u_1 &= -\frac{\lambda_0}{2\pi}\, arctg\, \frac{z}{y} \\ u_2 &= 0 \\ u_3 &= 0 \end{aligned}\right\} \quad \ldots \ldots \quad (27)$$

12. *Expressions for the stresses.* — Now that the expressions for the displacement components u_k have been found, the components σ_{kl} of the elastic stresses can be calculated by means of the equations:

$$\sigma_{kl} = \mu \left(\frac{\partial u_l}{\partial x_k} + \frac{\partial u_k}{\partial x_l} \right) + \delta_{kl}\, \lambda\, \theta. \quad \ldots \ldots \quad (28)$$

The quantities occurring in these equations can be obtained by means of line-integrals along σ: the terms depending upon the quantities g_k by means of eq. (8); the quantities depending upon Ψ by means of (20); from (11) and (14) it follows that:

$$\theta = -\frac{\mu}{\lambda + \mu}\, \triangle\, \Psi \quad \ldots \ldots \quad (29)$$

which leads to the equation:

$$\theta = + \frac{\mu}{2\pi(\lambda + 2\mu)} \int d\sigma \left(\frac{h_1}{r^2}\frac{d\xi_1}{d\sigma} + \frac{h_2}{r^2}\frac{d\xi_2}{d\sigma} + \frac{h_3}{r^2}\frac{d\xi_3}{d\sigma} \right) \quad . \, . \quad (30)$$

finally, the derivatives of the potential φ, which in the corresponding hydrodynamic problem represent the components of the velocity, can be

by way of example, *l.c.* p. 428. When we take $l = \lambda_0$, $m = n = p = q = r = 0$ in those formulae we obtain:

$$\left.\begin{aligned} u_1 &= \frac{\lambda_0}{2\pi}\, arctg\, \frac{y}{x} \\ u_2 &= \frac{\lambda_0}{2\pi}\, ln \sqrt{x^2 + y^2} \\ u_3 &= 0. \end{aligned}\right\}.$$

However, the expressions (25) given in the text above are in substantial agreement with the result given by VOLTERRA *l.c.* p. 465, eqs. (I), when in the latter we take $R_1 = 0$, $R_2 = \infty$, and interchange x and y. — The formulae for the stresses given by TAYLOR, Proc. Roy. Soc. London **A 145**, p. 376, 1934, correspond to the expressions given by VOLTERRA at p. 428.

[16]) This result is in accordance with the formulae given by VOLTERRA at p. 428, if we take $n = -\lambda_0$, $l = m = p = q = r = 0$.

calculated by means of the so-called "formula of BIOT and SAVART" [17]). This is especially convenient when the singular line σ consists of straight segments.

It can be shown that when the line σ is closed in itself and embraces a finite area, the stresses σ_{kl} with increasing distances r become of the order of magnitude at most of r^{-3}.

13. It may be useful to come back to the question of the balancing of the force system alluded to at the end of section **6** and also in section **8**. For this purpose we should consider the resultant force acting upon a cylindrical surface described with the (small) radius a around an element $d\sigma$ of the singular line as axis. However, instead of treating the general problem, we will consider the case of an infinitely extended straight singular line, tangent to the given line at the element $d\sigma$, and having the same cyclic constants as the latter, assuming that when the radius of curvature of the given singular line is great compared with the length of the cylindrical surface to be considered, the stresses for the two cases will differ at most by quantities of a finite order of magnitude, so that the difference between the resultant forces will vanish simultaneously with the radius a.

We first consider the stresses connected with the quantities u_k^* and that part Ψ^* of the complementary function Ψ which can be directly associated with them. In virtue of the relation $\partial u_k^* / \partial x_k = \partial u_k^{**} / \partial x_k$ — see eq. (15) — we have:

$$\Psi^* = \tfrac{1}{2}\, \Psi \quad \ldots \ldots \ldots \quad (31)$$

Taking the singular line along the z-axis (the direction being that of $-z$), we again find:

$$\varphi = \frac{1}{2\pi}\, arctg\, \frac{y}{x} \quad \ldots \ldots \ldots \quad (32)$$

while, if f_1, f_2, f_3 all are different from zero, we shall have:

$$\left.\begin{aligned} \Psi^* = \tfrac{1}{2}\,\Psi &= \frac{-(\lambda+\mu)}{8\pi(\lambda+2\mu)} \int d\sigma \left(f_1 \frac{y}{r} - f_2 \frac{x}{r} \right) \simeq \\ &\simeq \frac{\lambda+\mu}{4\pi(\lambda+2\mu)} (f_1\, y - f_2\, x)\, ln \sqrt{x^2+y^2} \end{aligned}\right\} \quad . \, . \, . \quad (33)$$

From this expression for Ψ^* we obtain:

$$\theta = -\frac{\mu}{\lambda+\mu}\, \triangle\, \Psi^* = -\frac{\mu}{2\pi(\lambda+2\mu)} \frac{f_1\, y - f_2\, x}{x^2+y^2} \quad . \, . \, . \quad (34)$$

Writing:

$$r = \sqrt{x^2+y^2}; \qquad \alpha = arctg\, y/x \quad . \, . \, . \quad (35)$$

it is found that the components σ_{rl} of the stresses acting upon an element of the cylindrical surface can be calculated from the equation:

$$\left.\begin{aligned} \sigma_{rl} &= \sigma_{1l} \cos\alpha + \sigma_{2l} \sin\alpha = \\ &= \mu\,(f_1 \cos\alpha + f_2 \sin\alpha)\, \frac{\partial\varphi}{\partial x_l} + 2\,\mu\, \frac{\partial^2\, \Psi^*}{\partial r\, \partial x_l} + \lambda\,\theta\, \cos(rl) \end{aligned}\right\} \quad . \, . \quad (36)$$

[17]) See H. LAMB, Hydrodynamics (Cambridge 1932), p. 211, eqs. (2) and (3).

which gives:

$$\left.\begin{aligned}
\sigma_{rx} &= -\frac{\mu}{2\pi a}(f_1 \cos a \sin a + f_2 \sin^2 a) - \frac{\mu(\lambda+\mu)}{2\pi(\lambda+2\mu)a} f_2 - \\
&\qquad - \frac{\mu\lambda}{2\pi(\lambda+2\mu)a}(f_1 \cos a \sin a - f_2 \cos^2 a), \\
\sigma_{ry} &= +\frac{\mu}{2\pi a}(f_1 \cos^2 a + f_2 \cos a \sin a) + \frac{\mu(\lambda+\mu)}{2\pi(\lambda+2\mu)a} f_1 - \\
&\qquad - \frac{\mu\lambda}{2\pi(\lambda+2\mu)a}(f_1 \sin^2 a - f_2 \cos a \sin a), \\
\sigma_{rz} &= 0.
\end{aligned}\right\} \quad . \quad . \quad (37)$$

The resultant force acting from the elastic medium upon the cylindrical element consequently has the components (per unit length):

$$\left.\begin{aligned}
&\text{in the } x\text{-direction:} && -\mu f_2 \\
&\text{in the } y\text{-direction:} && +\mu f_1 \\
&\text{in the } z\text{-direction:} && 0
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

14. We must next consider the stresses connected with the quantities u_k^{**} and the second part Ψ^{**} of the complementary function. We have:

$$\left.\begin{aligned}
\mu g_1 &= -\mu f_2 \\
\mu g_2 &= +\mu f_1 \\
\mu g_3 &= 0
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

while, from (8):

$$u_k^{**} = \frac{g_k}{4\pi}\int \frac{d\sigma}{r} \cong -\frac{g_k}{2\pi} \ln r \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

As $\Psi^{**} = \Psi^*$, $\theta^{**} = \theta^*$, the calculation is not difficult; instead of (36) we obtain:

$$\sigma_{rl} = -\frac{\mu g_l}{2\pi r} - \frac{\mu(g_1 \cos a + g_2 \sin a)}{2\pi}\frac{\partial}{\partial x_l}(\ln r) + 2\mu\frac{\partial^2 \Psi^*}{\partial r\, \partial x_l} + \lambda\theta \cos(rl)\,. \quad (41)$$

and the components of the resultant force become:

$$\left.\begin{aligned}
&\text{in the } x\text{-direction:} && +\mu f_2 \\
&\text{in the } y\text{-direction:} && -\mu f_1 \\
&\text{in the } z\text{-direction:} && 0
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

These quantities are equal and opposite to those given in (38); hence there is no resultant force acting upon our cylindrical element, as had been required in section **6**.

It may be remarked that the stress component σ_{zz} in general will not be zero, so that there may be tensions and pressures in the direction of the singular line, connected with the dilatation θ. The mean value of σ_{zz} over a cross section of the element, however, vanishes.

15. *Further geometrical considerations. Migration of a dislocation through the atomic lattice.* — The passage of a two-dimensional unit

dislocation across a crystal has been described by TAYLOR as the basic phenomenon in the explanation of the process of slipping [18]. A schematical picture of a case in which the singular line σ is parallel to the y-axis has been given in fig. 6; it will be seen that in the situation of fig. 6a the atom 1 can jump over to another equilibrium position; next the atom 2 can

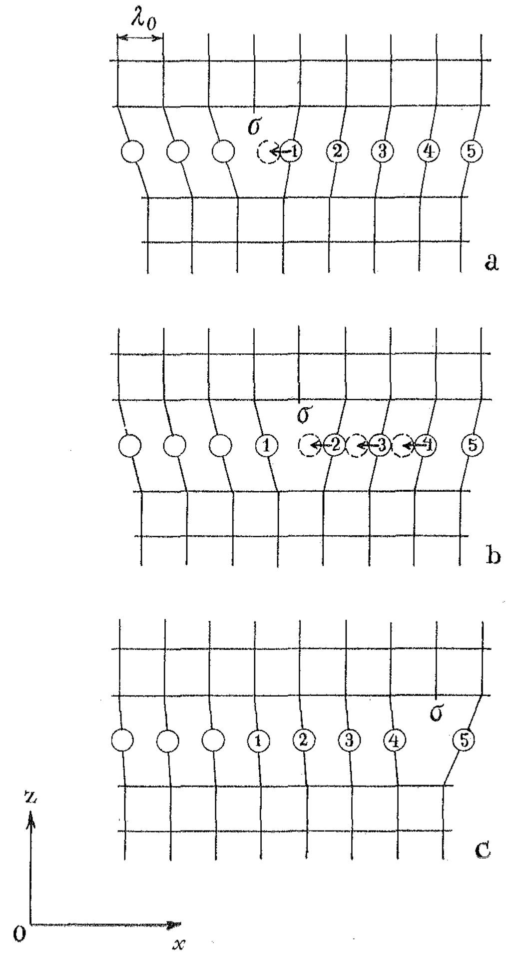

Fig. 6. Schematical picture of the migration of an elementary dislocation through the lattice.

make a similar jump, then 3, 4, ..., and in consequence of these jumps the singular line σ moves to the right. In a crystal of finite dimensions this process will be accompanied by a shift in the relative position of the upper and the lower parts of the crystal; this shift is of such a magnitude that it becomes equal to the amount λ_0 when the dislocation has moved across the whole crystal from the left hand boundary to the right hand boundary.

[18]) G. I. TAYLOR, *l.c.* p. 368. — See also "First Report on Viscosity and Plasticity", p. 199.

When the dislocation has migrated only over a distance L, in a crystal the dimension of which in the same direction is a, then the relative shift of the two parts will be given by $\lambda_0 L/a$.

When the dislocations are characterized by closed singular lines of finite extent, embedded in a lattice of indefinite, or at any rate of very large extent, it is to be noted that at large distances from the dislocation the lattice must be wholly regular. When we imagine a closed surface surrounding the characteristic lines of all dislocations, outside of this surface the u_k will be single-valued functions, which with increasing distances either will become zero or will approach to constant values. It will be evident that in a region with regular structure no dislocation can be generated "out of nothing": dislocations either must have been originated during the process of growth of the crystal, or they must have been derived from other already existing regions of irregular structure. When a dislocation of the type considered by us takes its birth from some unspecified region of irregular structure, then — the same as in any other case — the condition of never having an open end in the interior of the lattice always will remain valid: the singular line characteristic of the dislocation either must be closed in itself, or else its ends must be situated at the boundary of the irregular region (or eventually perhaps in the interior of this region). It would appear probable that a given dislocation can be displaced through the lattice over an arbitrary distance, possibly in various directions. The character of being a closed line will not be lost during such a displacement, although the singular line perhaps may change of form; further the strength of the dislocation, or more exactly the values of the cyclic constants $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3$ associated with the singular line, will not change. When a singular line in its migration through the lattice should meet another singular line, then it is to be expected that the simple type of migration, determined by jumps of the atoms of the kind as described above, cannot be continued. Hence we may assume that two singular lines in their process of migration in general cannot cross each other, or at least will have a certain difficulty in crossing each other. (It may be that the approach of the two dislocations leads to the formation of a region of irregular structure of larger extent, from which, under suitable circumstances, a new dislocation may take its birth; a more detailed investigation of such a process will be useful, but probably may be difficult). At any rate we may suppose that the easy migration of a dislocation is impeded when it meets other dislocations; this is one of the features which serve as a basis for the explanation of the fact that the plastic deformation of a crystal gradually becomes more difficult (*i.e.* requires the application of greater forces) with increasing values of the shear.

It will be evident that in the points mentioned there is an analogy with the properties of vortices in an ideal liquid. We might even go further and ask whether *e.g.* processes in which there is a change in the area enclosed by the singular line (or more exactly in the area's A_1, A_2, A_3 of its three

projections upon the coordinate planes; compare section **8**) will require the application of exterior forces to the lattice. We shall come back to this point in section **19**.

However, although a certain analogy with vortex lines exists, we must not forget that the migration of a dislocation, as pictured schematically in fig. 6, is intimately connected with the geometry of the atomic lattice. Consequently there may be restrictions on the possibilities for the displacement of the singular line, which have no analogy in the hydrodynamic case.

16. There are a few examples of migrations which can be discussed in a simple way.

Consider the case pictured in fig. 7. In the first place this may be

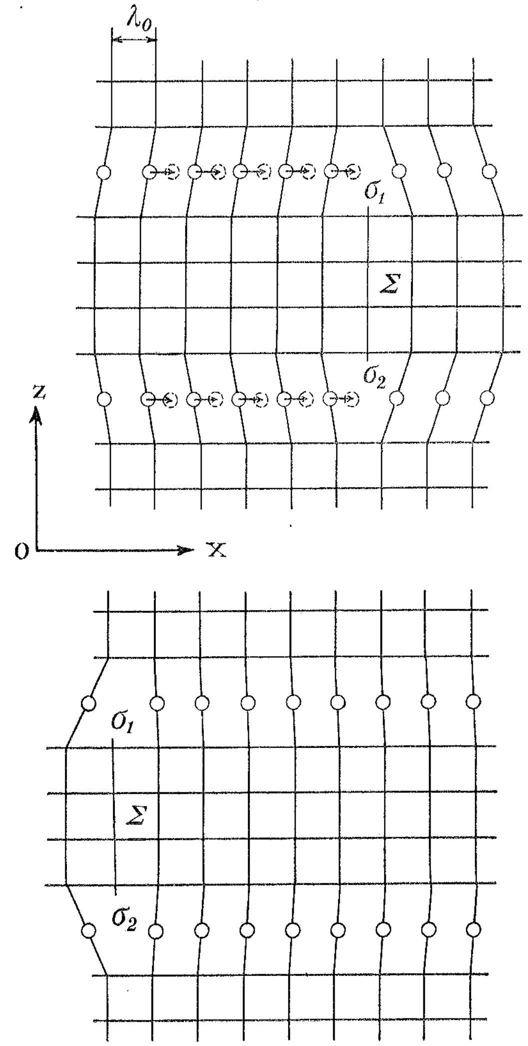

Fig. 7. Schematical picture of the migration of a dislocation, bounded by two parallel singular lines, when these lines move simultaneously with equal steps.

considered as representing a section of a lattice in which a dislocation has been introduced by inserting into it an extra layer of atoms along a surface Σ, perpendicular to the x-axis and bounded by two parallel singular lines σ_1 and σ_2 perpendicular to the plane of the paper (*i.e.* parallel to the y-axis), both of infinite extent. As indicated schematically, it is possible in such a case that the two singular lines migrate simultaneously over the same distance, *e.g.* to the left. It will be seen that, although there is a relative shift of the central portion of the lattice (*i.e.* the portion situated between the planes described by the singular lines in their movement) with respect to the rest, there is no resulting shift of the uppermost portion with respect to the undermost.

Fig. 7 may be considered also as representing a section of a three-dimensional field, in which the surface Σ is of finite extent and is bounded by a closed line σ (Σ in this case still being plane and perpendicular to the x-axis). There seems to be no objection to the assumption that in such a case a similar migration is possible, jumps of atoms now taking place with equal frequency at all points of σ; the singular line then migrates without change of form through the lattice in the direction perpendicular to its plane.

Turning back to the original conception in which we had two parallel singular lines σ_1 and σ_2 of infinite extent, it will be evident that the parallel and equal migration of σ_1 and σ_2 is not a necessary feature: these lines may just as well move independently of each other, *e.g.* in the way as indicated in fig. 8. The dislocation then of course obtains a different character in so far as the surface Σ of fig. 7 does no longer exist.

Is it possible to imagine something to be compared with the latter case taking place when the surface Σ is bounded by a closed line σ?

Referring to fig. 9, where the line σ originally had the form of a rectangle *ABDE* (the plane of the rectangle being perpendicular to the x-axis), we may imagine that jumps of atoms take place only along the parts *CD*, *DE* and *EF* of σ, producing a displacement of *CDEF* parallel to itself towards a new position *C'D'E'F'*, whereas the part *FABC* remains where it was. An irregularity in the arrangement of the atoms then will be produced along the whole course of the lines *CC'* and *FF'*, and these lines in fact will become parts of the singular line characteristic of the dislocation in its new form, joining up the parts *FABC* and *C'D'E'F'*. The nature of the singularity in the immediate neighbourhood of the segments *CC'* and *FF'* is of the character indicated in fig. 3 (and also in fig. 5 for the parts of the singular line which are parallel to the x-axis), while in the neighbourhood of the point F it can be compared to the case indicated in fig. 4.

17. The picture arrived at in the preceding section can be of help in discussing a point which had been raised in some sections devoted to the phenomena of plastic deformation in crystalline substances of the "Second Report on Viscosity and Plasticity". In connection with views brought forward by TAYLOR it had been assumed in the "First Report" that when

in the course of a shearing process applied to a crystal, a certain number of dislocations have started from already existing flaws, and have moved

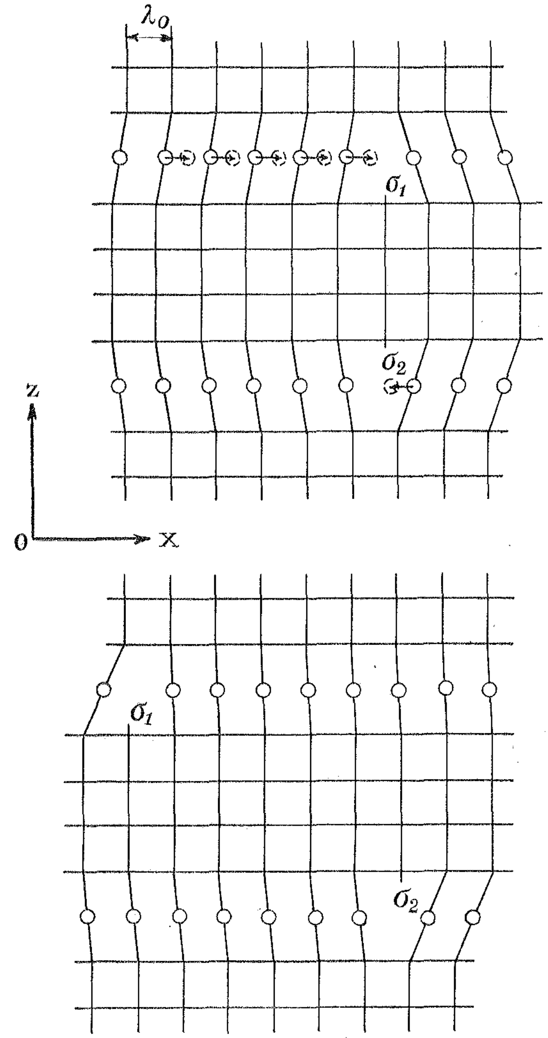

Fig. 8. Schematical picture of the change of character of a dislocation bounded by two parallel singular lines, when these lines move in opposite directions.

through the lattice until they are stopped by encountering other regions of irregular structure, there is produced a field of stresses, formed by the resultant effect of the fields connected with these dislocations, which field counteracts the stress due to the exterior forces causing the shearing process [19]. In making an estimate of the magnitude of the average shearing stress derived from the fields of the dislocations, TAYLOR's picture and calculations were used, referring to the two-dimensional type of dislo-

[19]) "First Report on Viscosity and Plasticity", p. 209.

cations, all singular lines being parallel to each other and extending right across the crystal. The average shearing stress then is found to be inversely proportional to the mean distance between the singular lines; consequently it is directly proportional to the square root of the number of singular lines per unit area.

The picture of a system of parallel dislocation lines, all extending in the lateral direction right across the crystal, however, has a degree of regularity which appears greater than may be expected in a crystal with flaws, and it is therefore that in the "Second Report a "three-dimensional" picture with dislocations of finite lateral extent had been suggested [20]).

We may now attempt to consider the process indicated schematically in fig. 9 as a possible case of the migration of such a three-dimensional dislocation. The case of fig. 7, which as stated, likewise can be taken as

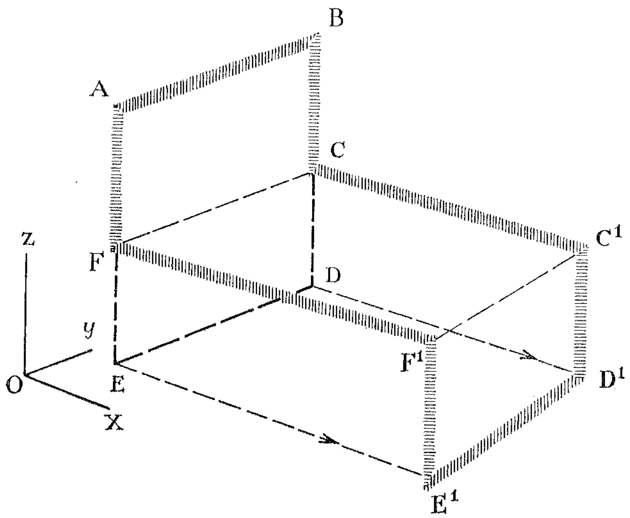

Fig. 9. Schematical picture of the transformation of a dislocation, originally characterized by a rectangular singular line situated in a plane parallel to the y, z-plane.

representing a dislocation of finite extent, is of no use, as in this case no resultant average shear will appear in the crystal. In the case of fig. 9 on the other hand, there is a contribution to the average shear, depending in magnitude upon the area of the rectangle $FF'C'CF$. The lateral extension of the dislocation, determined by the length AB ($= FC = ED$), like the dimension AE or BD, will depend upon the dimensions of the disturbed region from which this dislocation originated, and thus will correspond to the quantity l in the equations of the "Second Report"; whereas the distance CC' or FF' represents the length L of the path described by the dislocation [21]).

[20]) "Second Report on Viscosity and Plasticity", pp. 200 seq.

[21]) Compare "Second Report", p. 202, eq. (5. 1b), where λ corresponds to λ_0 in the present communication.

In calculating the field of stress associated with the type of singular line pictured in fig. 9 we may — in a similar way as can be done in all problems relating to vortex lines — separately consider the three circuits $AFCBA$, $F'E'D'C'F'$ and $FF'C'CF$. When we ask for the magnitude of the stresses in points situated at distances, say r_1 from $AFCBA$ and r_2 from $F'E'D'C'F'$, which are large in comparison with the sides AB and AF or $F'E'$ etc., then according to what has been remarked at the end of section **12**, the contributions of these circuits into the stresses will be of the orders $(r_1)^{-3}$ and $(r_2)^{-3}$ respectively. The contribution of the circuit $FF'C'CF$, however, will be of a different order when the length $CC' = L$ itself is great compared with $F'C' = l$.

18. *Calculation of the field of stress connected with a singular line of a form as given in fig. 9.* — With reference to fig. 10 the calculation for the circuit $FF'C'CF$ can be given as follows: Let φ, as before, represent the

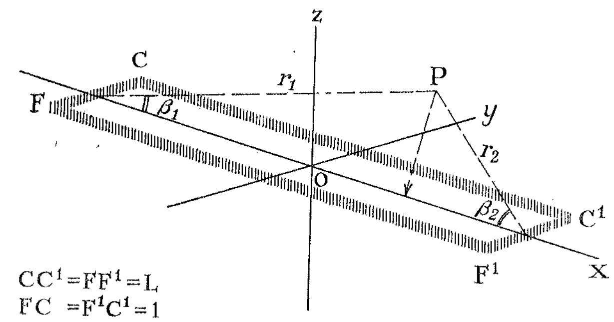

Fig. 10. The rectangular circuit $FF'C'CF$ of fig. 9 considered separately.

solid angle, divided by 4π, which the rectangle $FF'C'CF$ subtends at the point P. The positive direction of the normal to the rectangle is the direction Oz; the same as with the dislocation in its original form, characterized by $AEDBA$, there is a discontinuity in the component u_1 only, so that $f_2 = f_3 = 0$, while $f_1 = \lambda_0$. Hence, according to (4):

$$u_1^* = \lambda_0\,\varphi, \quad u_2^* = u_3^* = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Along FF' and CC' we have $g_1 = g_2 = g_3 = 0$ by (7), so that these segments do not give a contribution to u^{**}. From (19) and (20) it follows that they neither give a contribution to the value of Ψ.

Along $F'C'$ and CF (the positive direction of integration along the circuit being $FF'C'CF$) we have $g_1 = g_2 = 0$; along $F'C'$ we have: $d\xi_2/d\sigma = +1$; $g_3 = +\lambda_0$; $h_2 = -\lambda_0 z/r_2$, and along FC: $d\xi_2/d\sigma = -1$; $g_3 = -\lambda_0$; $h_2 = -\lambda_0 z/r_1$. Hence we obtain:

$$\text{from (8):} \quad u_1^{**} = u_2^{**} = 0; \quad u_3^{**} = \frac{\lambda_0\, l}{4\pi}\left(\frac{1}{r_2} - \frac{1}{r_1}\right). \quad . \quad . \quad (44)$$

$$\text{from (20):} \quad \Psi = -\frac{\lambda+\mu}{\lambda+2\mu}\,\frac{\lambda_0\, l}{4\pi}\left(\frac{z}{r_2} - \frac{z}{r_1}\right). \quad . \quad . \quad . \quad . \quad (45)$$

It has been assumed in these expressions that r is large compared with l.

The stress τ which had been considered in the "First" and "Second Reports" corresponds to the component σ_{13} in the notation used here, which is given by:

$$\sigma_{13}=\mu\left(\frac{\partial u_1}{\partial z}+\frac{\partial u_3}{\partial x}\right)=\mu\left(\lambda_0\frac{\partial\varphi}{\partial z}+\frac{\partial u_3^{**}}{\partial x}+2\frac{\partial^2\Psi}{\partial x\,\partial z}\right)\quad . \quad . \quad (46)$$

In this expression $\partial^2\Psi/\partial x\partial z$ is to be calculated from (45); $\partial\varphi/\partial z$ is obtained by calculating the velocity field associated with the rectangular vortex line. With sufficient approximation we have:

$$\mu\left(\lambda_0\frac{\partial\varphi}{\partial z}+\frac{\partial u_3^{**}}{dx}\right)=\frac{\mu\,\lambda_0 l}{4\pi}\,\frac{y^2-z^2-l^2/4}{(y^2+z^2+l^2/4)^2-y^2\,l^2}(\cos\beta_1+\cos\beta_2)\quad . \quad (47)$$

the angles β_1 and β_2 being defined in fig. 10.

Even without calculating the value of $\partial^2\Psi/\partial x\partial z$ (which for $r\gg l$ becomes of the order r^{-3}, as mentioned before), it will be seen from the expression (47) that so long as $|x|$ does not greatly surpass $L/2$, so that $\cos\beta_1+\cos\beta_2$ is either >1, or not much below 1, this result leads to values of σ_{13} of an order of magnitude decreasing inversely proportional to the square of the distance from the x-axis, *i.e.* it leads to values which for constant y/z practically are inversely proportional to (y^2+z^2). The dimensions of the region in which this result holds of course depend upon the magnitude of L.

19. It is of interest also to consider the stresses σ_{12} and σ_{13} in the points of the plane determined by the rectangle $F'E'D'C'$. We must then work out the calculations for the singular line $FF'E'D'C'C$; when L is sufficiently great, we may neglect the contributions due to the part $CBAF$. It will be superfluous to give the calculations in detail; and we mention only the following points:

It is found again that $u_1^{**}=0$, whereas:

$$\left.\begin{aligned} u_2^{**}&=\frac{\lambda_0}{4\pi}\left(\int_{E'}^{F'}\frac{d\zeta}{r}-\int_{D'}^{C'}\frac{d\zeta}{r}\right)\\ u_3^{**}&=\frac{\lambda_0}{4\pi}\int_{E'}^{D'}\frac{d\eta}{r}\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Further:

$$\Psi=\frac{(\lambda+\mu)\,\lambda_0}{4\pi\,(\lambda+2\mu)}\left\{\int_{D'}^{C'}d\zeta\,\frac{y-\eta}{r}-\int_{E'}^{F'}d\zeta\,\frac{y-\eta}{r}-\int_{E'}^{D'}d\eta\,\frac{z-\zeta}{r}\right\}\quad . \quad (49)$$

In the equations for the stress components σ_{12} and σ_{13} the derivatives

$\partial^2\Psi/\partial x\partial y$ and $\partial^2\Psi/\partial x\partial z$ occur; as $\partial\Psi/\partial x=0$ for points in the plane of the rectangle $F'E'D'C'$, the contributions to be derived from Ψ vanish here.

The values of $\partial\varphi/\partial y$ and $\partial\varphi/\partial z$ again can be obtained by means of the formulae for the velocity field associated with a vortex line. It is found that the sum

$$\lambda_0\frac{\partial\varphi}{\partial y}+\frac{\partial u_2^{**}}{\partial x},$$

occurring in the expression for σ_{12}, is determined exclusively by the contributions to the value of $\partial\varphi/\partial y$ derived from the lines FF' and $C'C$; whereas the sum

$$\lambda_0\frac{\partial\varphi}{\partial z}+\frac{\partial u_3^{**}}{\partial x},$$

occurring in the expression for σ_{13}, in the same way is determined by the contributions to the value of $\partial\varphi/\partial z$ derived from these lines.

The values of the stress components σ_{12} and σ_{13} in the points of the plane determined by $F'E'D'C'$ finally become:

$$\left.\begin{aligned}\sigma_{12}&=\frac{\mu\,\lambda_0\,l}{4\pi}\,\frac{-2\,y\,z}{(y^2+z^2+l^2/4)^2-y^2\,l^2}\\ \sigma_{13}&=\frac{\mu\,\lambda_0\,l}{4\pi}\,\frac{y^2-z^2-l^2/4}{(y^2+z^2+l^2/4)^2-y^2\,l^2}\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad (50)$$

Let us give attention to the value of σ_{13} at the points of the segment $E'D'$. It is not difficult to calculate the mean value of σ_{13} at the points of this segment; this mean value is found to be:

$$(\sigma_{13})_m=-\frac{\mu\,\lambda_0}{4\pi l}\ln\frac{l^2+h^2}{h^2}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

and thus appears to be *negative*.

In order to understand the meaning of the sign of σ_{13}, we remark that in the cases indicated in figs. 6 and 8 the application of a positive exterior shearing stress to the system, acting to the right at the upper surface of the crystal, and to the left at the lower surface, will promote the occurrence of the type of migration pictured in these diagrams. The case of fig. 9 has been derived from that of fig. 8 without change of signs, so that the same result will apply to it. Hence we may conclude that the appearance of a negative value of $(\sigma_{13})_m$ along $E'D'$ will act in the opposite way, and consequently will drive back the migration process, or at any rate impede its further progress in the original direction.

Here thus we have an instance of the "counteracting" effect of the field of stress connected with the dislocation itself. It will be evident that this "counteracting" effect can be overcome by the application to the crystal of an exterior shearing stress τ of sufficient positive magnitude.

We now are in a position to construct an expression for the work that apparently must be performed in order to displace the part $F'E'D'C'$ of the singular line in the positive x-direction, or in other words to increase the length L of the lines FF' and CC'. From eq. (5. 1*b*), p. 202 of the "Second Report", we see that the increase of the mean shear of a rectangular crystal with sides a, b, c due to an increase of L is given by:

$$d\gamma_1/dL = \lambda_0\, l/a\, b\, c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

The work done in this process by the exterior shearing force τ is determined by the product:

$$\tau \, . \, (d\gamma_1/dL) \, . \, (\text{volume});$$

hence, with $\tau = -(\sigma_{13})_m$, we obtain:

$$\text{work per unit increase of } L = \frac{\mu\,\lambda_0^2}{4\pi}\, ln\, \frac{l^2+h^2}{h^2} \quad . \quad . \quad . \quad (53)$$

It must be remarked that these considerations are given only in order to illustrate some of the concepts which have arisen in considering the phenomena accompanying the migration of a dislocation through the lattice; no great value can be attached to the exact form of the equations derived. For instance we have left aside the effect which the stress σ_{12} may have at the segments $F'E'$ and $C'D'$; it is true that the values of σ_{12} at these two segments are of opposite sign, and thus do not call for compensation by the application of an exterior shearing stress which can store work in the system. It will be a matter for further speculation to find out what is the meaning of the relations which have turned up here.

20. The impression will have been obtained that the introduction of dislocations of three-dimensional type leads to a picture possessing at least some features which point in the direction of the assumptions of which use was made in the "Second Report" [22]. Consequently we might imagine that in a crystal subjected to shear in the x-direction, along planes parallel to the x, y-plane, there would appear a number of disturbed strips of the nature of the rectangle $FF'C'CF$ in fig. 9 or fig. 10, extending over various lengths L in the x-direction, and having breadths l in the y-direction. A schematical picture of such a system of strips has been presented in fig. 11.

Nevertheless, the picture does not give results ready for immediate use in further calculations. In the preceding section we had arrived at an instance of a counteracting field, impeding the progress of a dislocation. From eqs. (47) and (50) it can be derived that in a region of the lattice surrounding the dislocation the value σ_{13} remains negative so long as $|y|$ is smaller than $|z|$. Hence the counteracting effect is also to be observed in the regions situated directly above and below the dislocation

[22] "Second Report", p. 204 in connection with eq. (5. 12*b*).

considered. However, in regions situated further away in the lateral direction, where $|y|$ becomes greater than $|z|$, the effect is of opposite sign,

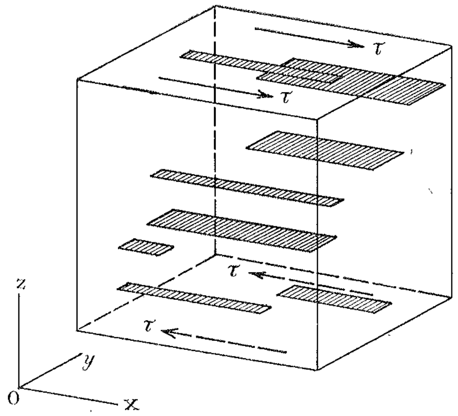

Fig. 11. Crystal block with a number of rectangular dislocation lines parallel to the x, y-plane.

and with the aid of eq. (46) it is not difficult to prove that the mean value of σ_{13} over a plane $z = const.$ is equal to zero.

This result is connected with the circumstance that the mathematical considerations developed above refer to the case of dislocations of finite extent embedded in a lattice which at great distances ultimately approaches to perfect regularity. Clearly such a picture is idealized too far, and cannot represent the state of things in an actual crystal, exhibiting irregularities of growth, etc. We must assume that in an actual crystal many of the irregularities, whatever be their nature, will extend over great distances, and that they will meet each other at various places, so that the crystal is divided up into more or less separate regions of regular structure. The boundary between two adjacent regions of regularity in many cases may be formed by systems of irregularities arranged so as to form a surface, and the dislocations migrating through the crystal in a process of plastic deformation often will have their endpoints moving over such surfaces. At these surfaces moreover the displacement components u_k may be subjected to certain conditions, which will react upon the field of stress.

21. *Systems of dislocations forming a "surface of misfit" between two regions of a lattice.* — An adequate treatment of the problems touched upon in the preceding section is difficult, and will not be given here. We shall restrict to the investigation of a surface of irregularity built up from a system of simple rectilinear dislocations of the type indicated in fig. 1. We assume a two-dimensional field, all singular lines being perpendicular to the x, y-plane. The problem to be considered is closely related to one treated by TAYLOR in order to obtain an example of a "surface of misfit" [23].

[23] G. I. TAYLOR, *l.c.* p. 400.

Making use of the equations developed in section **11 A**, we shall write, with a slight change of notation:

$$\left.\begin{aligned} u &= u^* + \partial\Psi/\partial x \\ v &= v^* + \partial\Psi/\partial y \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

When the complex variable $x + i\,y$ (with $i = \sqrt{-1}$) is introduced, eqs. (24*a*), (24*b*) give:

$$u^* + i\,v^* = \frac{-i\,\lambda_0}{2\pi}\,ln\,(x + i\,y) + const. \quad . \quad . \quad . \quad . \quad (55)$$

which is a form convenient for generalisation.

We take the case in which there are singular lines at the infinite series of points:

$$x = n\,l, \qquad y = n\,h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

where l and h are two arbitrary constants (both being equal to some multiple of λ_0), while n takes all integer values from $-\infty$ to $+\infty$. By giving special values to l and h various subcases can be constructed; with $h = 0$ all singular lines are situated in a horizontal plane (in the x, z-plane), with $l = 0$ all are situated in a vertical plane (in the y, z-plane). — Every singular line gives a contribution into u^* and v^* which can be expressed by:

$$(u^* + i\,v^*)_n = -\frac{i\,\lambda_0}{2\pi}\,ln\,\frac{x + i\,y - n\,l - i\,n\,h}{n\,(l + i\,h)} \quad . \quad . \quad . \quad . \quad (57)$$

According to well known procedures applied in the theory of functions of a complex variable, we consequently may construct the solution of our problem with its infinite number of singular lines by writing:

$$u^* + i\,v^* = -\frac{i\,\lambda_0}{2\pi}\,ln\,sin\,\frac{\pi\,(x + i\,y)}{l + i\,h} \quad . \quad . \quad . \quad . \quad . \quad (58)$$

For convenience in notation we introduce the auxiliary variables:

$$\frac{\pi\,(l\,x + h\,y)}{l^2 + h^2} = \xi, \qquad \frac{\pi\,(l\,y - h\,x)}{l^2 + h^2} = \eta \quad . \quad . \quad . \quad . \quad . \quad (59)$$

The separation of real and imaginary parts then can be effected by writing [24]):

$$sin\,(\xi + i\,\eta) = M\,e^{i\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60a)$$

where:

$$M = \sqrt{sin^2\,\xi + sinh^2\,\eta}, \quad tg\,\alpha = \frac{tgh\,\eta}{tg\,\xi} \quad . \quad . \quad . \quad . \quad (60b)$$

so that:

$$u^* = \frac{\lambda_0\,\alpha}{2\pi}; \qquad v^* = -\frac{\lambda_0}{2\pi}\,ln\,M \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

[24]) Compare *e.g.* E. JAHNKE u. F. EMDE, Funktionentafeln, 1st Ed. (Leipzig u. Berlin 1909), p. 11; 2nd Ed. (1933), p. 60.

From the theory of functions of a complex variable it follows that:

$$\frac{\partial u^*}{\partial x} = \frac{\partial v^*}{\partial y}; \qquad \triangle\,u^* = \triangle\,v^* = 0.$$

Equation (14) thus becomes:

$$\triangle\,\Psi = -\frac{2\,(\lambda + \mu)}{\lambda + 2\mu}\,\frac{\partial u^*}{\partial x} = \frac{\lambda + \mu}{\lambda + 2\mu}\,\frac{\lambda_0}{2\,(l^2 + h^2)}\,\frac{h\,sin\,2\,\xi + l\,sinh\,2\,\eta}{M^2}. \quad (62)$$

It is difficult to find the expression for Ψ itself, but the problem is satisfied if we take [25]):

$$\left.\begin{aligned} \frac{\partial\Psi}{\partial\xi} &= \frac{\lambda + \mu}{\lambda + 2\mu}\,\frac{\lambda_0}{2\,\pi^2}\left\{\frac{\eta\,(l\,sin\,2\,\xi - h\,sinh\,2\,\eta)}{2\,M^2} + h\,ln\,M\right\} \\ \frac{\partial\Psi}{\partial\eta} &= \frac{\lambda + \mu}{\lambda + 2\mu}\,\frac{\lambda_0}{2\,\pi^2}\left\{\frac{\eta\,(l\,sinh\,2\,\eta + h\,sin\,2\,\xi)}{2\,M^2} + l\,ln\,M\right\} \end{aligned}\right\} \quad . \quad (63)$$

From these equations $\partial\Psi/\partial x$ and $\partial\Psi/\partial y$ can be obtained without difficulty.

22. It is of interest first to investigate the meaning of these equations for points at large distances from the row of singular lines. As the row itself is situated at the line $\eta = 0$, these points are obtained by considering large values of η. We must distinguish between positive and negative values, and it is useful to observe that η is positive on the left hand side and negative on the right hand side.

For *positive* η we find:

$$M \simeq sinh\,\eta\,; \quad ln\,M = const. + \eta$$

$$tg\,\alpha = cot\,\xi \quad ; \qquad \alpha = const. - \xi\,;$$

hence, neglecting constant amounts:

$$\left.\begin{aligned} u^* &= -\frac{\lambda_0\,\xi}{2\pi} = -\frac{\lambda_0\,(l\,x + h\,y)}{2\,(l^2 + h^2)} \\ v^* &= -\frac{\lambda_0\,\eta}{2\pi} = -\frac{\lambda_0\,(l\,y - h\,x)}{2\,(l^2 + h^2)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

Further:

$$\frac{\partial\Psi}{\partial\xi} \simeq 0\,; \qquad \frac{\partial\Psi}{\partial\eta} = \frac{\lambda + \mu}{\lambda + 2\mu}\,\frac{\lambda_0\,l\eta}{\pi^2} \quad . \quad . \quad . \quad . \quad . \quad (65)$$

[25]) Equation (62) does not wholly determine the function Ψ, as a function Ψ', satisfying the equation $\triangle\,\Psi' = 0$, always can be added. The value of Ψ' can be fixed only by having recourse to the conditions assumed at infinity. In the expressions (63) a form of Ψ has been chosen, which leads to the most satisfactory behaviour of u and v at infinity, as will be seen from the results given in section **22**.

from which:

$$\left.\begin{aligned}\frac{\partial \Psi}{\partial x} &\simeq -\frac{\lambda+\mu}{\lambda+2\mu}\,\frac{\lambda_0\, l\, h\,(l y - h x)}{(l^2+h^2)^2}\\ \frac{\partial \Psi}{\partial y} &\simeq +\frac{\lambda+\mu}{\lambda+2\mu}\,\frac{\lambda_0\, l^2\,(l y - h x)}{(l^2+h^2)^2}\end{aligned}\right\} \quad \ldots\ldots \quad (66)$$

For *negative* η the values of u^*, v^*; $\partial\Psi/\partial\xi$ and $\partial\Psi/\partial\eta$; $\partial\Psi/\partial x$ and $\partial\Psi/\partial y$ change sign.

In order to get an insight into these results it is of advantage separately to consider the cases $l=0$ and $h=0$.

A. In the case $l=0$ (singular lines in a vertical row; compare fig. 12) we have: $\partial\Psi/\partial x=0$, $\partial\Psi/\partial y=0$; and consequently:

$$\left.\begin{aligned}&\text{for } x<0: \quad u=u^*=-\frac{\lambda_0\, y}{2h}, \quad v=v^*=+\frac{\lambda_0\, x}{2h}\\ &\text{for } x>0: \quad u=+\frac{\lambda_0\, y}{2h}, \quad v=-\frac{\lambda_0\, x}{2h}\end{aligned}\right\} \quad \ldots \quad (67)$$

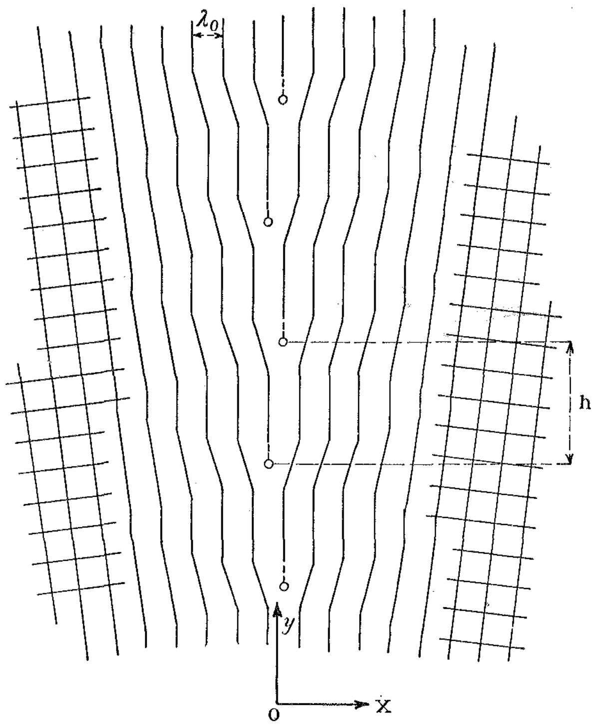

Fig. 12. Surface of misfit formed by parallel dislocation lines situated in the plane $x=0$.

These expressions make:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0; \quad \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=0. \quad \ldots\ldots \quad (68)$$

Hence they state that at large distances from the y, z-plane the lattices are inclined over the constant angle $+\lambda_0/2h$ on the left hand side, and $-\lambda_0/2h$ on the right hand side, without change of form. The deformations existing in the region near the y, z-plane completely disappear far away from this plane.

B. In the case $h=0$ (singular lines in a horizontal row; see fig. 13) we have:

$$\text{for } y>0: \quad u^*=-\frac{\lambda_0\, x}{2l}; \quad v^*=-\frac{\lambda_0\, y}{2l}$$

$$\frac{\partial \Psi}{\partial x}=0; \quad \frac{\partial \Psi}{\partial y}=+\frac{\lambda+\mu}{\lambda+2\mu}\,\frac{\lambda_0\, y}{l}$$

and hence:

$$\left.\begin{aligned}&u=-\frac{\lambda_0 x}{2l}; \quad v=+\frac{\lambda}{\lambda+2\mu}\,\frac{\lambda_0 y}{2l}\\ &\text{while for } y<0:\\ &u=+\frac{\lambda_0 x}{2l}; \quad v=-\frac{\lambda}{\lambda+2\mu}\,\frac{\lambda_0 y}{2l}\end{aligned}\right\} \quad \ldots\ldots \quad (69)$$

In this case above the row of singular lines there is a lateral compression,

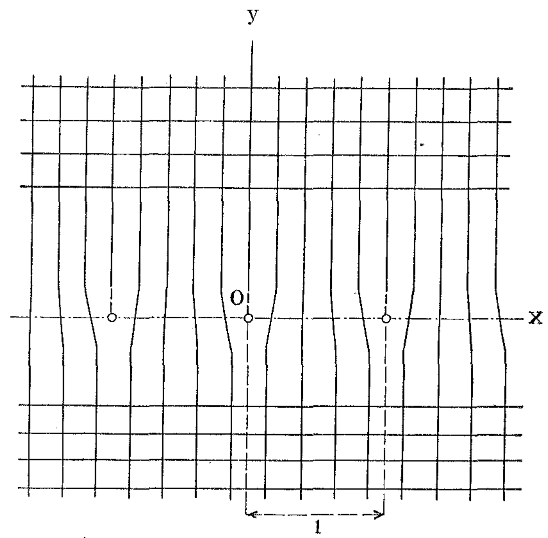

Fig. 13. Surface of misfit formed by parallel dislocation lines situated in the plane $y=0$.

accompanied by an extension in the y-direction, while below the row the reverse situation is found.

The stress components become:

$$\left.\begin{array}{l} \text{for } y>0: \quad \sigma_{xx} = -\dfrac{2\mu(\lambda+\mu)}{\lambda+2\mu}\,\dfrac{\lambda_0}{l}; \quad \sigma_{xy}=0; \quad \sigma_{yy}=0 \\ \text{and for } y<0: \\ \qquad \sigma_{xx} = +\dfrac{2\mu(\lambda+\mu)}{\lambda+2\mu}\,\dfrac{\lambda_0}{l}; \quad \sigma_{xy}=0; \quad \sigma_{yy}=0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad (70)$$

The fact that we obtain a value for σ_{xx} which does not vanish at infinity, shows that in a block of finite extension in the x-direction the state of deformation described by our formulae can exist only if suitable pressures and tractions are applied at the lateral boundaries. When this is not the case, another deformation will be superposed upon the one calculated here, the precise nature of which will depend upon the form of the boundary.

23. The system obtained by taking $l=0$ (fig. 12) clearly is of similar nature as the case considered by TAYLOR and pictured schematically in fig. 2*a* (p. 392) of the paper mentioned in footnote 2) above. It seems plausible to suppose that the "surfaces of misfit" occurring in actual crystals give rise to lattice inclinations of small amount, generally less than a degree, and often measuring a few minutes of arc only. Such cases are obtained when the distance h between two consecutive dislocations is of the order of 100 λ_0 to 1000 λ_0. There is, however, a difference between the conception introduced here and TAYLOR's picture: TAYLOR appears to assume that the disturbance of the lattice in passing from one block to the other is small only in relatively small regions, represented in his fig. 2*a* mentioned above by the gaps in the line AB, where the distance of atoms on one side from the nearest atoms on the other is the same as that which belongs to the perfect crystal structure. In our picture on the other hand the regions where the two lattices are united in a regular way are of much larger extent than the regions where there is a disturbance; in particular when h is of the order 100 λ_0—1000 λ_0, the parts of the "surface of misfit" where there is an actual disturbance are of the order of a few percent only, perhaps even less, of the whole area. The two parts of the lattice in our picture are united so to say "in the best way possible" for a given angle of inclination between them. The "surface of misfit" in our conception therefore would be still less "opaque" than in TAYLOR's picture, so long as we restrict to the consideration of dislocations the singular lines of which are all parallel to the z-axis. Evidently the "surface of misfit" of our picture can be "opaque" to dislocations with singular lines in other directions, provided these lines are of sufficient length.

From the equations obtained in section **21** we can calculate the values

of σ_{xx} and σ_{xy} at the points of the "surface of misfit", *i.e.* of the y, z-plane. We have:

$$\sigma_{xx} = 2\mu\frac{\partial u}{\partial x} + \lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

$$\sigma_{xy} = \mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right).$$

As in fig. 12: $\xi = \pi\, y/h$, $\eta = -\pi\, x/h$, we find, for $x=0$:

$$\left.\begin{array}{l} \dfrac{\partial u^*}{\partial x} = \dfrac{\partial v^*}{\partial y} = -\dfrac{\lambda_0}{2h}\cot\dfrac{\pi y}{h} \\ \dfrac{\partial u^*}{\partial y} = -\dfrac{\partial v^*}{\partial x} \quad \text{(everywhere in the field)} \end{array}\right\} \quad . \quad . \quad . \quad . \quad (71)$$

$$\left.\begin{array}{l} \dfrac{\partial^2 \Psi}{\partial x^2} = \dfrac{\partial^2 \Psi}{\partial y^2} = \dfrac{(\lambda+\mu)\lambda_0}{2(\lambda+2\mu)h}\cot\dfrac{\pi y}{h} \\ \dfrac{\partial^2 \Psi}{\partial x \partial y} = 0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

Hence in the plane $x=0$:

$$\left.\begin{array}{l} \sigma_{xx} = -\dfrac{\mu(\lambda+\mu)\lambda_0}{(\lambda+2\mu)h}\cot\dfrac{\pi y}{h} \\ \sigma_{xy} = 0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

When x is different from zero we obtain:

$$\sigma_{xy} = 2\mu\frac{\partial^2 \Psi}{\partial x \partial y} = \frac{\mu(\lambda+\mu)\lambda_0}{(\lambda+2\mu)h}\,\frac{\pi x}{h}\,\frac{(\cosh 2\pi x/h)(\cos 2\pi y/h)-1}{2(\sinh^2 \pi x/h + \sin^2 \pi y/h)^2} \quad . \quad . \quad (74)$$

These expressions can be used for making estimates of the magnitude of the stresses to be expected in given cases, in a similar way as is done by TAYLOR.

Other examples can be constructed, giving rise to a multitude of possible cases. For instance the "surface of misfit" can be repeated periodically at a distance L_0, so that a series of flat blocks is obtained. Or fields containing a finite number of dislocations may be investigated, and cases where the singular lines are not parallel straight lines, but lines of other form. However, it seems preferable for the present to leave the matter here.