wo C eine positive, nur von der Wahl des Quaders $\mathfrak{Q}$ abhängige Konstante bedeutet.

Also auch: die Menge $\mathfrak{M}'(x)$ sämtlicher Punkte A in $\mathfrak{Q}$, für die bei irgend einem Gitterpunkt X der Höhe $x \geqq 1$ aus $\mathfrak{R}$ und irgend einem Gitterpunkt Y aus R_r die sämtlichen Ungleichungen (3) gelten, hat das äussere Mass

$$\overline{\mathfrak{m}}\,\mathfrak{M}'(x) \leqq 2\,m\,(2x+1)^{m-1}\{2^n K(x) \prod_{\nu=1}^{n} \varphi_\nu(x) + C\,Max(\varphi_1(x), \ldots, \varphi_n(x))\}$$

$$\leqq m \, . \, 3^{m+n} \, . \, x^{m-1} K(x) \prod_{\nu=1}^{n} \varphi_\nu(x) + m \, . \, 3^m \, . \, C \, . \, x^{m-1}\, Max(\varphi_1(x), \ldots, \varphi_n(x)).$$

Wegen der Konvergenz der Reihen (2) konvergiert also die Reihe

$$\sum_{x=1}^{\infty} \overline{\mathfrak{m}}\,\mathfrak{M}'(x),$$

so dass fast alle A aus $\mathfrak{Q}$ höchstens einer endlichen Anzahl der Mengen $\mathfrak{M}'(x)$ angehören. Q.e.d.

Mathematics. — *Tauberian theorems for Cesaro-summability of double series.* By H. D. KLOOSTERMAN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of January 27, 1940.)

In two papers (to appear shortly in the "Journal of the London Mathematical Society" and the "Mathematische Zeitschrift") I have given a new method of proof for Tauberian theorems for Cesaro-summability. This method depends on some formulae, which appear to be new and which belong to the theory of finite differences. The Tauberian theorems for Cesaro-summability are immediate consequences of these formulae, and the proofs of these theorems thus obtained are considerably simpler than the proofs already known. Now in a paper entitled "Limitierungs-Umkehrsätze für Doppelfolgen", Math. Zeitschr. **45** (1939), p. 573—589, K. KNOPP has proved Tauberian theorems for double series. However his theorem on Cesaro-summability treats summability of the first order only. The object of the present paper is to show, that the method used in my two papers mentioned above, also gives proofs of Tauberian theorems on double series for Cesaro-summability *of any order.*

The following notations will be used. Let

$$\sum_{m,n=1}^{\infty} a_{m,n} \quad \ldots\ldots\ldots\ldots \quad (1)$$

be a double series with real terms. We write

$$s_{m,n}^{(-1,-1)} = a_{m,n} \qquad (m, n = 1, 2, \ldots)$$

and if t and r are integers $\geqq -1$:

$$s_{m,n}^{(t+1,r)} = \sum_{\mu=1}^{m} s_{\mu,n}^{(t,r)} \quad , \quad s_{m,n}^{(t,r+1)} = \sum_{\nu=1}^{n} s_{m,\nu}^{(t,r)} \quad (m, n = 1, 2, \ldots).$$

If t and r are non-negative, the double series (1) is called summable $(C; t, r)$, if the double limit

$$\lim_{m,n \to \infty} \frac{s_{m,n}^{(t,r)}}{\binom{m+t-1}{t}\binom{n+r-1}{r}} \quad \ldots\ldots \quad (2)$$

exists. Summability $(C; 0, 0)$ is identical with convergence. If the limit (2) is s, the series (1) is said to be summable $(C; t, r)$ to the sum s. It can be easily proved (as in the case of single series), that convergence implies summability $(C; t, r)$, if t and r are $\geqq 0$ ("theorem of consistency").

For any function $u_{m,n}$ of the two positive integral variables m and n (which are written as lower indices) let

$$D_h u_{m,n} = u_{m+h,n} - u_{m,n} \quad \text{and} \quad \triangle_k u_{m,n} = {}_{m,n+k} - u_{m,n}$$

be the differences of $u_{m,n}$ with respect to the first and second variable respectively, if h and k are positive integers. If however $-h$ and $-k$ are negative integers, we write

$$D_{-h} u_{m,n} = u_{m,n} - u_{m-h,n} \quad \text{and} \quad \triangle_{-k} u_{m,n} = u_{m,n} - u_{m,n-k} \qquad (h<m; k<n).$$

The higher differences are defined in the usual way. Thus, if t and r are positive integers and $h>0$, $k>0$:

$$D_h^t u_{m,n} = D_h(D_h^{t-1} u_{m,n}) = \sum_{\tau=0}^{t} (-1)^\tau \binom{t}{\tau} u_{m+(t-\tau)h,\, n},$$

$$\triangle_k^r u_{m,n} = \triangle_k(\triangle_k^{r-1} u_{m,n}) = \sum_{\varrho=0}^{r} (-1)^\varrho \binom{r}{\varrho} u_{m,\, n+(r-\varrho)k}$$

and

$$D_h^t \triangle_k^r u_{m,n} = \sum_{\tau=0}^{t} \sum_{\varrho=0}^{r} (-1)^{\tau+\varrho} \binom{t}{\tau}\binom{r}{\varrho} u_{m+(t-\tau)h,\, n+(r-\varrho)k} \quad . \quad (3)$$

Also, if $-h$ and $-k$ are negative integers:

$$D_{-h}^t u_{m,n} = D_{-h}(D_{-h}^{t-1} u_{m,n}) = \sum_{\tau=0}^{t} (-1)^\tau \binom{t}{\tau} u_{m-\tau h,\, n},$$

$$\triangle_{-k}^r u_{m,n} = \triangle_{-k}(\triangle_{-k}^{r-1} u_{m,n}) = \sum_{\varrho=0}^{r} (-1)^\varrho \binom{r}{\varrho} u_{m,\, n-\varrho k}$$

and

$$D_{-h}^t \triangle_{-k}^r u_{m,n} = \sum_{\tau=0}^{t} \sum_{\varrho=0}^{r} (-1)^{\tau+\varrho} \binom{t}{\tau}\binom{r}{\varrho} u_{m-\tau h,\, n-\varrho k} \quad . \quad . \quad . \quad (4)$$

Clearly the symbols D and $\triangle$ are commutative.

Lemma 1. If m, n, h, k, t, r are positive integers, then

$$D_h^t \triangle_k^r s_{m,n}^{(t,r)} = h^t k^r s_{m,n}^{(0,0)} + k \sum_{\omega=1}^{h} (h-\omega+1) \sum_{\tau=0}^{t-1} h^{t-\tau-1} D_h^\tau \triangle_k^{r-1} s_{m+\omega,n}^{(\tau-1,r-1)} +$$

$$+ \sum_{\varkappa=1}^{k} (k-\varkappa+1) D_h^t \triangle_k^{r-1} s_{m,n+\varkappa}^{(t,r-2)} + h^t \sum_{\varkappa=1}^{k} (k-\varkappa+1) \sum_{\varrho=0}^{r-2} k^{r-\varrho-1} \triangle_k^\varrho s_{m,n+\varkappa}^{(0,\varrho-1)}$$

(where $\sum\limits_{\varrho=0}^{r-2}$ means 0, if $r=1$).

Proof. If $m, n, \omega, \varkappa$ are positive integers, we have

$$s_{m+\omega,n+\varkappa}^{(0,0)} = s_{m+\omega,n}^{(0,0)} + \sum_{\lambda=1}^{\varkappa} s_{m+\omega,n+\lambda}^{(0,-1)}.$$

Summing over $\varkappa$ from 1 to k, we get

$$\triangle_k s_{m+\omega,n}^{(0,1)} = k\, s_{m+\omega,n}^{(0,0)} + \sum_{\varkappa=1}^{k} (k-\varkappa+1) s_{m+\omega,n+\varkappa}^{(0,-1)} \quad . \quad . \quad . \quad . \quad (5)$$

Now summing over ω from 1 to h, we get

$$D_h \triangle_k s_{m,n}^{(1,1)} = k\, D_h s_{m,n}^{(1,0)} + \sum_{\varkappa=1}^{k} (k-\varkappa+1) D_h s_{m,n+\varkappa}^{(1,-1)}. \quad . \quad . \quad . \quad (6)$$

We have also

$$s_{m+\omega,n}^{(0,0)} = s_{m,n}^{(0,0)} + \sum_{\lambda=1}^{\omega} s_{m+\lambda,n}^{(-1,0)}.$$

Summing over ω from 1 to h, we get

$$D_h s_{m,n}^{(1,0)} = h\, s_{m,n}^{(0,0)} + \sum_{\omega=1}^{h} (h-\omega+1) s_{m+\omega,n}^{(-1,0)} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If we substitute this result in (6), we get

$$D_h \triangle_k s_{m,n}^{(1,1)} = h k\, s_{m,n}^{(0,0)} + k \sum_{\omega=1}^{h} (h-\omega+1) s_{m+\omega,n}^{(-1,0)} + \sum_{\varkappa=1}^{k} (k-\varkappa+1) D_h s_{m,n+\varkappa}^{(1,-1)}$$

and thus the lemma is proved in the special case $t=r=1$.

We now apply induction with respect to t, in order to prove the lemma in the case $r=1$. Suppose that for some integer $t \geqq 1$ the formula

$$\left.\begin{aligned} D_h^t \triangle_k s_{m,n}^{(t,1)} = h^t k\, s_{m,n}^{(0,0)} + k \sum_{\omega=1}^{h} (h-\omega+1) \sum_{\tau=0}^{t-1} h^{t-\tau-1} D_h^\tau s_{m+\omega,n}^{(\tau-1,0)} + \\ + \sum_{\varkappa=1}^{k} (k-\varkappa+1) D_h^t s_{m,n+\varkappa}^{(t,-1)} \end{aligned}\right\} \quad . \quad (8)$$

is already proved. We then replace m by $m+\lambda$ and sum over λ from 1 to h. We then get:

$$\left.\begin{aligned} D_h^{t+1} \triangle_k s_{m,n}^{(t+1,1)} = h^t k\, D_h s_{m,n}^{(1,0)} + \\ + k \sum_{\omega=1}^{h} (h-\omega+1) \sum_{\tau=0}^{t-1} h^{t-\tau-1} D_h^{\tau+1} s_{m+\omega,n}^{(\tau,0)} + \sum_{\varkappa=1}^{k} (k-\varkappa+1) D_h^{t+1} s_{m,n+\varkappa}^{(t+1,-1)}. \end{aligned}\right\} (9)$$

Hence, substituting (7) in (9):

$$D_h^{t+1} \triangle_k s_{m,n}^{(t+1,1)} = h^{t+1} k\, s_{m,n}^{(0,0)} + k \sum_{\omega=1}^{h} (h-\omega+1)\, h^t s_{m+\omega,n}^{(-1,0)} +$$

$$+ k \sum_{\omega=1}^{h} (h-\omega+1) \sum_{\tau=1}^{t} h^{t-\tau} D_h^{\tau} s_{m+\omega,n}^{(\tau-1,0)} + \sum_{\varkappa=1}^{k} (k-\varkappa+1)\, D_h^{t+1} s_{m,n+\varkappa}^{(t+1,-1)} =$$

$$= h^{t+1} k\, s_{m,n}^{(0,0)} + k \sum_{\omega=1}^{h} (h-\omega+1) \sum_{\tau=0}^{t} h^{t-\tau} D_h^{\tau} s_{m+\omega,n}^{(\tau-1,0)} +$$

$$+ \sum_{\varkappa=1}^{k} (k-\varkappa+1)\, D_h^{t+1} s_{m,n+\varkappa}^{(t+1,-1)}.$$

Thus the induction from t to $t+1$ is achieved and therefore (8) is proved for all integral values $\geqq 1$ of t. This is the special case $r=1$ of the statement of the lemma. We now apply induction with respect to r, in order to prove the lemma for all integral $r \geqq 1$. We then replace n by $n+\lambda$ and sum over λ from 1 to k. Then it follows, that

$$\left.\begin{aligned} D_h^t \triangle_k^{r+1} s_{m,n}^{(t,r+1)} &= h^t k^r \triangle_k s_{m,n}^{(0,1)} + \\ &+ k \sum_{\omega=1}^{h} (h-\omega+1) \sum_{\tau=0}^{t-1} h^{t-\tau-1} D_h^{\tau} \triangle_k^r s_{m+\omega,n}^{(\tau-1,r)} + \\ + \sum_{\varkappa=1}^{k} (k-\varkappa+1)\, D_h^t \triangle_k^r s_{m,n+\varkappa}^{(t,r-1)} &+ h^t \sum_{\varkappa=1}^{k} (k-\varkappa+1) \sum_{\varrho=0}^{r-2} k^{r-\varrho-1} \triangle_k^{\varrho+1} s_{m,n+\varkappa}^{(0,\varrho)}. \end{aligned}\right\} \quad (10)$$

We now substitute (5) (with $\omega=0$) in (10) and then get:

$$D_h^t \triangle_k^{r+1} s_{m,n}^{(t,r+1)} = h^t k^{r+1} s_{m,n}^{(0,0)} +$$

$$+ k \sum_{\omega=1}^{h} (h-\omega+1) \sum_{\tau=0}^{t-1} h^{t-\tau-1} D_h^{\tau} \triangle_k^r s_{m+\omega,n}^{(\tau-1,r)} + \sum_{\varkappa=1}^{k} (k-\varkappa+1)\, D_h^t \triangle_k^r s_{m,n+\varkappa}^{(t,r-1)} +$$

$$+ h^t \sum_{\varkappa=1}^{k} (k-\varkappa+1) \sum_{\varrho=0}^{r-1} k^{r-\varrho} \triangle_k^{\varrho} s_{m,n+\varkappa}^{(0,\varrho-1)}.$$

Thus the induction from r to $r+1$ is achieved and lemma 1 is completely proved.

Lemma 2. If m, n, h, k, t, r are positive integers and $m > th$, $n > rk$, then

$$D_{-h}^t \triangle_{-k}^r s_{m,n}^{(t,r)} = h^t k^r s_{m,n}^{(0,0)} - k \sum_{\omega=0}^{h-1} (h-\omega-1) \sum_{\tau=0}^{t-1} h^{t-\tau-1} D_{-h}^{\tau} \triangle_{-k}^{r-1} s_{m-\omega,n}^{(\tau-1,r-1)} +$$

$$- \sum_{\varkappa=0}^{k-1} (k-\varkappa-1)\, D_{-h}^t \triangle_{-k}^{r-1} s_{m,n-\varkappa}^{(t,r-2)} - h^t \sum_{\varkappa=0}^{k-1} (k-\varkappa-1) \sum_{\varrho=0}^{r-2} k^{r-\varrho-1} \triangle_{-k}^{\varrho} s_{m,n-\varkappa}^{(0,\varrho-1)}$$

(where $\sum\limits_{\varrho=0}^{r-2}$ means 0, if $r=1$).

Proof. Starting from the formulae

$$\triangle_{-k} s_{m-\omega,n}^{(0,1)} = k\, s_{m-\omega,n}^{(0,0)} - \sum_{\varkappa=0}^{k-1} (k-\varkappa-1)\, s_{m-\omega,n-\varkappa}^{(0,-1)}$$

and

$$D_{-h} s_{m,n}^{(1,0)} = h\, s_{m,n}^{(0,0)} - \sum_{\omega=0}^{h-1} (h-\omega-1)\, s_{m-\omega,n}^{(-1,0)}$$

instead of (5) and (7), the proof is very much like the proof of lemma 1 and it may therefore be left to the reader.

Lemma 3. Suppose, that the inequalities

$$s_{m,n}^{(0,-1)} < \frac{K}{n} \quad , \quad s_{m,n}^{(-1,0)} < \frac{K}{m} \quad \ldots\ldots \quad (11)$$

are valid for all positive integers m and n. Here K is a positive constant, independant of m and n. Then if $h \geqq 1$, $k \geqq 1$, $r \geqq 0$, $t \geqq 0$ are integers, we have

$$D_h^t \triangle_k^r s_{m,n}^{(t,r-1)} < K \frac{h^t k^r}{n} \quad , \quad D_h^t \triangle_k^r s_{m,n}^{(t-1,r)} < K \frac{h^t k^r}{m}. \quad . \quad (12)$$

If, in addition $m > th$ and $n > rk$, then

$$D_{-h}^t \triangle_{-k}^r s_{m,n}^{(t,r-1)} < K \frac{h^t k^r}{n-rk} \quad , \quad D_{-h}^t \triangle_{-k}^r s_{m,n}^{(t-1,r)} < K \frac{h^t k^r}{m-th}. \quad (13)$$

Proof. If $t=r=0$ the first inequality (12) is true, since it then is identical with the first inequality (11). Suppose it to be true for $r=0$ and some integer $t \geqq 0$. Then

$$D_h^t s_{m,n}^{(t,-1)} < K \frac{h^t}{n} \ldots\ldots \quad (14)$$

If we replace m by $m+\omega$ and sum over ω from 1 to h, we get

$$D_h^{t+1} s_{m,n}^{(t+1,-1)} = \sum_{\omega=1}^{h} D_h^t s_{m+\omega,n}^{(t,-1)} < K \sum_{\omega=1}^{h} \frac{h^t}{n} = K \frac{h^{t+1}}{n}.$$

Therefore, by induction, the inequality (14) is true for any integer $t \geqq 0$. Again, suppose, that the first inequality (12) is true for all integers $t \geqq 0$ and some integer $r \geqq 0$. Then, replacing n by $n+\varkappa$ and summing over $\varkappa$ from 1 to k, we get

$$D_h^t \triangle_k^{r+1} s_{m,n}^{(t,r} = \sum_{\varkappa=1}^{k} D_h^t \triangle_k^r s_{m,n+\varkappa}^{(t,r-1)} < K \sum_{\varkappa=1}^{k} \frac{h^t k^r}{n+\varkappa} < K \frac{h^t k^{r+1}}{n}.$$

Thus the first inequality (12) is proved by induction. The second inequality (12) and the inequalities (13) can be proved in the same way.

Theorem 1. Let the double series (1) be summable $(C; t, r)$ to the sum s, where t and r are positive integers. Then if

$$s^{(0,-1)}_{m,n} < \frac{K}{n} \quad , \quad s^{(-1,0)}_{m,n} < \frac{K}{m}$$

for all positive integers m, n, where K is a positive constant, independant of m and n, then the double series is convergent with the sum s.

Proof. Let first $s = 0$. Then it follows from

$$\lim_{m,n\to\infty} \frac{s^{(t,r)}_{m,n}}{\binom{m+t-1}{t}\cdot\binom{n+r-1}{r}} = 0,$$

that also

$$\lim_{m,n\to\infty} \frac{s^{(t,r)}_{m,n}}{m^t n^r} = 0.$$

Therefore, ε being a positive number, we have

$$|s^{(t,r)}_{m,n}| < \varepsilon m^t n^r \qquad (m \geqq N(\varepsilon),\ n \geqq N(\varepsilon)). \quad . \quad . \quad . \quad . \quad (15)$$

Now let h and k be positive integers, such that

$$h < \frac{m}{2t} \quad , \quad k < \frac{n}{2r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

From (3), (15) and (16) it follows that

$$\left.\begin{aligned} |D^t_h \Delta^r_k s^{(t,r)}_{m,n}| &< \varepsilon \sum_{\tau=0}^{t} \sum_{\varrho=0}^{r} \binom{t}{\tau}\binom{r}{\varrho} (m+th)^t (n+rk)^r \\ &< 2^{t+r} m^t n^r \varepsilon \sum_{\tau=0}^{t} \sum_{\varrho=0}^{r} \binom{t}{\tau}\binom{r}{\varrho} = 2^{2t+2r} m^t n^r \varepsilon \\ &\qquad\qquad (m \geqq N,\ n \geqq N) \end{aligned}\right\} (17)$$

In the same way (4) and (15) give

$$|D^t_{-h} \Delta^r_{-k} s^{(t,r)}_{m,n}| \leqq m^t n^r \varepsilon \sum_{\tau=0}^{t} \sum_{\varrho=0}^{r} \binom{t}{\tau}\binom{\tau}{\varrho} = 2^{t+r} m^t n^r \varepsilon \quad . \quad (18)$$

Using the inequality (17) and lemma 3 it now follows from lemma 1, that for $m \geqq N$, $n \geqq N$:

$$h^t k^r s^{(0,0)}_{m,n} > -2^{2t+2r} m^t n^r \varepsilon - k \sum_{\omega=1}^{h} h \sum_{\tau=0}^{t-1} h^{t-\tau-1} . K \frac{h^\tau . k^{r-1}}{m} +$$

$$- \sum_{\varkappa=1}^{k} k . K \frac{h^t k^{r-1}}{n} - h^t \sum_{\varkappa=1}^{k} k \sum_{\varrho=0}^{r-2} k^{r-\varrho-1} . K \frac{k^\varrho}{n}.$$

Hence

$$h^t k^r s^{(0,0)}_{m,n} > -2^{2t+2r} m^t n^r \varepsilon - Kt \frac{h^{t+1} k^r}{m} - Kr \frac{h^t k^{r+1}}{n} \ (m \geqq N, n \geqq N). \ (19)$$

We now choose

$$h = \left[\varepsilon^{\frac{1}{4t}} m\right] + 1 \quad , \quad k = \left[\varepsilon^{\frac{1}{4r}} n\right] + 1. \quad . \quad . \quad . \quad . \quad (20)$$

Then, if m and n are sufficiently large ($m \geqq N_1$, $n \geqq N_1$) we have $h \geqq 2$, $k \geqq 2$ and

$$\varepsilon^{\frac{1}{4t}} m < h \leqq 2 \varepsilon^{\frac{1}{4t}} m \quad , \quad \varepsilon^{\frac{1}{4r}} n < k \leqq 2 \varepsilon^{\frac{1}{4r}} n.$$

Also, if ε is sufficiently small, the conditions (16) are satisfied and it now follows from (19), that

$$s^{(0,0)}_{m,n} > -2^{2t+2r} \varepsilon^{\frac{1}{2}} - 2Kt\varepsilon^{\frac{1}{4t}} - 2Kr\varepsilon^{\frac{1}{4r}} \qquad (m \geqq N_1, \quad n \geqq N_1).$$

Therefore, since ε is arbitrary, if η is a given positive number, we have

$$s^{(0,0)}_{m,n} > -\eta \qquad (m \geqq N_2(\eta), \quad n \geqq N_2(\eta)). \quad . \quad . \quad . \quad (21)$$

In the same way, using the inequality (18) and lemma 3, it follows from lemma 2, that for $m \geqq 2N$, $n \geqq 2N$ (then $m - ht > \frac{1}{2}m \geqq N$ and $n - kr > \frac{1}{2}n \geqq N$) we have

$$h^t k^r s^{(0,0)}_{m,n} < 2^{t+r} m^t n^r \varepsilon + k \sum_{\omega=0}^{h-1} h \sum_{\tau=0}^{t-1} h^{t-\tau-1} . K \frac{h^\tau k^{r-1}}{m-\omega-\tau h} +$$

$$+ \sum_{\varkappa=0}^{k-1} k . K \frac{h^t k^{r-1}}{n-\varkappa-(r-1)k} + h^t \sum_{\varkappa=0}^{k-1} k \sum_{\varrho=0}^{r-2} k^{r-\varrho-1} . K \frac{k^\varrho}{n-\varkappa-\varrho k}.$$

Hence

$$h^t k^r s^{(0,0)}_{m,n} < 2^{t+r} m^t n^r \varepsilon + 2Kt \frac{h^{t+1} k^r}{m} + 2Kr \frac{h^t k^{r+1}}{n} \qquad (m \geqq 2N,\ n \geqq 2N).$$

With the same choice (20) of h and k, we find

$$s^{(0,0)}_{m,n} < 2^{t+r} \varepsilon^{\frac{1}{2}} + 4Kt\varepsilon^{\frac{1}{4t}} + 4Kr\varepsilon^{\frac{1}{4r}} < \eta \quad (m \geqq N_3(\eta);\ n \geqq N_3(\eta)), \quad . \quad (22)$$

since ε is arbitrary.

From (21) and (22) it now follows, that

$$\lim_{m,n\to\infty} s^{(0,0)}_{m,n} = 0,$$

which proves theorem 1, if $s = 0$.

If $s \neq 0$, we replace the term $a_{1,1}$ of the double series (1) by $a_{1,1} - s$. Then $s_{m,n}^{(t,r)}$ must be replaced by

$$s_{m,n}^{(t,r)} - s\binom{m+t-1}{t}\cdot\binom{n+r-1}{r}$$

(if $t \geqq 0$, $r \geqq 0$). Therefore the modified series is summable to sum 0. Since $s_{m,n}^{(0,-1)}$ remains unaltered, if $n > 1$ and $s_{m,n}^{(-1,0)}$ remains unaltered, if $m > 1$, we can apply the result just proved to the modified series. Therefore the modified double series converges to 0 and the original double series is convergent with sum s.

Theorem 2. Let the double series (1) be summable $(C; t, r)$ to the sum s, where t and r are positive integers. Then if

$$a_{m,n} < \frac{K}{m^2+n^2} \quad \dots\dots\dots \quad (23)$$

for all positive integers m, n, where K is a positive constant, independant of m and n, then the double series is convergent with the sum s.

Proof. It is sufficient to prove, that (23) implies (11) with some K. Now we have

$$s_{m,n}^{(0,-1)} = \sum_{\mu=1}^{m} s_{\mu,n}^{(-1,0)} = \sum_{\mu=1}^{m} a_{\mu,n} < K \sum_{\mu=1}^{m} \frac{1}{\mu^2+n^2} < K \sum_{\mu=1}^{m} \int_{\mu-1}^{\mu} \frac{dx}{x^2+n^2} =$$

$$= K\int_0^m \frac{dx}{x^2+n^2} < K\int_0^\infty \frac{dx}{x^2+n^2} = \frac{K\pi}{2n},$$

and in the same way:

$$s_{m,n}^{(-1,0)} < \frac{K\pi}{2m}.$$

This proves theorem 2.

Finely it may be remarked, that theorem 1 and theorem 2 remain valid, if the conditions (11) and (23) are replaced by

$$s_{m,n}^{(0,-1)} > -\frac{K}{n}, \quad s_{m,n}^{(-1,0)} > -\frac{K}{m}$$

and

$$a_{m,n} > -\frac{K}{m^2+n^2}$$

respectively. These theorems are also true for double series with complex coefficients, if the conditions (11) and (23) are replaced by

$$|s_{m,n}^{(0,-1)}| < \frac{K}{n}, \quad |s_{m,n}^{(-1,0)}| < \frac{K}{m}$$

and

$$|a_{m,n}| < \frac{K}{m^2+n^2}$$

respectively. In order to prove this, it is sufficient, to apply the theorems already proved to the series of the real and imaginary parts.