Geophysics. — *On the vibrations of an elastic sphere with central core.* By J. G. SCHOLTE. (Communicated by Prof. J. D. VAN DER WAALS Jr.)

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§ 1. A homogeneous elastic body of unlimited extent can transmit two kinds of waves, the longitudinal and the transversal. When the body is limited, both kinds of waves are reflected by its boundary and give rise to other waves of the same kind. Any vibratory motion of the body can be represented as the result of superposing longitudinal and transversal waves. In case of an infinite solid with an infinite horizontal surface these waves can combine to form a displacement that does not penetrate far beneath the surface: the Rayleigh wave. Assuming that this body is covered by a layer with other elastic properties, it is possible to construct another kind of surface waves: the Love waves. Further the effect, due to gravity and the surface layer, on the Rayleigh waves — and various similar corrections — can be taken into account.

There is however another method of investigating the movements of a limited body: the method of normal functions. The solution of the equation of wave propagation on a sphere is then written as a combination of some spherical harmonics, satisfying the boundary conditions. The theory of the superficial waves must be included in this theory of the vibrations of a sphere, as has been pointed out by RAYLEIGH. In case of a homogeneous sphere the deduction of the equation, giving the velocity of the Rayleigh waves from the period equation for the vibrations of a sphere, has been effected by BROMWICH and LOVE.

In this paper we shall investigate the oscillations of a sphere with central core, taking gravity into account, and are to arrive at period equations, from which numerous known equations can be deduced.

§ 2. According to the theory of LOVE [3]), concerning the oscillations of a homogeneous sphere, the equations of vibratory motion are three of the type

$$\varrho\,\frac{\partial^2 u}{\partial t^2}=(\lambda+\mu)\frac{\partial\triangle}{\partial x}+\mu\nabla^2 u+\varrho\,\frac{\partial}{\partial x}\left(A\,\frac{\partial V}{\partial r}\right)-\varrho\triangle\,\frac{\partial V}{\partial x}+\varrho\,\frac{\partial W}{\partial x}$$

where u = the x-component of the displacement; with similar equations for the y and z components (v and w). We have

ϱ = the density; λ and μ = Lamé's constants.

A = the radial component of the displacement.

$\triangle=\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}$.

V = the potential if no disturbation.

W = the potential due to a distribution of mass ($-\varrho\triangle$ in the medium and $+\varrho A$ at its boundary) together with that due to possible external bodies (e.g. the moon); hence

$$\nabla^2 W=4\pi\gamma\varrho\triangle.\quad(\gamma=\text{gravitation constant } 6.7\times10^{-8}).$$

These equations can be transformed into:

$$\left.\begin{aligned}
&\varrho\,\frac{\partial^2(r\,rot_r)}{\partial t^2}=\mu\nabla^2(r\,rot_r)\\
&\varrho\,\frac{\partial^2\triangle}{\partial t^2}=(\lambda+2\mu)\nabla^2\triangle+8\pi\gamma\varrho^2\triangle-\varrho\,\frac{\partial V}{\partial r}\cdot\frac{\partial\triangle}{\partial r}+\varrho\nabla^2\left(A\,\frac{\partial V}{\partial r}\right)\\
&\varrho\,\frac{\partial^2(rA)}{\partial t^2}=\mu\nabla^2(rA)+\varrho\,r\,\frac{\partial}{\partial r}\left(A\,\frac{\partial V}{\partial r}\right)+\\
&\qquad+(\lambda+\mu)\,r\,\frac{\partial\triangle}{\partial r}-2\mu\triangle-\varrho\triangle\,r\,\frac{\partial V}{\partial r}+\varrho\,r\,\frac{\partial W}{\partial r}
\end{aligned}\right\}\quad.\quad(I)$$

where $r=\sqrt{x^2+y^2+z^2}$, rot_r = the radial rotation component.

We shall write the solutions of this set of equations in the form $F(r).W_n.e^{ipt}$; W_n is a spherical solid harmonic of degree n.

From the form of the equations (*I*) it will be seen, that there are two possible types of vibrations:

1°. those, which involve no dilatation and no radial displacement; these correspond with the oscillations called by LAMB [1]) "vibrations of the first class".

2°. the second type, which LAMB described as being of the "second class", are those which cause no radial rotation component.

§ 3. Beginning with the vibrations of the first class, we put $\triangle=0$ and $A=0$; equations (*I*) are then reduced to

$$(\varrho p^2+\mu\nabla^2)(r\,rot_r)=0.$$

The general solution is

$$r\,rot_r=n(n+1)(a_1\psi_n+b_1\pi_n)\,.\,W_n\,.\,e^{ipt},$$

where

$$\psi_n=\left(\frac{1}{x}\frac{d}{dx}\right)^n\frac{\sin x}{x},\quad \pi_n=\left(\frac{1}{x}\frac{d}{dx}\right)^n\frac{\cos x}{x},$$

and the argument $x=\dfrac{\varrho p^2}{\mu}\,r$ or αr.

As $\triangle = 0$ and $A = 0$:

$$\frac{\partial C}{\partial \varphi} + \frac{\partial (B \sin \psi)}{\partial \psi} = 0,$$

(B and C are the meridional and the azimuthal components of the displacement); hence

$$C = \frac{\partial M}{\partial \psi}, \quad B = -\frac{1}{\sin \psi}\,\frac{\partial M}{\partial \varphi},$$

which gives

$$r \, rot_r = r^2 \left(\nabla^2 - \frac{\partial^2}{\partial r^2} - \frac{2}{r}\,\frac{\partial}{\partial r} \right) . M$$

or

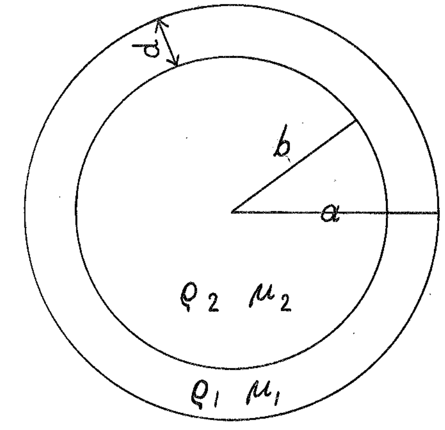
$$M = -(a_1 \psi_n + b_1 \pi_n) \,.\, W_n.$$

The components of the displacement are therefore:

$$A = 0, \; B = (a_1 \psi_n + b_1 \pi_n) \,.\, \frac{\partial W_n}{\sin \psi \,.\, \partial \varphi}, \; C = -(a_1 \psi_n + b_1 \pi_n) \,.\, \frac{\partial W_n}{\partial \psi}$$

(the time factor e^{ipt} being omitted for brevity).

We apply this solution to a sphere with central core; the movement of the first kind is perpendicular to the radius and must therefore be related to the Love waves, as those movements are also perpendicular to the vertical.



The solution is now:

$$\left\{ \begin{aligned} B &= (a_1 \psi_n + b_1 \pi_n) \,.\, \frac{\partial W_n}{\sin \psi \,.\, \partial \varphi}, \\ C &= -(a_1 \psi_n + b_1 \pi_n) \,.\, \frac{\partial W_n}{\partial \psi}, \end{aligned} \right.$$

if $b \leqq r \leqq a$; and

$$\left\{ \begin{aligned} B &= a_2 \psi'_n \,.\, \frac{\partial W_n}{\sin \psi \,.\, \partial \varphi}, \\ C &= -a_2 \psi'_n \,.\, \frac{\partial W_n}{\partial \psi}, \end{aligned} \right.$$

if $r \leqq b$

(the argument of ψ'_n is $\frac{\varrho_2 p^2}{\mu_2} r$ or $\alpha_2 r$).

The boundary conditions are:

1. at $r = b$ continuity of movement:

$$a_1 \psi_n^b + b_1 \pi_n^b = a_2 \psi'^b_n$$

and continuity of tension:

$$\mu_1 \frac{\partial}{\partial b} \left\{ (a_1 \psi_n^b + b_1 \pi_n^b) \,.\, \frac{W_n}{b} \right\} = \mu_2 \frac{\partial}{\partial b} \left\{ a_2 \psi'^b_n \,.\, \frac{W_n}{b} \right\}$$

2. the vanishing of the traction across the free surface $r = a$:

$$\frac{\partial}{\partial a} \left\{ (a_1 \psi_n^a + b_1 \pi_n^a) \,.\, \frac{W_n}{a} \right\} = 0.$$

Hence

$$b_1 = -\frac{(n-1)\, \psi_n^a + \alpha_1 a\, \overline{\psi}_n^a}{(n-1)\, \pi_n^a + \alpha_1 a\, \overline{\pi}_n^a} \,.\, a_1 \quad \text{and} \quad a_2 = \frac{\psi_n^b + \frac{b_1}{a_1} \,.\, \pi_n^b}{\psi'^b_n} \,.\, a_1$$

(a horizontal line means differentiation to the argument).

Substituting in the second equation we obtain the period equation:

$$\frac{\mu_2}{\mu_1} \,.\, \left\{ (n-1) + \alpha_2 b \,.\, \frac{\overline{\psi}'^b_n}{\psi'^b_n} \right\} = (n-1) + \alpha_1 b \,.\, \frac{1 + \frac{b_1}{a_1} \,.\, \frac{\overline{\pi}_n^b}{\overline{\psi}_n^b}}{1 + \frac{b_1}{a_1} \,.\, \frac{\pi_n^b}{\psi_n^b}} \,.\, \frac{\overline{\psi}_n^b}{\psi_n^b}$$

or:

$$\left\{ \left(1 - \frac{\mu_1}{\mu_2} \right) . (n-1) \,.\, \psi'^b_n + \alpha_2 b \,.\, \overline{\psi}'^b_n \right\} .$$
$$. \left\{ (n-1) (\psi_n^b \pi_n^a - \psi_n^a \pi_n^b) + \alpha_1 a \, (\psi_n^b \overline{\pi}_n^a - \overline{\psi}_n^a \pi_n^b) \right\} =$$
$$= \frac{\varrho_1}{\varrho_2} \,.\, \alpha_2 b \,.\, \psi'^b_n \,.\, \left\{ (n-1) (\overline{\psi}_n^b \pi_n^a - \psi_n^a \overline{\pi}_n^b) + \alpha_1 a \, (\overline{\psi}_n^b \overline{\pi}_n^a - \overline{\psi}_n^a \overline{\pi}_n^b) \right\} .$$

By putting $a = b$ we arrive at an equation, identical with that found by LAMB [1]) for the vibrations of the 1st class of a homogeneous sphere:

$$\alpha_2 a \,.\, \frac{\overline{\psi}'^a_n}{\psi'^a_n} + (n-1) = 0.$$

a. To explain seismometric data it is generally assumed, that the thickness ($d = a - b$) of the surface layer is small compared with the earth's radius. As a first approximation we have: $\psi_n^b = \psi_n^a - \alpha d \overline{\psi}_n^a$; then $\overline{\psi}_n^b = \overline{\psi}_n^a - \alpha d \,.\, \overline{\overline{\psi}}_n^a$, and as ψ_n^a satisfies $\overline{\overline{\psi}}_n^a + \frac{2(n+1)}{\alpha a} \overline{\psi}_n^a + \psi_n^a = 0$:

$$\overline{\psi}_n^b = \overline{\psi}_n^a \,.\, \left\{ (1 + 2(n+1) \frac{d}{a} \right\} + \alpha d \,.\, \psi_n^a.$$

After some reduction we find the period equation in the form:

$$\alpha_2\, a \cdot \frac{\bar{\psi}'^a_n}{\psi^a_n} + (n-1) = \frac{\left(1-\frac{\mu_1}{\mu_2}\right)(n-1)(n+2) - \left(1-\frac{\mu_2}{\mu_1}\cdot\frac{\varrho_1^{\,2}}{\varrho_2^{\,2}}\right)\cdot(\alpha_2\, a)^2}{(2n+1)\frac{d}{a}+1}\cdot\frac{d}{a} \quad (II)$$

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if $n d \langle\langle\, a$.

In case of a sectorial harmonic $\frac{\pi a}{n}$ is the spherical distance between two points on the surface of the earth where $W_n = 0$, or $\frac{2\pi a}{n}$ is the "wave length". Therefore the condition $n d \langle\langle\, a$ means, that the wave length should be great compared with the depth of the layer; the vibrations of the earth are then those given by the period equation of a homogeneous sphere (consisting of the material of the earth's core), with a correction term due to the surface layer.

b. This approximation is no longer applicable, when the degree n of the spherical solid harmonic is great. Taking a, b and n very great, we shall find the period equation of the oscillations, possible in an infinite solid with a plane surface, covered by a surface layer with depth d, viz. the equation of the Love-waves. We suppose: a, b and n are infinite, so that $a-b=d$ and $\frac{n}{a}=l$ are finite. It is now necessary to obtain the form of ψ_n (or π_n), when both the order and the argument are very great and $r = b \pm z$, $z \langle\langle\, b$. Using the method of BROMWICH [2]):

ψ_n satisfies the equation:

$$\frac{d^2y}{dr^2} + \frac{2(n+1)}{r}\frac{dy}{dr} + \alpha_1^{\,2}\, y = 0$$

write $r = b + z$:

$$\frac{d^2y}{dz^2} + \frac{2(n+1)}{z+b}\frac{dy}{dz} + \alpha_1^{\,2}\, y = 0;$$

as $n \rangle\rangle\, 1$, $r \langle\langle\, b$:

$$\frac{d^2y}{dz^2} + 2l\frac{dy}{dz} + \alpha_1^{\,2}\, y = 0; \text{ hence } y = e^{-lz \pm s_1 z}, \text{ where } s_1 = \sqrt{l^2-\alpha_1^{\,2}}.$$

The functions must then be of the form:

$$\psi_n = e^{-lz}\,(c_1\, e^{s_1 z} + d_1\, e^{-s_1 z})$$
$$\pi_n = e^{-lz}\,(c_2\, e^{s_1 z} + d_2\, e^{-s_1 z}).$$

We have now (c_1, d_1, c_2, d_2 are unknown constants):

$$b_1 = -\frac{(n-1)\,\psi^a_n + \alpha_1\, a\, \bar{\psi}^a_n}{(n-1)\,\pi^a_n + \alpha_1\, a\, \bar{\pi}^a_n}\cdot a_1 \approx -\frac{l\,\psi^a_n + \left(\frac{d\bar{\psi}_n}{dz}\right)_{z=d}}{l\,\pi^a_n + \left(\frac{d\bar{\pi}_n}{dz}\right)_{z=d}}\, a_1 = -\frac{c_1\, e^{s_1 d} - d_1\, e^{-s_1 d}}{c_2\, e^{s_1 d} - d_2\, e^{-s_1 d}}\cdot a_1$$

and

$$\frac{n-1}{b} + \alpha_1 \cdot \frac{1+\frac{b_1}{a_1}\cdot\frac{\bar{\pi}^b_n}{\bar{\psi}^b_n}}{1+\frac{b_1}{a_1}\cdot\frac{\pi^b_n}{\psi^b_n}}\cdot\frac{\bar{\psi}^b_n}{\psi^b_n} = l + \frac{\left(\frac{d\psi_n}{dz}\right)_{z=0} + \frac{b_1}{a_1}\cdot\left(\frac{d\pi_n}{dz}\right)_{z=0}}{(\psi_n)_{z=0} + \frac{b_1}{a_1}\,(\pi_n)_{z=0}} \approx$$

$$\approx s_1 \cdot \frac{(c_1-d_1) + \frac{b_1}{a_1}(c_2-d_2)}{(c_1+d_1) + \frac{b_1}{a_1}(c_2+d_2)} = s_1 \cdot \frac{e^{s_1 d} - e^{-s_1 d}}{e^{s_1 d} + e^{-s_1 d}} = -\, s_1 \,.\, tgh\, s_1 d.$$

BROMWICH found on this method:

$$\psi_n' = c_3\,.\, e^{(l-s_2)z}, \text{ where } z = b - r \text{ and } s_2 = \sqrt{l^2-\alpha_2^{\,2}};$$

the left-hand member of the period equation can now be reduced to $\frac{\mu_2 s_2}{\mu_1}$. Hence:

$$\frac{\mu_2\, s_2}{\mu_1} = -\, s_1 \,.\, tgh\, s_1 d.$$

This equation has only roots, if s_1 is imaginary; we assume therefore $\alpha_1 > l$ and put $s_1 = \sqrt{\alpha_1^{\,2} - l^2}$. The period equation becomes

$$\frac{\mu_2\, s_2}{\mu_1\, s_1} = tg\; s_1 d.,$$

which is the equation of the Love-waves, which are only possible if

$$\alpha_1 > l > \alpha_2, \text{ or } \frac{\varrho_1}{\mu_1} > \left(\frac{l}{p}\right)^2 > \frac{\varrho_2}{\mu_2}.$$

§ 4. Proceeding to the movements of the second class we narrow down the problem to that of a core, surrounded by an incompressible liquid; this is to the seismologist one of the most interesting of these problems, viz. the interaction between the movements of the ocean and the earth's core.

Since the liquid is incompressible: $\triangle = 0$, but as $\lambda = \infty$, $\lambda\triangle$ will be finite. Putting $\lambda\triangle = D$, we have the modified equations

$$\begin{cases} \nabla^2\left(D + \varrho_1\, A\,\frac{\partial V}{\partial r}\right) = 0 \quad , \quad \nabla^2\, W = 0 \\ \varrho_1\, p^2\, A + \frac{\partial}{\partial r}\left(D + \varrho_1\, A\,\frac{\partial V}{\partial r}\right) + \varrho_1\,\frac{\partial W}{\partial r} = 0. \end{cases}$$

Suppose $W = (a + b\, r^{-(2n+1)})\, W_n$ and $D + \varrho_1\, A\,\frac{\partial V}{\partial r} = (c + d\, r^{-(2n+1)})\, W_n$, where $W_n = r^n \times$ spherical surface harmonic S_n.

Whe have then: $A = \frac{\partial F}{\partial r}$, and, with $rot_r = 0$ and $\Delta = 0$, we find

$$B = \frac{1}{r}\frac{\partial F}{\partial \psi},\quad C = \frac{1}{r\sin\psi}\frac{\partial F}{\partial \varphi},\ \text{with}\ F = -\frac{a\varrho_1 + c}{\varrho_1 p^2}\,.\,r^n S_n - \frac{b\varrho_1 + d}{\varrho_1 p^2} r^{-(n+1)}\,.\,S_n.$$

Assuming for simplicity's sake, that the core is also incompressible, the equations of motion of the core are

$$\left\{\begin{aligned} &\nabla^2\left(D + \varrho_2 A \frac{\partial V}{\partial r}\right) = 0 \quad , \quad \nabla^2 W = 0 \\ &(\varrho_2 p^2 + \mu\nabla^2)\, r A + r\frac{\partial}{\partial r}\left(D + \varrho_2 A \frac{\partial V}{\partial r}\right) + \varrho_2 r \frac{\partial W}{\partial r} = 0. \end{aligned}\right.$$

Put $W = a' W_n$ and $D + \varrho_2 A \frac{\partial V}{\partial r} = c' W_n$; then we have

$$(\varrho_2 p^2 + \mu\nabla^2)\, r A + r \frac{\partial}{\partial r}(a'\varrho_2 + c')\, W_n = 0,$$

whence $A = -\frac{a'\varrho_2 + c'}{\varrho_2 p^2} n W_n + n e \psi_n W_n$, where the argument of ψ_n is $\frac{\varrho_2 p^2}{\mu} r$ or $k^2 r$. Again we have $rot_r = 0$ and $\Delta = 0$, therefore:

$$B = \left\{-\frac{a'\varrho_2 + c'}{\varrho_2 p^2} + e\,.\left(\frac{1}{n+1} kr\overline{\psi}_n + \psi_n\right)\right\}.\,\frac{1}{r}\,\frac{\partial W_n}{\partial \psi}$$

$$C = \text{the same factor} \times \frac{1}{r\sin\psi}\,\frac{\partial W_n}{\partial \varphi}.$$

The boundary conditions at the bottom of the ocean ($r = r_2$) are the continuity of W, A and of the tensions T_{RR} and $T_{R\psi}$, or, in the same order:

1. $a' = a + b\, r_2^{-(2n+1)}$.

2. $-n.\frac{a'\varrho_2 + c'}{\varrho_2 p^2} + n e \psi_n = -n\,.\,\frac{a\varrho_1 + c}{\varrho_1 p^2} + (n+1)\,.\,\frac{b\varrho_1 + d}{\varrho_1 p^2} r_2^{-(2n+1)}$.

3. $c' + \left({}^4/_3 \pi\gamma\varrho_2^2 + 2\mu\frac{n-1}{r_2^2}\right).\,r_2 A + 2\mu\,.\,\frac{n}{r_2^2}\,.\,e k r_2\,.\,\overline{\psi}_n =$

$$= c + d r_2^{-(2n+1)} + {}^4/_3\pi\gamma\varrho_1\varrho_2 r_2 A.$$

4. $2(n-1)\,.\left(-\frac{a'\varrho_2 + c'}{\varrho_2 p^2} + e\psi_n\right) - \frac{k^2 r_2^2}{n+1} e\psi_n - \frac{2}{n+1} e k r_2 \overline{\psi}_n = 0.$

The equations 2. and 4. give

$$e = \frac{2(n^2 - 1)}{n k r_2 (k r_2 \psi_n + 2\overline{\psi}_n)}\,.\,r_2 A,$$

substituting in 4.:

$$\frac{c'}{\varrho_2 p^2} = -\frac{a'}{p^2} + \left\{\frac{2(n^2-1)\psi_n}{n k r_2 (k r_2 \psi_n + 2\overline{\psi}_n)} - \frac{1}{n}\right\}.\,r_2 A$$

with 3.

$$c + d r_2^{-(2n+1)} = \left\{{}^4/_3\pi\gamma\varrho_2(\varrho_2 - \varrho_1) + 2\mu\,.\,\frac{n-1}{r_2^2} + \right.$$

$$\left. + \frac{2(n^2-1)}{k^2 r_2^2}\,.\,\varrho_2 p^2\,.\,\frac{x + \frac{1}{n}}{x+1} - \frac{\varrho_2 p^2}{n}\right\}.\,r_2 A$$

where we have put $2\frac{\overline{\psi}_n}{k r_2 \psi_n} = x$ for brevity.

With 1. we find

$$\frac{\varrho_1}{\varrho_2}\,.\,\frac{c + d r_2^{-(2n+1)} + \varrho_2 a + \varrho_2 b r_2^{-(2n+1)}}{\varrho_1 p^2\,.\,r_2 A} = \frac{{}^4/_3\pi\gamma(\varrho_2 - \varrho_1)}{p^2} - \frac{1}{n} +$$

$$+ \frac{2(n-1)}{k^2 r_2^2} + \frac{2(n^2-1)}{k^2 r_2^2}\,.\,\frac{x + \frac{1}{n}}{x+1}$$

or

$$2\frac{\overline{\psi}_n}{k r_2 \psi_n} = -\frac{\frac{k^2 r_2^2}{2(n-1)} H - (2n+1)}{\frac{k^2 r_2^2}{2(n-1)} H - n(n+2)}, \quad \ldots\ldots \quad (IIIa)$$

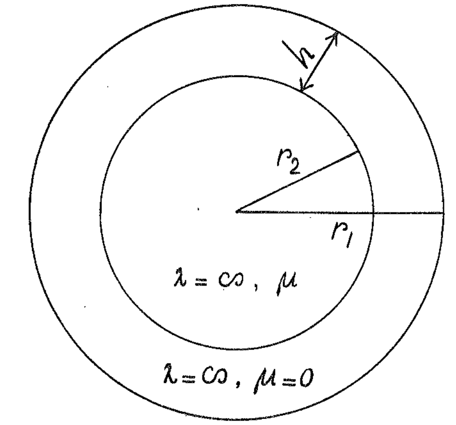
where

$$H = 1 - \frac{{}^4/_3\pi\gamma(\varrho_2 - \varrho_1)}{p^2} + n\,.\,\frac{\varrho_1}{\varrho_2}\,.\,\frac{c + d r_2^{-(2n+1)} + \varrho_2 a + \varrho_2 b r_2^{-(2n+1)}}{-(c + \varrho_1 a) n + (d + \varrho_1 b)(n+1) r_2^{-(2n+1)}}.$$

The boundary condition at the free surface ($r = r_1$) is $T_{RR} = 0$; hence:

$c + d r_1^{-(2n+1)} + {}^4/_3\pi\gamma\varrho_1\varrho_0 r_1 A = 0$, where we have put $\varrho_0 = \varrho_1 + (\varrho_2 - \varrho_1)\,.\left(\frac{r_2}{r_1}\right)^3$.

Again there are two boundary conditions due to the surface mass-



distribution $\varrho_1 A$ at $r = r_1$ and $(\varrho_1 - \varrho_2) A$ at $r = r_2$. The potential W is

$$(a + b\, r^{-(2n+1)}) \,.\, r^n S_n, \text{ if } r_2 \leqq r \leqq r_1$$

$$a'\, r^n S_n \text{ or } (a + b\, r_2^{-(2n+1)}) \,.\, r^n S_n, \text{ if } r \leqq r_2$$

and at external points $(r \geqq r_1)$

$$(a\, r_1^{+(2n+1)} + b) \,.\, r^{-(n+1)} \,.\, S_n.$$

The surface characteristic equations for the potential are therefore:

$$\frac{\partial}{\partial r} \{(a + b\, r^{-(2n+1)})\, W_n\} - \frac{\partial}{\partial r} \{(a\, r_1^{2n+1} + b)\, r^{-(2n+1)} \,.\, W_n\} = 4\pi\gamma\varrho_1 A \text{ at } r = r_1$$

$$\frac{\partial}{\partial r} \{(a + b\, r^{-(2n+1)})\, W_n\} - \frac{\partial}{\partial r} \{(a + b\, r_2^{-(2n+1)})\, W_n\} = 4\pi\gamma(\varrho_1 - \varrho_2) A \text{ at } r = r_2.$$

Or

$$(2n+1) \,.\, b\, r_2^{-(2n+1)} = 4\pi\gamma(\varrho_2 - \varrho_1) \,.\, r_2 A \text{ and } (2n+1)\, a = 4\pi\gamma\varrho_1 r_1 A.$$

These equations give with 6.

$$b = a \,.\, \frac{\varrho_2 - \varrho_1}{\varrho_1} \,.\, \frac{1 + \dfrac{n(n+1)(1-s)}{2n+1} \,.\, \dfrac{f^\alpha}{\alpha s}}{1 + \dfrac{n(n+1)(1-s)}{2n+1} \,.\, \dfrac{1}{\beta}} \,.\, r_2^{2n+1},$$

where

$$\alpha = \frac{(2n+1)\, p^2}{4\pi\gamma\varrho_1}, \quad \beta = \frac{(2n+1)\, p^2}{4\pi\gamma\,(\varrho_2 - \varrho_1)}, \quad s = \left(\frac{r_2}{r_1}\right)^{2n+1},$$

$$f^\alpha = 1 - \frac{2n+1}{3} \frac{\varrho_0}{\varrho_1} + \frac{\alpha}{n}, \quad f^\beta = \frac{\varrho_2 - \varrho_1}{\varrho_1} - \frac{2n+1}{3} \frac{\varrho_2}{\varrho_1} + \frac{\alpha}{n}.$$

Substituting in H we obtain:

$$H = n \frac{\varrho_1}{\varrho_2} \,.\, \frac{\dfrac{f^\beta}{\beta} + \dfrac{f^\alpha}{\alpha s} + \dfrac{2}{\beta} - \dfrac{n(n+1)(1-s)}{2n+1} \,.\, \dfrac{1}{\beta^2}}{1 + \dfrac{n(n+1)(1-s)}{2n+1} \,.\, \dfrac{f^\alpha}{\alpha s}} \quad . \quad . \quad . \quad (IIIb)$$

Equations $(IIIa)$ and $(IIIb)$ give the period equation of a sphere, covered by an ocean of uniform depth.

a. It is obvious, that we can consider the periods p, determined by a known value of n, as the periods of the vibrations of the core, altered by the surface layer, or as those of the ocean with a variation due to the core. Equation (III) must therefore include as limiting cases the period equations, both of a free oscillating homogeneous sphere and of a free oscillating ocean.

Firstly: we put $r_1 = r_2$ or $s = 1$; then

$$H = n \frac{\varrho_1}{\varrho_2} \left(\frac{f^\beta}{\beta} + \frac{f^\alpha}{\alpha} + \frac{2}{\beta} \right)$$

$$H = 1 - \frac{2n(n-1)}{(2n+1)\, p^2} \,.\, \tfrac{4}{3}\pi\gamma\varrho_2.$$

Substituting in $(IIIa)$ we obtain

$$2 \frac{\overline{\psi}_n}{k\, r_2\, \psi_n} = - \frac{\dfrac{k^2 r_2^2}{2(n-1)} - \dfrac{n\, g\, \varrho_2 r_2}{(2n+1)\, \mu} - (2n+1)}{\dfrac{k^2 r_2^2}{2(n-1)} - \dfrac{n\, g\, \varrho_2 r_2}{(2n+1)\, \mu} - n(n+2)}, \text{ where } g = {}^4/_3\, \pi\gamma\varrho_2 r_2,$$

the equation found by BROMWICH [2]) in case of a homogeneous incompressible sphere.

Secondly: to deduce the period equation of an ocean with uniform depth, covering a non-vibrating core, we must assume that $\mu = \infty$. Then $k = 0$ and $H = \infty$, or

$$1 + \frac{n(n+1)(1-s)}{2n+1} \,.\, \frac{f^\alpha}{\alpha s} = 0;$$

this gives:

$$p^2 = \frac{n(n+1)(1-s)}{n+1+ns} \,.\, \left(1 - \frac{3}{2n+1} \,.\, \frac{\varrho_1}{\varrho_0}\right) \,.\, \tfrac{4}{3}\pi\gamma\varrho_0,$$

being the equation in question. (LAMB. Hydrodynamics).

Note. Taking this solution of LAMB we find that the tension, due to the movement of the ocean, on the surface of the core

$$T_{RR} = \left(\frac{r_2}{r_1}\right)^n \,.\, g \,.\, \frac{2n+1}{n+1+s} \,.\, \left(\varrho_1 \,.\, \frac{3n(1-s)}{(2n+1)^2} - \varrho_0\right) \,.\, S_n.$$

This periodic tension causes a vibration of the core, which can only be neglected, if the wave length is small compared with the depth of the ocean.

b. In case of a liquid core we put $\mu=0$; then $k=\infty$ and $H=0$, or

$$a^2\left(1+\frac{\varrho_2-\varrho_1}{\varrho_1}\cdot\frac{n+1+ns}{2n+1}\right)+na\left\{1-\frac{2n+1}{3}\frac{\varrho_0}{\varrho_1}+\right.$$

$$\left.+\frac{\varrho_2-\varrho_1}{\varrho_1}s-(n+1)(1-s)\frac{\varrho_2-\varrho_1}{\varrho_1}\cdot\frac{1}{3}\frac{\varrho_0}{\varrho_1}-2(n-1)\cdot\frac{\varrho_2-\varrho_1}{\varrho_1}\cdot\frac{n+1+ns}{2n+1}\cdot\frac{1}{3}\frac{\varrho_2}{\varrho_1}\right\}$$

$$+\frac{n^2(n+1)(1-s)}{2n+1}\cdot\frac{\varrho_2-\varrho_1}{\varrho_1}\cdot\left\{\left(1-\frac{2n+1}{3}\frac{\varrho_0}{\varrho_1}\right)\left(\frac{\varrho_2-\varrho_1}{\varrho_1}-\frac{2n+1}{3}\frac{\varrho_2}{\varrho_1}\right)-\frac{\varrho_2-\varrho_1}{\varrho_1}s\right\}=0.$$

If n is great, the two roots of this equation correspond with 1. the frequency of an ocean of uniform depth, and 2. the frequency of two infinite liquids with a plane interface, as has been remarked by SEZAWA [8]).

c. BROMWICH (and later LOVE) has proved, that the Rayleigh-waves are vibrations of the 2nd class of a homogeneous sphere (if r_1 and n are very great). We can therefore expect, that the period equation of the corresponding vibrations of a heterogeneous sphere include as a limiting case the equation which determines the propagation of the Rayleigh-waves, with correction terms due to gravity and the liquid surface layer.

When r_1, r_2 and n are infinite, so that $r_1-r_2=h$ and $\frac{n}{r_1}=l$ are finite, then:

$$s=\left(\frac{r_2}{r_1}\right)^{2n+1}\approx e^{-2hl}\quad,\quad \varrho_0=\varrho_1+(\varrho_2-\varrho_1)\left(\frac{r_2}{r_1}\right)^3\approx\varrho_2.$$

$$a=\frac{(2n+1)\,p^2}{4\pi\gamma\varrho_1}\approx\frac{{}^2/_3\,n^2p^2}{gl}\cdot\frac{\varrho_2}{\varrho_1},\quad\text{where } g={}^4/_3\,\pi\gamma\varrho_2 r_1.$$

After some reduction, we find

$$\frac{f^\alpha}{\alpha}\approx\frac{1}{n}\left(1-\frac{gl}{p^2}\right)\quad\text{and}\quad\frac{f^\beta}{\beta}\approx\frac{\varrho_2-\varrho_1}{\varrho_1}\cdot\frac{1}{n}\left(1-\frac{gl}{p^2}\right).$$

Hence

$$H\approx\left(1+\frac{\varrho_1}{\varrho_2}\cdot\frac{1+\dfrac{gl}{p^2}}{ctgh\,hl-\dfrac{gl}{p^2}}\right)\left(1-\frac{gl}{p^2}\right),\quad\text{or}\quad H=(1+K)\left(1-\frac{gl}{p^2}\right).$$

Applying the method of BROMWICH to equation (*IIIa*) we find

$$4\sqrt{1-\zeta}-(2-\zeta)^2+\zeta^2(1-H)=0,\quad\text{where }\zeta=\frac{k^2}{l^2}.$$

Substituting H, the period equation becomes

$$4\sqrt{1-\zeta}-(2-\zeta)^2+\frac{g\varrho_2}{\mu l}\zeta(1+K)-\zeta^2K=0,\quad K=\frac{\varrho_1}{\varrho_2}\cdot\frac{1+\dfrac{gl}{p^2}}{ctgh\,hl-\dfrac{gl}{p^2}}.\qquad(IV)$$

This equation determines the velocity of the Rayleigh-waves on the bottom of the ocean; it is obvious, that it includes the equation of the Rayleigh-waves ($g=0$ and $h=0$) and the equation

$$4\sqrt{1-\zeta}-(2-\zeta)^2+\frac{g\varrho_2}{\mu l}\zeta-\zeta^2\,tgh\,hl=0,$$

found by BROMWICH in case $hl \ll 1$.

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