Hydrodynamics. — *Application of a model system to illustrate some points of the statistical theory of free turbulence.* By J. M. BURGERS. (Mededeeling No. 37 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hoogeschool te Delft.)

(Communicated at the meeting of December 30, 1939.)

1. *Introduction.* — In recent years several authors, in particular G. I. TAYLOR and TH. VON KARMAN, have given much attention to the investigation of the correlation coefficients characteristic of the turbulent motion which is found in an air stream of constant mean velocity [1]). The turbulence in the cases considered usually is produced by a screen or a honeycomb through which the air stream has passed (such a case often presents itself with the air flow in a wind channel), and it is assumed that in the region to be considered the motion of the air is not influenced by guiding walls etc.

The assumption that the mean velocity has a constant value throughout the current implies that there is no transfer of energy from the mean motion to the turbulent motion; in consequence there will be a gradual decay of the turbulence, and the object of the investigations is to find the laws of this decay, and of the correlation phenomena associated with it.

An important conception introduced by TAYLOR into this work is that of isotropic turbulence, which is characterized by the circumstance that the average value of any function of the velocity components, defined in relation to a given set of axes, is unaltered if the axes of reference are rotated in any manner [2]).

Various results arrived at in the theoretical developments have been compared with the results of numerous accurate experimental observations,

[1]) G. I. TAYLOR, Statistical theory of turbulence, Proc. Roy. Soc. (London) A **151**, pp. 421—478, 1935; A **156**, pp. 307—317, 1936; Journ. Aeron. Sciences **4**, p. 311, 1937; Some recent developments in the study of turbulence, Proc. Vth Intern. Congr. for Applied Mechanics, Cambridge, Mass., 1938, p. 294.

TH. VON KARMAN, The fundamentals of the statistical theory of turbulence, Journ. Aeron. Sciences **4**, p. 131, 1937; On the statistical theory of turbulence, Proc. Nat. Acad. of Sciences (Washington) **23**, p. 98, 1937; Some remarks on the statistical theory of turbulence, Proc. Vth Intern. Congress for Applied Mechanics, Cambridge, Mass., 1938, p. 347; TH. DE KARMAN and L. HOWARTH, On the statistical theory of isotropic turbulence, Proc. Roy. Soc. (London) A **164**, pp. 192—215, 1938.

H. L. DRYDEN, Turbulence investigations at the National Bureau of Standards, Proc. Vth Intern. Congress for Applied Mechanics, Cambridge, Mass., 1938, p. 362; Turbulence and the boundary layer, Journ. Aeron. Sciences **6**, p. 85, 1938; Turbulence and diffusion, Journ. Industrial and Engineering Chemistry **31**, p. 416, 1939 (all with extensive references to the literature of the subject).

L. PRANDTL, Beitrag zum Turbulenzsymposium, Proc. Vth Intern. Congress for Applied Mechanics, Cambridge, Mass., 1938, p. 340.

[2]) G. I. TAYLOR, Proc. Roy. Soc. (London) A **151**, p. 430, 1935.

made possible by the application of electrical hot wire anemometers, and the insight obtained in this way is continually developing [3]).

Now in a previous communication the present author has described some mathematical model systems, by means of which several features could be illustrated which play a part in the behaviour of turbulent motion [4]). Although these model systems had been constructed with a view to illustrate the development of a dissipative secondary motion, which grows by detracting energy from a given primary phenomenon until a balance is obtained between energy detracted and energy dissipated, the question can be brought forward whether these same model systems also may be used in order to illustrate some of the relations found in the theory of the decay of free turbulence. It is true that most of the geometrical relations which are of importance in the theory of isotropic turbulence cannot find a counterpart in the model referred to; nevertheless it is possible to illustrate the conception of correlation and the equations describing the decay of free turbulence.

This will be shown in the following sections (**2**—**3**), while in **4** and **5** those properties of the model system which are operative in the propagation of "elementary regions of turbulence" have been considered in more detail.

2. *Application of the assumptions of the theory of uniform isotropic turbulence to the model system.* — It is convenient to take as a guide the exposition of the statistical theory of isotropic turbulence given by VON KARMAN and HOWARTH [5]), and to indicate which of the ideas developed in their paper can be applied to our model system.

The model system is defined by the equations [6]):

$$\left.\begin{aligned} \frac{\partial v}{\partial t} &= U\,(v-w) + \nu\,\frac{\partial^2 v}{\partial y^2} - 2v\,\frac{\partial v}{\partial y} + 2w\,\frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial t} &= U\,(v+w) + \nu\,\frac{\partial^2 w}{\partial y^2} + 2w\,\frac{\partial v}{\partial y} + 2v\,\frac{\partial w}{\partial y} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

[3]) Apart from the papers already mentioned in footnote 1) see: G. I. TAYLOR, Correlation measurements in a turbulent flow through a pipe, Proc. Roy. Soc. (London) A **157**, pp. 537—546, 1936; G. I. TAYLOR and A. E. GREEN, Mechanism of the production of small eddies from large ones, *ibid.* A **158**, pp. 499—521, 1937; G. I. TAYLOR, Production and dissipation of vorticity in a turbulent field, *ibid.* A **164**, pp. 15—23, 1938; The spectrum of turbulence, *ibid.* A **164**, pp. 476—490, 1938.

For the experimental investigations themselves the reader is referred to the articles mentioned by Prof. TAYLOR in these papers, and to those of DRYDEN and his co-workers.

The reader is also referred to a great number of papers on turbulence in the Proc. Vth Intern. Congress for Applied Mechanics, Cambridge, Mass., 1938. References to the experimental investigations are given in many of these papers.

[4]) J. M. BURGERS, Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion, Verhand. Kon. Nederl. Akad. v. Wetenschappen, Afd. Natuurk. (Ie sectie) **17**, No. 2, 1939.

[5]) TH. DE KARMAN and L. HOWARTH, Proc. Roy. Soc. (London) A **164**, p. 192, 1938.

[6]) J. M. BURGERS, *l.c.* p. 16, eqs. (8. 2), (8. 3).

In order to adapt the equations to the conditions of the present subject it is necessary to assume that the domain of the coordinate y extends from $-\infty$ to $+\infty$, so that there are no boundary conditions to be fulfilled by v and w. It is supposed that the initial distributions (for $t=0$) of v and w are given as functions of y, and it is asked to find some general rules governing their development in the course of time. A special case *e.g.* might be represented by a distribution in which the y-axis is divided into alternating segments of lengths a and b, the initial values of v and w being assumed to be zero in the segments of length a, whereas they follow some prescribed course, may be of similar type but not necessarily the same for all segments, in the segments of length b.

To obtain a full analogy with the "free turbulence" of the air stream referred to before, in which there is no transfer of energy from the mean motion to the turbulent motions, we must take U equal to zero in the model system. However, when in the model system U is supposed to be a constant, its presence does not interfere with the condition of isotropy (in this respect the model system differs from the actual hydrodynamical equations); hence provisionally we shall retain the terms with U, as it is possible to drop them afterwards at any time we may like.

In analogy with the assumptions stated by VON KARMAN and HOWARTH, *l.c.* p. 193, we shall suppose that in our model system it is possible to consider *average values with respect to the time* of quantities like v, w, v^2, ..., assuming at the same time that the fluctuations actually occurring in these quantities are so rapid, that the variations of the average values are negligible throughout the period of time required for averaging. The average values consequently shall be treated as slowly varying functions of the time.

It will be assumed that the turbulence is statistically *uniform*, so that the average values are invariant against a translation along the y-axis of the points or systems of points with reference to which they are defined.

The assumption of statistical *isotropy* further induces us to suppose that the average values are invariant with regard to a "rotation" and a "reflection" of the axes for the variables v and w. This implies that we take:

$$\left.\begin{array}{ll} \overline{v}=0, & \overline{w}=0 \\ \overline{v^2}=\overline{w^2} & \\ \overline{vw}=0, & \overline{v^2w}=0, \text{ etc.} \end{array}\right\} \quad \ldots \ldots \quad (2)$$

We now define a correlation function $f(r, t)$ by considering two points P_1 and P_2, with coordinates y_1, y_2 lying at a distance $r=y_2-y_1$ from each other; then we calculate the average value of $v_1(t) \cdot v_2(t)$ and write:

$$\overline{v_1 \cdot v_2}=f \cdot \overline{v^2} \quad \ldots \ldots \quad (3)$$

The condition of isotropy implies that also:

$$\overline{w_1 \cdot w_2}=f \cdot \overline{w^2}=f \cdot \overline{v^2} \quad \ldots \ldots \quad (4)$$

whereas on the other hand we shall have:

$$\overline{v_1 \cdot w_2}=\overline{w_1 \cdot v_2}=0 \quad \ldots \ldots \quad (5)$$

The correlation function f will always have the value unity for $r=0$, while in general it will decrease to zero when r increases indefinitely. It may be a slowly varying function of the time.

The introduction of the correlation function f makes it possible to define a linear quantity l by means of the integral:

$$l=\int_0^\infty f\,dr \quad \ldots \ldots \quad (6)$$

This quantity is the analogue of the "average size of an eddy" as defined by TAYLOR [7]). In the present case it may be termed the "average size of a domain of coherence".

Following section 4 of VON KARMAN and HOWARTH's paper, we now can deduce an expression for the correlation coefficients between the derivatives of v or w. Making use of the condition of uniformity, mentioned before, we have:

$$\overline{\frac{\partial v_1}{\partial y_1} \cdot v_2}=\frac{\partial}{\partial y_1}(\overline{v_1 v_2})=-\overline{v^2}\,\frac{\partial f}{\partial r} \quad \ldots \ldots \quad (7)$$

and:

$$\overline{\frac{\partial v_1}{\partial y_1} \cdot \frac{\partial v_2}{\partial y_2}}=\frac{\partial}{\partial y_2}\left(\overline{\frac{\partial v_1}{\partial y_1} \cdot v_2}\right)=-\overline{v^2}\,\frac{\partial^2 f}{\partial r^2} \quad \ldots \ldots \quad (8)$$

from which, when the points P_1 and P_2 are made to coalesce:

$$\overline{\left(\frac{\partial v}{\partial y}\right)^2}=-\overline{v^2}\left(\frac{\partial^2 f}{\partial r^2}\right)_{r=0} \quad \ldots \ldots \quad (9)$$

In analogy with TAYLOR then a second linear quantity λ can be defined by means of the formula [8]):

$$(\partial^2 f/\partial r^2)_{r=0}=-2/\lambda^2 \quad \ldots \ldots \quad (10)$$

TAYLOR takes λ to be a measure of the smallest eddies which are responsible for the dissipation.

When the considerations stated in section 6 of VON KARMAN and HOWARTH's paper are adapted to our system, we arrive at the result that all quantities of the types: $\overline{v_1^2 v_2}$, $\overline{v_1^2 w_2}$, $\overline{v_1 w_1 v_2}$, etc., must be zero, so that apparently in our case there are no triple correlations to be retained in the equations. It must be admitted, however, that here we have to do

[7]) G. I. TAYLOR, Proc. Roy. Soc. (London) **A 151**, p. 426, 1935.
[8]) G. I. TAYLOR, Proc. Roy. Soc. (London) **A 151**, p. 437, 1935.

with a rather dangerous point, and a further analysis of its applicability to the present case would be desirable. Such an analysis then ought to proceed along other lines. Provisionally, therefore, we shall make use of the assumption that the triple correlations can be neglected, as the purpose of this section and of the following one is no more than to point out various analogies with the equations of VON KARMAN and HOWARTH. In section **4** and **5**, however, we shall give attention to the effect of the terms of the second degree in the equations for the model system, and we shall see that they play an important part in the propagation of elementary regions of turbulence.

3. *The equation for the propagation of the correlation* (VON KARMAN and HOWARTH, section 8). From the first one of the equations (1) we deduce the relation:

$$\frac{\partial}{\partial t}(v_1 v_2) = U(2 v_1 v_2 - v_2 w_1 - v_1 w_2) + \nu \frac{\partial^2}{\partial y_1^2}(v_1 v_2) + \nu \frac{\partial^2}{\partial y_2^2}(v_1 v_2) -$$

$$- \frac{\partial}{\partial y_1}(v_1^2 v_2 - w_1^2 v_2) - \frac{\partial}{\partial y_2}(v_1 v_2^2 - v_1 w_2^2),$$

from which, by means of the process of averaging:

$$\frac{\partial}{\partial t}(f\overline{v^2}) = 2 U f \overline{v^2} + 2 \nu \overline{v^2} \frac{\partial^2 f}{\partial r^2}. \quad \dots \dots \quad (11)$$

The same result could have been obtained from the second equation of the system (1). — When the term with U is dropped eq. (11) is the analogue of equation (51), p. 206, of VON KARMAN and HOWARTH's paper.

A) We first take $r=0$. Then $f=1$, while $(\partial^2 f/\partial r^2)_{r=0} = -2/\lambda^2$, according to (10); λ evidently can be a function of the time. Equation (11) reduces to:

$$d\overline{v^2}/dt = 2\, U \overline{v^2} - 4 \nu \overline{v^2}/\lambda^2. \quad \dots \dots \quad (12)$$

With $U=0$ this becomes:

$$d\overline{v^2}/dt = -4 \nu \overline{v^2}/\lambda^2 \quad \dots \dots \quad (12a)$$

which is the analogue of eq. (55), p. 207, of VON KARMAN and HOWARTH's paper (section 9).

B) Returning to eq. (11) and expanding the left hand side, we can eliminate $d\overline{v^2}/dt$ by means of (12), and obtain:

$$\frac{\partial f}{\partial t} = 2 \nu \left[\frac{\partial^2 f}{\partial r^2} - f\left(\frac{\partial^2 f}{\partial r^2}\right)_{r=0}\right] \quad \dots \dots \quad (13)$$

This equation, which is independent of the value of U, is the analogue of eq. (63), p. 209, of VON KARMAN and HOWARTH's paper.

A particular solution of this equation, in which f is a function of $r/\sqrt{\nu t}$ only, is given by:

$$f = e^{-r^2/8 \nu t}. \quad \dots \dots \quad (14)$$

This solution is the analogue of the "self-preserving correlation functions", considered by VON KARMAN and HOWARTH in section 10. When f is of this special type, eq. (10) gives:

$$\lambda^2 = 8 \nu t \quad \dots \dots \quad (15)$$

so that eq. (12) becomes:

$$d\overline{v^2}/dt = 2\, U \overline{v^2} - \overline{v^2}/2t,$$

from which

$$\overline{v^2} = \overline{v_0^2}\, e^{2Ut} (t_0/t)^{1/2}.$$

With $U=0$ this reduces to:

$$\overline{v^2} = \overline{v_0^2}\, (t_0/t)^{1/2}.$$

It can also be tried to construct a solution analogous to those considered in section 11 of the paper of VON KARMAN and HOWARTH. In these solutions it is assumed that the correlation function f is independent of ν, with the exception of small values of the distance r. Hence in eq. (13) we shall neglect $\nu(\partial^2 f/\partial r^2)$, but retain the term $\nu(\partial^2 f/\partial r^2)_{r=0}$. Then it is assumed that f is a function of $\eta = r/L$, where L is a provisionally unknown function of the time. Substitution into (13) leads to:

$$-f' \frac{r}{L^2} \frac{dL}{dt} = -2 \nu f \left(\frac{\partial^2 f}{\partial r^2}\right)_{r=0},$$

which equation can be satisfied only provided:

$$\frac{1}{L}\frac{dL}{dt} = \text{const.}\, \nu \left(\frac{\partial^2 f}{\partial r^2}\right)_{r=0} \quad \dots \dots \quad (16)$$

If, in connection with eq. (15), we assume $(\partial^2 f/\partial r^2)_{r=0} = -2/\lambda^2 = -1/4 \nu t$, we should find (taking the constant in eq. (16) equal to $-C$):

$$\frac{1}{L}\frac{dL}{dt} = +\frac{C}{4t},$$

from which:

$$L = \text{const.}\, t^{C/4}.$$

At the same time we obtain:

$$-f'\eta = \frac{2}{C} f.$$

so that:

$$f = \text{const.} \left(\frac{1}{\eta}\right)^{2/C}.$$

The assumption $C = 2$ *e.g.* would give:

$$L \backsim \sqrt{t}; \qquad f \backsim 1/\eta \backsim L/r.$$

However, we cannot come much further in this way, and also VON KARMAN and HOWARTH's paper does not give more than a set of possibilities.

4. *Direct investigation of the propagation of an elementary region of turbulence in the model systems.* — It may be asked whether a direct investigation of the equations defining the model systems can give us a closer insight into the laws governing the propagation of disturbances. The question is particularly interesting as this propagation depends upon the terms of the second degree in these equations, which had been eliminated in the process of averaging.

In considering this problem we shall simplify the equations by dropping the terms with U. Moreover we shall give attention mainly to the far more tractable case of the model system with only one variable v, which (with $U = 0$) is governed by the equation [9]):

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial y^2} - 2v \frac{\partial v}{\partial y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

We assume ν to be small, and we shall neglect the term $\nu \partial^2 v / \partial y^2$, except when a discontinuity threatens to arise in the solution of the equation. By means of a reasoning of similar kind as was given for the system with two variables [10]) it is found that a discontinuity propagates itself with a velocity c, determined by:

$$c = v_l + v_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

while at the same time it gives rise to a dissipation of energy of amount:

$$\frac{1}{6}(v_l - v_r)^3.$$

(It is to be observed that in a discontinuity we always have $v_l > v_r$). When these results are observed, we may restrict to the consideration of the equation:

$$\frac{\partial v}{\partial t} = -2v \frac{\partial v}{\partial y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

A typical example of a solution of this equation is indicated schematically in fig. 1. It has been assumed that for $t = 0$ we have $v = v_0$ in a domain

[9]) See J. M. BURGERS, *l.c.* p. 14, eq. (7.2), and the reference to RIEMANN given p. 27, footnote 12).

[10]) J. M. BURGERS, *l.c.* pp. 30, 40—43.

of length L_0 (say, in the domain $0 < y < L_0$); outside of this domain $v = 0$. The values of v for all interior points of the domain propagate them-

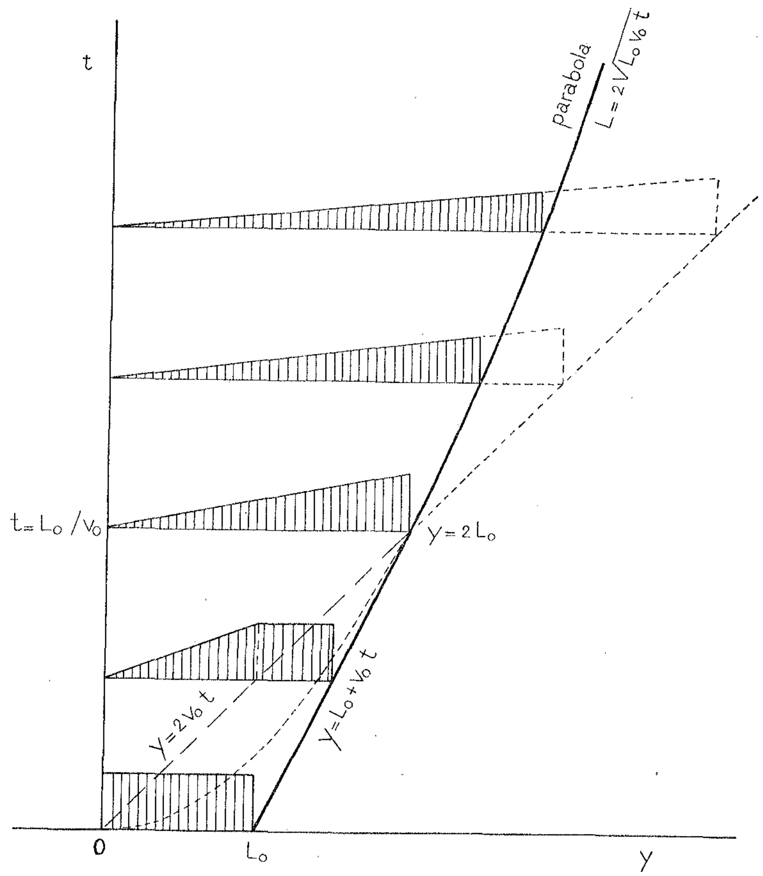

Fig. 1.

selves to the right with the velocity $2v$, which at first for all these points has the value $2v_0$. Such a propagation would leave open a space at the left hand side of the domain, of breadth $2v_0 t$; in this space we obtain the solution $v = y/2t$, whereas for $y \geqq 2v_0 t$ we have $v = v_0$. The discontinuity at the right hand side of the domain, however, displaces itself with the velocity v_0. Hence the region where $v = v_0$ becomes continuously narrower, until it vanishes for $t = L_0/v_0$. From then onward we have the solution $v = y/2t$ in a domain extending from $y = 0$ until $y = L$, where L is a function of t; at the right hand end of this domain there is a steep front (a discontinuity), moving with the velocity $dL/dt = L/2t$. It is easily found that we have:

$$L = 2\sqrt{L_0 v_0 t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

With increasing values of t the front decreases in height; at the same time its velocity decreases.

When the influence of the frictional term $\nu\partial^2 v/\partial y^2$ on the form of the steep front is investigated, it is found that the maximum value of $\partial v/\partial y$ is of the order v^2/ν.

It is to be observed that the integral:

$$\Omega = \int v\, dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

remains constant during the whole process; this is true also when the frictional term is not neglected, and represents an analogy to the conservation of the impulse in the hydrodynamical case.

The equation for the motion of the front can be written:

$$L = 2\sqrt{\Omega t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

5. It is possible to develop solutions of similar nature from other initial conditions, and it can be expected that after a sufficient lapse of time we shall always obtain a solution of the type:

$$v = (y - y_0)/2\,(t - t_0). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where y_0 and t_0 are two constants, depending upon the initial conditions. This solution will be valid in a domain extending from $y = y_0$ until $y = y_0 + L$, the quantity L satisfying the equation:

$$L = 2\sqrt{\Omega(t - t_0)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where Ω is the constant value of the integral $\int v\, dy$. The quantity L may be either positive or negative. The steep front at $y = y_0 + L$ is rounded off by the influence of the viscosity, and the maximum value of $\partial v/\partial y$ again is of the order v^2/ν.

The asymmetrical character of the growth of the domain where v differs from zero, is an accidental circumstance due to the particular form of equation (17). When we return to the system formed by eqs. (1), (taking $U = 0$), in which there are two variables, there is found a more symmetrical growth of the domains where v and w are different from zero, but the exact investigation will be far more difficult.

Returning to the case of eq. (17) or (19) it is still to be observed that various of the growing domains may meet each other. For instance, when we have two domains with positive values of v, and the left hand one has the greater Ω, this one will overtake the other. After a sufficient lapse of time there will be left a single domain only, again of the type described in the beginning of this section, and characterized by a value of Ω which is the sum of the Ω's of the original domains.

Another case may be formed by two domains of different sign, situated in such a way that the one with v positive is to the left of the one with v

negative. When they have met, then after a sufficient lapse of time again only one domain will be left, characterized by a value of Ω which is the difference of the Ω's of the original regions.

We thus arrive at the general result that there is a tendency for all domains where some disturbance is present (that is, where at $t = 0$, v, or both v and w in the system with two variables, are different from zero), to increase in size, at a rate approximately proportional to $\sqrt{t}$. In this process the larger domains overrun the smaller ones, and consequently more and more of the details of the original pattern will be gradually eliminated. Hence it is to be expected that the average size of the "domains of coherence" to be found in the field will increase as well, and also roughly proportionally to $\sqrt{t}$.

A particularly interesting feature of the process is the tendency to form steep fronts at the advancing edges of the domains. The steep fronts are the sources of intensive dissipation of energy. When ν is small, the dissipation in these fronts is far more important than the dissipation due to the values $\partial v/\partial y$ assumes in the rest of the domains.

This result is remarkable as again it affords an analogy with what is observed in fluid motion, although the geometrical features of the field are different in the hydrodynamical case. It has been pointed out by TAYLOR that in turbulent motion the intensity of the vorticity is always increased in those parts of the field, where vortex filaments are extended, as according to the law of the conservation of circulation the absolute value of the vorticity changes proportionally to the length of a filament having the same direction as the vorticity vector. As TAYLOR remarks turbulent motion is found to be diffusive, so that particles which were originally neighbours move apart as the motion proceeds; consequently there must be a continuous increase of the vorticity [11]). In TAYLOR's view

[11]) G. I. TAYLOR, Journ. Aeron. Sciences **4**, p. 315, 1937; G. I. TAYLOR and A. E. GREEN, Proc. Roy. Soc. (London) A **158**, p. 501, 1937; G. I. TAYLOR, *ibid.* A **164**, p. 15, 1938.

In order to obtain an estimate of the minimum thickness to which a vortex can be drawn out, we may consider a field of fluid motion, which, when described with reference to cylindrical coordinates r, ϑ, z, possesses the velocities:

$$v_r = -Ur; \quad v_\vartheta = u\,(t, r); \quad v_z = +2\,Uz,$$

where U is a constant. This field satisfies the equation of continuity. As u has been assumed to be independent of z, the only vorticity component is:

$$\gamma_z = \frac{1}{r}\,\frac{\partial(ur)}{\partial r}.$$

When the pressure is taken equal to:

$$p = -\frac{1}{2}\,\varrho\, U^2\,(r^2 + 4\,z^2) + \varrho \int dr\,(u^2/r)$$

this process represents the fundamental mechanical cause which controls the dissipation of energy in turbulent motion.

It will be evident that TAYLOR's result, connected as it is with the laws governing vortex motion, is a typical effect due to the presence of terms of the *second degree* in the hydrodynamical equations. Notwithstanding the difference in geometrical character, the model system in this respect shows a similar behaviour: also here there are present terms of the second degree, which bring about the tendency to produce steep fronts, and these are the loci of high values of the gradient $\partial v/\partial y$ and consequently of intensive dissipation.

6. *Conclusive remark.* — The results of the preceding discussion, taken together with the investigations of the previous paper, can be summarized by saying that the model system in a simplified way possesses the essential features which are governing the energetical relations of the hydrodynamical system. It would appear therefore that a further investigation of the statistical character of the solutions of eqs. (1), in particular in the case where the domain of the coordinate y is bounded and where boundary conditions are applied to the variables v and w, certainly will be worth while; should it have success, then it is to be expected that it will bring out features which will be helpful in the analysis of some of the still existing riddles of turbulent fluid motion.

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the equations of motion for the directions r and z are satisfied. We write $\sigma = u\,r$; then the equation for the ϑ-direction takes the form:

$$\frac{\partial \sigma}{\partial t} = U r \frac{\partial \sigma}{\partial r} + \nu \left(\frac{\partial^2 \sigma}{\partial r^2} - \frac{1}{r} \frac{\partial \sigma}{\partial r} \right).$$

It must be expected that the solution of this equation asymptotically will approach to a form in which σ is a function of r only. This function then must satisfy the equation:

$$\nu \frac{d^2 \sigma}{dr^2} + \left(Ur - \frac{\nu}{r} \right) \frac{d\sigma}{dr} = 0.$$

The solution appropriate to our case is: $\sigma = C(1 - e^{-r^2 U/2\nu})$, where $2\pi C$ represents the strength (circulation) of the vortex. We then obtain for the vorticity:

$$\gamma = \frac{CU}{\nu} e^{-r^2 U/2\nu}.$$

The dissipation, calculated per unit of height in the z-direction, is found to have the value:

$$\mu \int_0^\infty 2\pi\, r \gamma^2\, dr = \pi \varrho C^2 U.$$

It is interesting to observe that here again — the same as in the case of the discontinuities of the model system (see J. M. BURGERS, *l.c.* pp. 26, 43) — the dissipation is given by an expression which is independent of the viscosity, and which is of the third degree with respect to the velocity components of the motion ($v_r = -Ur$; $v_\vartheta = u = C/r$ outside of the vortex proper).

Mathematics. — *Ein Satz über assoziierte Geraden im R_4.* Von R. WEITZENBÖCK.

(Communicated at the meeting of December 30, 1939.)

Zu je vier Geraden allgemeiner Lage im R_4 lässt sich auf lineare Weise eine fünfte Gerade konstruieren, was zu der bekannten Figur von fünf assoziierten Geraden führt.

Ich beweise hier zweierlei. Erstens, dass fünf assoziierte Geraden allgemeiner Lage nicht Erzeugende derselben Quadrik F_2 im R_4 sein können und zweitens, dass die fünfte Gerade der Ort der Kegelspitzen ist für alle dreidimensionalen Kegel zweiter Ordnung, die die vier ersten Geraden enthalten.

§ 1. *Die assoziierte Gerade.*

Sind 1, 2, 3 und 4 vier Geraden im R_4 mit den Koordinaten

$$a_{ik},\ \alpha_{ik},\ p_{ik} \text{ und } m_{ik},$$

so ist

$$(x S'_{12}) = (x a^2 \alpha^2) = 4 \,.\, \Sigma\, x_1 a_{23} \alpha_{45} = 0$$

die Gleichung des R_3, der 1 und 2 verbindet. Die drei Räume S'_{12}, S'_{23} und S'_{31} schneiden sich in einer Geraden 4*, die mit 4 verbunden den Raum 4′ liefert. Dann gehen die vier Räume 1′, 2′, 3′ und 4′ durch dieselbe Gerade: die assoziierte zu 1, 2, 3, 4. Ihre Gleichung lautet, wenn

$$G_1 = (\pi^3 a^2) = 0\ ,\ G_2 = (\pi^3 \alpha^2) = 0, \ldots$$

die Gleichungen der vier Geraden 1, 2, 3 und 4 sind [1]):

$$G_5 = \frac{G_1}{A_1} + \frac{G_2}{A_2} + \frac{G_3}{A_3} + \frac{G_4}{A_4} = 0. \qquad (1)$$

Die Invarianten A_i sind hier die vier unabhängigen Invarianten der Geraden 1 bis 4 und gegeben durch:

$$\left.\begin{aligned} A_1 &= (m S'_{13})\ (m S'_{12}) \\ A_2 &= (a S'_{23})\ (a S'_{24}) \\ A_3 &= (\alpha S'_{13})\ (\alpha S'_{34}) \\ A_4 &= (p S'_{14})\ (p S'_{24}) \end{aligned}\right\} \qquad (2)$$

Die Bezeichnung ist hier so gewählt, dass in (1) alle Glieder das

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1) Vgl. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **42**, 248 (1939).