ensuite de (78), par dérivation, et en posant $\alpha = \beta = -\frac{1}{2}$, le développement

$$H_{2n-1}(\sqrt{v})\, H_{2n}(\sqrt{v}) = 2 v^{\frac{1}{2}} \sum_{k=0}^{n-1} h'_k\, L_{2k+1}(2v), \quad . \quad . \quad . \quad (79a)$$

qui fournit, par application de la transformation de HANKEL de noyau $J_0(2\sqrt{u\,v})\, e^{-v}\, v^{-\frac{1}{2}}$, l'équation intégrale analogue à celle de MITRA, et trouvée par A. ERDÉLYI [35]. (Ces équations intégrales sont des cas particuliers de (22), et leur déduction précédente rentre dans la remarque faite au sujet de la formule (78)).

Remarquons, pour terminer, que (18) et (78) donnent lieu aux développements

$$H_{2n}(\sqrt{v}) = \sum_{k=0}^{n} h_k\, L_k^{(n-1)}(v) = (-1)^n \sum_{k=0}^{n} h_k \frac{v^k}{k!} \quad . \quad . \quad . \quad . \quad (80)$$

(avec les mêmes h_k dans les coefficients!), et

$$H_{2n}(\sqrt{v}) = \sum_{k=0}^{\left[\frac{n}{2}\right]} h'_k\, L_{2k}^{(n-1)}(2v) \quad . \quad . \quad . \quad . \quad . \quad . \quad (80')$$

L'application de la transformation de HANKEL à (80) (ainsi qu'à la relation que l'on en déduit par dérivation) ou à (80') conduira à des équations intégrales démontrées également par MITRA, et MEIJER [36]), et formant des cas particuliers de (22).

Budapest, le 12 février, 1940.

[35]) ERDÉLYI, [2].

[36]) MITRA, Math. Zeitschr. **43**, 205—211 (1938); C. S. MEIJER, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **41**, 744—755 (1938).

Mathematics. — *On the thermo-hydrodynamics of perfectly perfect fluids.* I. By D. VAN DANTZIG. (Communicated by Prof. J. A. SCHOUTEN).

(Communicated at the meeting of February 24, 1940.)

Summary.

The equations of motion of a perfectly perfect (in particular of a relativistically perfect) fluid are brought into a general invariant form, independent of metrical geometry (§ 1). They are shown to be derivable from a simple variational principle. It states that the integral of the pressure over an arbitrary fourdimensional domain in space time, under a deformation "dragging along" (cf. § 1) the chemical parameters λ^r and the temperature vector ϑ^h, hence also the congruence of macroscopic wordlines, is equal to δx^4 times the virtual heat of the deformation, flowing through the boundary into U (§ 2). In § 3 some other variational relations are derived. In § 4 a result due to EISENHART and used by SYNGE is obtained by metrical specialisation from a one dimensional variational principle.

§ 1. *The equations of motion.*

The equations of motion of continuously distributed matter are according to EINSTEIN [1])

$$\nabla_j\, \mathfrak{T}^j_{.i} = 0, \,^{2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where ∇_j is the symbol for covariant derivation, whereas for a relativistically perfect fluid [3])

$$\mathfrak{T}^h_{.i} = \mathfrak{K}^h_{.i} + \mathfrak{S}^h_{.i}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\mathfrak{K}^h_{.i} = -(\varrho + \mathfrak{p})\, i^h\, i_i + \mathfrak{p}\, A^h_i, \,^{4)}\,^{5)}\,^{6)} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\mathfrak{S}^h_{.i} = \sqrt{-\mathfrak{g}}\,(F^{hk} F_{ik} - \tfrac{1}{4} A^h_i F_{jk} F^{jk}) \quad . \quad . \quad . \quad (4)$$

[1]) E. g. A. EINSTEIN, [1], in particular § 17, 19, 20. The numbers between square brackets refer to the bibliography at the end of the paper.

[2]) The suffixes h, i, j, k, l run independently through the range $1, 2, 3, 4$, corresponding with the space-time coordinates x^h; r, s, t run through the range $5, \ldots, 4+n$, corresponding with the n chemical components Γ^r; $\varkappa, \lambda, \mu, \nu$ through the range $1, 2, 3, \ldots, 4+n$, and A, B, C through the range $0, 1, 2, \ldots, 4+n$.

[3]) The remark that (3) is valid for a perfect fluid in *adiabatic* motion only, was made by EINSTEIN [1] already; it seems to have been neglected by several later autors.

[4]) In order to avoid superfluous factors $\sqrt{-\mathfrak{g}}$ ($\mathfrak{g} = det\, g_{ij}$), ϱ itself instead of $\varrho \sqrt{-\mathfrak{g}}$ stands for the proper energy density. All densities except ϱ, ϱ_0, $\bar{\varrho}$ are denoted by Gothic letters.

[5]) A^h_i is the unit-tensor of space-time: $A^h_i = \begin{cases} 1, & h = i \\ 0, & h \neq i \end{cases}$. The unit-tensors Δ^r_s, $E^\varkappa_\lambda$, $\mathfrak{A}^A_B$ play the same rôle as A^h_i with regard to their respective ranges.

[6]) $i^h = \dfrac{dx^h}{ds}$ is a unit-vector along the macroscopic worldlines, $i^h\, i_h = +1$, $i^4 > 0$.

Usually $\mathfrak{K}^h_{.i}$ is called the material part and $\mathfrak{S}^h_{.i}$ the electromagnetic part of the total tensordensity $\mathfrak{T}^h_{.i}$ of stress, momentum, and energy. According to MAXWELL's equations

$$\nabla_j \mathfrak{S}^j_{.i} = -F_{ik}\,\mathfrak{s}^k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $\mathfrak{s}^h$ is the electric current-density. Hence (1) becomes

$$\nabla_j \mathfrak{K}^j_{.i} = F_{ik}\,\mathfrak{s}^k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In order to bring our notations in accordance with previous papers [7]) we add to $\mathfrak{K}^h_{.i}$ the potential momentum and energy of the matter with respect to the electromagnetic field, which is $\mathfrak{s}^h \varphi_i$. In fact, let the substance be a mixture of n different components, in relative equilibrium with each other, so that they all have the same velocity. Further let m^r and e^r be the mass and charge of one particle (molecule, ion, electron, etc.) of the r^{th} component, $N_r^{\mathrm{d}V}$ the number of particles [8]) of the r^{th} component in a volume-element $\mathrm{d}V$, $\mathfrak{N}^h_r$ the corresponding particle-current, so that $N_r^{\mathrm{d}V} =_{\mathrm{df}} \mathfrak{N}^h_r\,\mathrm{d}\mathfrak{V}_h$ [9]), $\mathrm{d}\mathfrak{V}_i$ being the components of $\mathrm{d}V$ [10]), and f^r_i the potential momentum and energy of each particle of the r^{th} component. Then $\mathfrak{s}^h = \mathfrak{N}^h_r\, e^r$, $f^r_i = e^r \varphi_i$. Hence the total potential momentum and energy in $\mathrm{d}V$ with respect to the field is

$$N_r^{\mathrm{d}V} f^r_i = \mathfrak{N}^h_r\, \mathrm{d}\mathfrak{V}_h \,.\, e^r \varphi_i = \mathfrak{s}^h\, \mathrm{d}\mathfrak{V}_h \,.\, \varphi_i.$$

The total stress-tensor of the matter is therefore

$$\mathfrak{P}^h_{.i} =_{\mathrm{df}} \mathfrak{K}^h_{.i} + \mathfrak{N}^h_r f^r_i = \mathfrak{K}^h_{.i} + \mathfrak{s}^h \varphi_i, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

whereas the remaining part is

$$\mathfrak{T}^h_{.i} - \mathfrak{P}^h_{.i} = \mathfrak{S}^h_{.i} - \mathfrak{s}^h \varphi_i. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If the mixture is neutral, the second term in the right members of (7) and (8) vanishes.

Introducing (7) into (6) we obtain

$$\nabla_j \mathfrak{P}^j_{.i} = \mathfrak{N}^j_r \nabla_i f^r_j, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where we used MAXWELL's equations in the form

$$F_{ik} = 2\nabla_{[i}\varphi_{k]} =_{\mathrm{df}} \nabla_i \varphi_k - \nabla_k \varphi_i,\ \nabla_j F^{hj}\sqrt{-\mathfrak{g}} = \mathfrak{s}^h, \text{ whence } \nabla_j \mathfrak{s}^j = 0.$$

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7) D. VAN DANTZIG, [2], [3], [4], referred to as Ph. Th., R. Th., and R. G. respectively.

8) If we wish to avoid any molecular model, we may read $N_r^{\mathrm{d}V}$ as the number of *gm. mol.* of the r^{th} component, contained in $\mathrm{d}V$, and $\mathfrak{N}^h_r$ as the corresponding current-density. Then λ^r, η^r, π^r_i, f^r_i, m^r, e^r also have to be reckoned pro gm. mol. instead of pro molecule. N.B. Summation convention, for r, s also.

9) By $=_{\mathrm{df}}$ and $_{\mathrm{df}}=$ we denote an equality defining its *left* and its ***right*** member respectively.

10) Cf. Ph. Th. p. 684.

Equation (9) remains valid if other than electromagnetic forces work upon the matter, provided they are derivable from a vector potential, which is the same for all particles of each component of the mixture, and if we take for f^r_i the sum of $e^r \varphi_i$ and this vector potential.

According to (3), (7) the mixture is what I called a "perfectly perfect fluid" [11]), the stress-tensor and particle-densities having the form [12])

$$\mathfrak{P}^h_{.i} = \vartheta^h\, \mathfrak{p}_i + \mathfrak{p}\, A^h_i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\mathfrak{N}^h_r = \mathfrak{p}_r\, \vartheta^h, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where $\vartheta^h = i^h\,\vartheta_0 = c\,i^h/k\,T_0$ [13]) is the "temperature-vector", T_0 being the proper temperature and k BOLTZMANNs constant, whereas [6])

$$\mathfrak{p}_i = -\,c^{-1}\,k\,T_0\,(\varrho + \mathfrak{p})\,i_i + \mathfrak{p}_r\, f^r_i. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

According to Ph. Th. p. 697, R. Th. p. 604, generalised for a mixture of n components. $\mathfrak{p}$ can for constant x^h (i.e. in each given point of space-time) be considered as a function of $n+4$ independent variables ϑ^h and λ^r with

$$\mathfrak{p}_s = \frac{\partial \mathfrak{p}}{\partial \lambda^s}, \qquad \mathfrak{p}_i = \frac{\partial \mathfrak{p}}{\partial \vartheta^i}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

If the dependance upon the coordinates is taken into account also, $\mathfrak{p}$ depends upon them: 1⁰. through the ϑ^h and λ^r, 2⁰. through the f^r_i, which enter into $\mathfrak{p}$ in the combination $\lambda^r + f^r_j\,\vartheta^j$ only, 3⁰. through the g_{ij}. If the curvature is negligible and no direction in empty space is given, $\mathfrak{p}$ will have the form [14])

$$\mathfrak{p} = \varphi\,(\vartheta_0,\ \lambda^r + f^r_i\,\vartheta^i)\sqrt{-\mathfrak{g}}, \qquad \vartheta_0 =_{\mathrm{df}} \vartheta^h\, i_h = \sqrt{g_{ij}\,\vartheta^i\,\vartheta^j} = c/kT_0. \quad (14)$$

We suppose φ to be a function of its $n+1$ arguments alone (which

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11) Ph. Th. p. 686; R. Th. p. 602.

12) A very thorough discussion of the relativistic stress-tensor on an axiomatic base was given by J. L. SYNGE, [1]. I find however some difficulty in accepting SYNGE's hypothesis of "elementary impulses". For either the "gas" consisting of these elementary impulses would behave in a way entirely different from any other gas with respect to its thermodynamic properties, or otherwise it could, like a photon-gas, be considered like one of the components of our mixture, admitted the alterations mentioned in R. G. § 10. In that case however it would be difficult to see, how the elementary impulses could have any appreciable influence, except at very high temperatures. In any case the complete independence of the interaction between molecules and the temperature would be hard to account for.

13) In order to avoid factors c in the general (non-metrical) part of the theory, where they have nothing to do, we may take with respect to orthogonal coordinates $x^4 = t$ (instead of $x^4 = ct$), $g_{44} = c^2$, $g^{44} = c^{-2}$, etc. This leads to some unusual factors c, but makes comparison with the non-metrical as well as the classical theory easier. This system of notations was introduced by DE DONDER [1].

14) Cf. e.g. for an ideal gas R. G. (51).

themselves are functions of x^h) and not to depend upon the x^h explicitly. Then, putting $\varphi' =_{\mathrm{df}} \partial\varphi/\partial\vartheta_0$, $\varphi_r =_{\mathrm{df}} \partial\varphi/\partial\lambda^r$,

$$\left.\begin{aligned} \mathfrak{p}_r &= \varphi_r \sqrt{-\mathfrak{g}}, \\ \mathfrak{p}_i &= (\varphi_r f_i^r + \varphi' i_i)\sqrt{-\mathfrak{g}}, \\ \frac{\partial\mathfrak{p}}{\partial f_j^r} &= \vartheta^j \varphi_r \sqrt{-\mathfrak{g}} = \mathfrak{N}_r^j, \\ \frac{\partial\mathfrak{p}}{\partial g_{ij}} &= \mathfrak{p}\, g^{ij} + \tfrac{1}{2}\varphi' \sqrt{-\mathfrak{g}}\,.\,\vartheta^i \vartheta^j/\vartheta_0. \end{aligned}\right\} \quad \ldots\ldots \quad (15)$$

Denoting by ∂_i^0 the operator $\partial/\partial x^i$ under constant ϑ^h and λ^r, and by ∂_i the complete operator $\partial/\partial x^i$, so that

$$\partial_i^0 =_{\mathrm{df}} \partial_i - (\partial_i \vartheta^h)\frac{\partial}{\partial\vartheta^h} - (\partial_i \lambda^r)\frac{\partial}{\partial\lambda^r}, \quad \ldots\ldots \quad (16)$$

a short calculation shows that (9) is equivalent with

$$\boxed{\partial_j \mathfrak{P}^j_{.i} - \partial_i^0 \mathfrak{p} = 0} \quad \ldots\ldots\ldots \quad (17)$$

In this equation the metrical quantities do not occur anymore. Though its terms themselves are not invariant under arbitrary transformations of coordinates in space-time, the equation (17) is invariant as a whole, $\mathfrak{P}^h_{.i}$ having the form (10). At the other hand it can be shown that it is also necessary that $\mathfrak{P}^h_{.i}$ has the form (10), i.e. that the fluid is perfectly perfect, in order that (17) be invariant, as long as the operator ∂_i^0 is defined by (16). We may therefore consider (17) as the equations of motion of an arbitrary perfectly perfect fluid, independent of metrical geometry.

From here on, except when the contrary is stated, all equations are independent of any special assumption concerning the relation between energy and momentum. Hence they are independent of the axioms of relativity theory and also of metrical geometry.

Inserting (10) and (16) into (17) we obtain

$$\vartheta^j \partial_j \mathfrak{p}_i + \mathfrak{p}_j \partial_i \vartheta^j + \mathfrak{p}_i \partial_j \vartheta^j = -\mathfrak{p}_r \partial_i \lambda^r. \quad \ldots\ldots \quad (18)$$

Both sides of this equation are invariant quantities.

The differential equations

$$\frac{dx^h}{d\theta} = \vartheta^h(x) \quad \ldots\ldots\ldots\ldots \quad (19)$$

define a one-parametric group over a part of space-time. Its infinitesimal

transformation is determined by $dx^h =_{\mathrm{df}} d\theta\,\vartheta^h$, $d\theta$ being an infinitesimal increasement of the parameter θ. Its LIE-symbol is $\frac{d}{d\theta} =_{\mathrm{df}} \vartheta^i \partial_i$.

The infinitesimal transformation transforms any *scalar function* f into $f + df/d\theta$. The corresponding invariant operator working upon a general geometric object $X^{\bullet}$ (where the big points stand for rows of indices) was first defined by ŚLEBODZIŃSKI [15]). It will be called the LIE-derivative and denoted by $\frac{d_{\mathrm{L}}}{d\theta}$. It is defined as follows. The components of $X^{\bullet} + d_{\mathrm{L}} X^{\bullet}$ in a point (with respect to the system of coordinates under consideration) are the components of the value, $X^{\bullet}$ takes in a point $x^h + dx^h = x^h + \vartheta^h d\theta$ with respect to the system of coordinates, obtained by "dragging along" the original coordinates along dx^h, i.e. by ascribing to each point y^h as new coordinates $y^{h'}$ the original coordinates of the point from which it was obtained by the infinitesimal transformation: $y^{h'} = y^h - \vartheta^h(y)\,d\theta$. By this definition $\frac{d_{\mathrm{L}}}{d\theta}$ becomes an invariant operator. Applied to a scalar f we get $\frac{d_{\mathrm{L}} f}{d\theta} = \frac{df}{d\theta}$. Applied to a scalar density (or W-density [16])) $\mathfrak{p}$ of weight $+1$, to a contravariant vector v^h and to a covariant vector w_i we obtain [17])

$$\frac{d_{\mathrm{L}}\mathfrak{p}}{d\theta} = \partial_j \mathfrak{p}\,\vartheta^j = \frac{d\mathfrak{p}}{d\theta} + \mathfrak{p}\,\partial_j \vartheta^j \quad \ldots\ldots \quad (20)\ ^{18)}$$

$$\frac{d_{\mathrm{L}} v^h}{d\theta} = \vartheta^j \partial_j v^h - v^j \partial_j \vartheta^h = \frac{dv^h}{d\theta} - v^j \partial_j \vartheta^h, \quad \ldots \quad (21)\ ^{19)}$$

$$\frac{d_{\mathrm{L}} w_i}{d\theta} = 2\vartheta^j \partial_{[j} w_{i]} + \partial_i \vartheta^j w_j = \frac{dw_i}{d\theta} + w_j \partial_i \vartheta^j. \quad \ldots \quad (22)$$

Evidently

$$\frac{d_{\mathrm{L}} \vartheta^h}{d\theta} = 0. \quad \ldots\ldots\ldots\ldots \quad (23)$$

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15) W. ŚLEBODZIŃSKI [1]. Cf. also J. A. SCHOUTEN and E. R. VAN KAMPEN [1], D. VAN DANTZIG [5]; J. A. SCHOUTEN and D. J. STRUIK [1], p. 142.

16) W-densities are quantities which under transformations of coordinates are multiplied by a power of the *absolute value* of the transformation-modulus. They were introduced by H. WEYL, e.g. [1] p. 98. Cf. also J. A. SCHOUTEN [1], J. A. SCHOUTEN and D. VAN DANTZIG [1].

17) Cf. J. A. SCHOUTEN and E. R. VAN KAMPEN, loc. cit., p. 4.

18) A differential operator (e.g. ∇_j, ∂_j, $\frac{d_{\mathrm{L}}}{d\theta}$, etc.) works upon all following quantities untill either a closing bracket belonging to an opening bracket preceding the operator, or a $+$, $-$ or $=$ sign, or the end of the formula is met. The symbol for a differential, a variation or an element (e.g. d, δ, d) belongs to the immediately following symbol alone.

19) Introducing the Lie-symbols $X_1 =_{\mathrm{df}} \frac{d_{\mathrm{L}}}{d\theta} =_{\mathrm{df}} \vartheta^j \partial_j$ and $X_2 =_{\mathrm{df}} v^j \partial_j$, we have $\left(\frac{d_{\mathrm{L}}}{d\theta} v^j\right)\partial_j = (X_1, X_2)$, where the right member is the ordinary LIE-bracket.

Applied to a covariant vectordensity (or vector-W-density) $\mathfrak{p}_i$ of weight $+1$ we obtain

$$\frac{d_{\mathrm{L}}\,\mathfrak{p}_i}{d\theta} = \vartheta^j\,\partial_j\,\mathfrak{p}_i + \mathfrak{p}_j\,\partial_i\,\vartheta^j + \mathfrak{p}_i\,\partial_j\,\vartheta^j, \qquad (24)$$

which is the left side of (18). Hence we have proved:

Theorem 1. *The equations of motion* (17) *are equivalent with the system of equations*

$$\boxed{\frac{d_{\mathrm{L}}\,\mathfrak{p}_i}{d\theta} = -\,\mathfrak{p}_r\,\partial_i\,\lambda^r} \qquad (25)$$

§ 2. *Derivation from a variational principle.*

In this paragraph we show that the equations of motion (17) for a perfectly perfect fluid can be derived from a simple variational principle.

Let U be a part of space-time, bounded by a differentiable threedimensional manifold B. The latter is provided with an exterior orientation [20]), viz the direction *outward* from U. An element of U will be denoted by $\mathrm{d}\,U$, its fourdimensional volume (as measured by the coordinates under consideration) by $\mathrm{d}\,\mathfrak{U}$; an element of B by $\mathrm{d}B$ or $\mathrm{d}\,V$; its components by $\mathrm{d}\mathfrak{B}_i$ or $\mathrm{d}\mathfrak{B}_i$. Let further a contravariant vectorfield $z^h = z^h(x^i)$ be defined in U, the components of which have continuous first derivatives with respect to the x^i.

In order to determine easily the variations of several quantities it is useful to introduce the LIE-variation δ_{L} with respect to the equations $dx^h/d\mu = z^h(x)$ in the same way as $d_{\mathrm{L}} = d\theta \dfrac{d_{\mathrm{L}}}{d\theta}$ was defined with respect to (19).

The variation of an integral $W =_{\mathrm{df}} \displaystyle\int_U \mathfrak{W}\,\mathrm{d}\,\mathfrak{U}$ under the infinitesimal transformation $\delta =_{\mathrm{df}} \delta x^j\,\partial_j =_{\mathrm{df}} \delta\mu\,.\,z^j\,\partial_j$ is then

$$\delta\,W = \delta_{\mathrm{L}}\,W = \int_U (\delta_{\mathrm{L}}\,\mathfrak{W})\,\mathrm{d}\,\mathfrak{U} = \int_U (\partial_j\,\mathfrak{W}\,\delta x^j)\,\mathrm{d}\,\mathfrak{U} = \int_B \mathfrak{W}\,\delta x^j\,\mathrm{d}\,\mathfrak{B}_j, \quad (26)$$

as by definition $\delta_{\mathrm{L}}\,\mathrm{d}\,\mathfrak{U} = 0$, the new element $\mathrm{d}\,U$ being obtained by "dragging along" the original element. It is of course trivial that $\delta\,W$ reduces to a boundary integral as the variation consists of a displacement alone.

20) An exterior orientation of a simplex Σ is an orientation for each simplex of complementary dimension having exactly one point in common with Σ; Σ itself need *not* be oriented. Cf. O. VEBLEN and J. H. C. WHITEHEAD [1], p. 56, and J. A. SCHOUTEN and D. VAN DANTZIG [1].

Now we take in particular $\mathfrak{W} =_{\mathrm{df}} \mathfrak{p} = \mathfrak{p}(x^h, \vartheta^h, \lambda^r)$. In U a congruence of curves is defined, viz the macroscopic wordlines of the motion, satisfying the equations (17). We now define the operator $\overset{0}{\delta}_{\mathrm{L}}$ working upon a quantity $X^{\bullet} =_{\mathrm{df}} X^{\bullet}(x^h, \vartheta^h, \lambda^r)$, depending upon ϑ^h and λ^r, by requiring that not only the coordinates, but also the values of ϑ^h and λ^r shall be "dragged along". More precisely: $X^{\bullet} + \overset{0}{\delta}_{\mathrm{L}}\,X^{\bullet}$ shall be the components with respect to the coordinates mentioned above of $X^{\bullet}('x^h, '\vartheta^h, '\lambda^r)$, where $'x^h = x^h + \delta x^h$, whereas $'\vartheta^h$ and $'\lambda^r$ are defined by requiring $\overset{0}{\delta}_{\mathrm{L}}\,\vartheta^h = 0$, $\overset{0}{\delta}_{\mathrm{L}}\,\lambda^r = 0$. Hence

$$\overset{0}{\delta}_{\mathrm{L}}\,X^{\bullet} =_{\mathrm{df}} \delta_{\mathrm{L}}\,X^{\bullet} - \frac{\partial X^{\bullet}}{\partial\vartheta^h}\,\delta_{\mathrm{L}}\,\vartheta^h - \frac{\partial X^{\bullet}}{\partial\lambda^r}\,\delta_{\mathrm{L}}\,\lambda^r. \qquad (27)$$

In particular

$$\left.\begin{aligned} \overset{0}{\delta}_{\mathrm{L}}\,\mathfrak{p} &= \delta_{\mathrm{L}}\,\mathfrak{p} - \mathfrak{p}_i\,\delta_{\mathrm{L}}\,\vartheta^i - \mathfrak{p}_r\,\delta_{\mathrm{L}}\,\lambda^r = \\ &= \delta\,\mathfrak{p} + \mathfrak{p}\,\partial_j\,\delta x^j - \mathfrak{p}_i\,\delta\,\vartheta^i + \mathfrak{p}_i\,\vartheta^j\,\partial_j\,\delta x^i - \mathfrak{p}_r\,\delta\lambda^r = \\ &= \delta^0\,\mathfrak{p} + \mathfrak{P}^{j}_{.i}\,\partial_j\,\delta x^i, \end{aligned}\right\} \qquad (28)$$

where $\delta^0 =_{\mathrm{df}} \delta x^i\,\overset{0}{\partial}_j$, $\overset{0}{\partial}_j$ being defined by (16).

Hence

$$\left.\begin{aligned} \overset{0}{\delta}_{\mathrm{L}} \int_U \mathfrak{p}\,\mathrm{d}\,\mathfrak{U} &= \int_U (\delta x^i\,\overset{0}{\partial}_i\,\mathfrak{p} + \mathfrak{P}^{j}_{.i}\,\partial_j\,\delta x^i)\,\mathrm{d}\,\mathfrak{U} = \\ &= \int_B \mathfrak{P}^{h}_{.i}\,\delta x^i\,\mathrm{d}\,\mathfrak{B}_h + \int_U \delta x^i\,(\overset{0}{\partial}_i\,\mathfrak{p} - \partial_j\,\mathfrak{P}^{j}_{.i})\,\mathrm{d}\,\mathfrak{U}. \end{aligned}\right\} \qquad (29)$$

Hence we have proved:

Theorem 2. *The system of equations of motion* (17) *is equivalent with the variational principle*

$$\overset{0}{\delta}_{\mathrm{L}} \int_U \mathfrak{p}\,\mathrm{d}\,\mathfrak{U} = \int_B P_i^{\mathrm{d}B}\,\delta x^i \qquad (30)\ ^{21)}$$

for any part U of space-time and any continuously differentiable variation δx^i, and also with the variational principle

$$\overset{0}{\delta}_{\mathrm{L}} \int_U \mathfrak{p}\,\mathrm{d}\,\mathfrak{U} = 0$$

for any part U of space-time and any continuously differentiable [22]) *variation δx^i, vanishing at the boundary B of U.*

21) $P_i^{\mathrm{d}B} =_{\mathrm{df}} \mathrm{d}\,\mathfrak{B}_h\,\mathfrak{P}^{h}_{.i}$ is the amount of energy and momentum contained in or flowing through the element $\mathrm{d}\,B$. Analogously $N_r^{\mathrm{d}V}$, $S^{\mathrm{d}V}$, etc. Cf. Ph. Th. p. 684.

22) These conditions will not be mentioned explicitly further on.

This theorem seems to be sufficiently remarkable, to restate it in more explicite and intuitive terms. Let $I_U[\mathfrak{p}]$ be the integral of the pressure $\mathfrak{p}$ over a fourdimensional domain U in space time. A small deformation $x^h \rightarrow {}'x^h =_{\mathrm{df}} x^h + \delta x^h$ deforms U into a new domain $'U$. Moreover the congruence of macroscopic worldlines in U is deformed into another congruence in $'U$. If the parameter θ, defined by $\vartheta^h(x) = dx^h/d\theta$, is kept constant, the vectorfield $\vartheta^h(x)$ passes into a new vectorfield ${}'\vartheta^h('x) =_{\mathrm{df}} \vartheta^h(x) + d(\delta x^h)/d\theta$, which differs from $\vartheta^h('x) = \vartheta^h(x) + \delta\vartheta^h$ by $d_{\mathrm{L}}\,\delta x^h$. Leaving the *values* of λ^r unaltered, i.e. taking ${}'\lambda^r('x) =_{\mathrm{df}} \lambda^r(x) = \lambda^r('x) - \delta\lambda_r$, the variation of the pressure integral is the difference of the integral of ${}'\mathfrak{p} =_{\mathrm{df}} \mathfrak{p}({}'\vartheta^h('x), ({}'\lambda^r('x), {}'x^h)$ over $'U$ and the original integral over U. This variation $\delta^0 I = I_{'U}['\mathfrak{p}] - I_U[\mathfrak{p}]$ is according to Theorem 2 (if the coordinates are such that x^4 may be interpreted as the time) equal to δx^4 times the virtual heat belonging to the deformation, flowing through te boundary *into* U, if and only if the equations of motion are satisfied.

§ 3. *Other variational relations.*

The variational principle (30) can be brought into another form. Therefore let S_0 be a part of a threedimensional manifold in space-time, which has at most one point in common with any (macroscopic) worldline, and which nowhere is tangent to such a worldline [22], and let O_0 be its twodimensional boundary. The threedimensional manifold S_θ will be defined as the locus of all points x which have a difference of "thermasy" with S_0 equal to θ, i.e. such that the worldline through x intersects S_0 in a point x_0 and that the integral

$$\int_{x_0}^{x} d\theta = \int k\,T\,dt = \int c^{-1}\,k\,T_0\,ds \quad . \quad . \quad . \quad . \quad . \quad (32)$$

taken along the worldline through x is equal to θ. Let U be the locus of all S_θ for $0 \leqq \theta \leqq \theta_1$. Its boundary consists of S_0, S_{θ_1} and the locus B' of the parts of all worldlines through O_0 between S_0 and S_{θ_1}.

We suppose U to be divided into elements $\mathrm{d}U$ of the same nature as U itself, i.e. bounded by parts of two hypersurfaces S_θ and $S_{\theta+d\theta}$ and by ∞^2 parts of worldlines passing through the boundary of an element $\mathrm{d}V$ of S_θ. Then

$$\mathrm{d}\mathfrak{U} = dx^h\,\mathrm{d}\mathfrak{V}_h = \vartheta^h\,\mathrm{d}\mathfrak{V}_h\,d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

and

$$\int_U \mathfrak{p}\,\mathrm{d}\mathfrak{U} = \int_0^{\theta_1} d\theta \int_{S_\theta} \mathrm{d}\mathfrak{V}_h\,\vartheta^h\,\mathfrak{p} = \int_0^{\theta_1} d\theta \int_{S_\theta} Z^{\mathrm{d}V}, \quad . \quad . \quad . \quad (34)$$

or

$$\boxed{\int_{S_\theta} Z^{\mathrm{d}V} = \frac{d}{d\theta} \int_{U_\theta} \mathfrak{p}\,\mathrm{d}\mathfrak{U}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

if U_θ is the part of space-time bounded by B' and S_0, S_θ.

Now let a deformation δx^h of S_0 *in itself* be given, so that

$$\delta x^i\,\mathrm{d}\mathfrak{V}_i = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

for any element $\mathrm{d}\mathfrak{V}_i$ of S_0 and let δx^h vanish at the boundary O_0 of S_0. Let every element $\mathrm{d}\mathfrak{V}_i$ of S_0 as well as the displacement δx^h be „dragged along" by the transformation d_L, so that

$$d_{\mathrm{L}}\,\delta x^h = -\,\delta_{\mathrm{L}}\,\vartheta^h\,d\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad (37)\ [23]$$

and $d_{\mathrm{L}}\,\mathrm{d}\mathfrak{V}_i = 0$. Then (36) remains valid for every element of every S_θ. Moreover $\delta x^h = 0$ on B'.

By (29) and (37) we have then $\delta^0_{\mathrm{L}}\,\mathfrak{p} = \delta_{\mathrm{L}}\,\mathfrak{p} - \mathfrak{p}_r\,\delta_{\mathrm{L}}\,\lambda^r$, hence

$$\mathrm{d}\mathfrak{U}\,\delta^0_{\mathrm{L}}\,\mathfrak{p} = \mathrm{d}\mathfrak{U}\,\delta_{\mathrm{L}}\,\mathfrak{p} - d\theta\,N_r^{\mathrm{d}V}\,\delta_{\mathrm{L}}\,\lambda^r \quad . \quad . \quad . \quad . \quad . \quad (38)$$

hence, using (37) again,

$$\delta^0 \int_U \mathfrak{p}\,\mathrm{d}\mathfrak{U} = \delta \int_U \mathfrak{p}\,\mathrm{d}\mathfrak{U} - \int_0^{\theta_1} d\theta \int_{S_\theta} N_r^{\mathrm{d}V}\,\delta\lambda^r. \quad . \quad . \quad . \quad (39)$$

By (26) and the boundary conditions however

$$\delta \int_U \mathfrak{p}\,\mathrm{d}\mathfrak{U} = \int_B \mathfrak{p}\,\delta x^i\,\mathrm{d}\mathfrak{V}_i = \left[\int_{S_\theta} \mathfrak{p}\,\delta x^i\,\mathrm{d}\mathfrak{V}_i\right]_0^{\theta_1} = 0. \quad . \quad . \quad (40)$$

The right member of (30) need also be extended over S_0 and S_{θ_1} only, so that we have proved:

Theorem 3. The equations of motion (17) *are equivalent with the variational principle*

$$-\int_0^{\theta_1} d\theta \int_{S_\theta} N_r^{\mathrm{d}V}\,\delta\lambda^r = \left[\int_{S_\theta} P_i^{\mathrm{d}V}\,\delta x^i\right]_0^{\theta_1} \quad . \quad . \quad . \quad . \quad (41)$$

for every variation δx^i of each S_θ in itself, vanishing at the boundary of each S_θ, and dragged along by d_L.

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[23]) According to [19]) this condition means simply that the deformation δ_{L} and the transformation d_{L} are interchangeable.

A variation of this type leaves the congruence of macroscopical world-lines invariant. If we assume each displaced tube of worldlines to be filled up with the same amount of matter of each kind as the original one, we have $\delta N_r^{\mathrm{d}V} = 0$. Then, putting

$$-N_r^{\mathrm{d}V}\,\lambda^r = L^{\mathrm{d}V} \quad \ldots\ldots\ldots \quad (42)$$

(41) becomes equivalent with

$$\boxed{\delta \int\limits_{\theta_0}^{\theta_1} d\theta \int\limits_{S_\theta} L^{\mathrm{d}V} = \Big[\int\limits_{S_\theta} P_i^{\mathrm{d}V}\,\delta x^i\Big]_0^{\theta_1}} \quad \ldots\ldots \quad (43)$$

This variational principle shows a great analogy with that of ordinary point mechanics, viz

$$\delta \int\limits_{t_0}^{t_1} L\,dt = [p_i\,\delta x^i]_{t_0}^{t_1}. \quad \ldots\ldots\ldots \quad (44)$$

From (33) and (28) we obtain by Ph. Th. (58), (60) for an *arbitrary* variation δx^h:

$$\delta^0 \int\limits_U \mathfrak{p}\,\mathrm{d}\mathfrak{U} = \int\limits_U (\delta_{\mathrm{L}}^0\,\mathfrak{p})\,\mathrm{d}\mathfrak{U} = \int\limits_0^{\theta_1} d\theta \int\limits_{S_\theta} \mathrm{d}\mathfrak{B}_h\,\vartheta^h\,\delta_{\mathrm{L}}^0\,\mathfrak{p} =$$

$$= \int\limits_0^{\theta_1} d\theta \int\limits_{S_\theta} \mathrm{d}\mathfrak{B}_h\,\vartheta^h\,(\delta_{\mathrm{L}}\,\mathfrak{p} - \mathfrak{p}_r\,\delta_{\mathrm{L}}\,\lambda^r - \mathfrak{p}_i\,\delta_{\mathrm{L}}\,\vartheta^i) =$$

$$= \int\limits_0^{\theta_1} d\theta \int\limits_{S_\theta} (\delta_{\mathrm{L}}\,Z^{\mathrm{d}V} - N_r^{\mathrm{d}V}\,\delta_{\mathrm{L}}\,\lambda^r - P_i^{\mathrm{d}V}\,\delta_{\mathrm{L}}\,\vartheta^i - \mathfrak{Z}^h\,\delta_{\mathrm{L}}\,\mathrm{d}\mathfrak{B}_h),$$

hence

$$\delta^0 \int\limits_U \mathfrak{p}\,\mathrm{d}\mathfrak{U} = \int\limits_0^{\theta_1} d\theta \int\limits_{S_\theta} (\delta_{\mathrm{L}}\,S^{\mathrm{d}V} + \lambda^r\,\delta_{\mathrm{L}}\,N_r^{\mathrm{d}V} + \vartheta^i\,\delta_{\mathrm{L}}\,P_i^{\mathrm{d}V} - \mathfrak{p}\,\vartheta^h\,\delta_{\mathrm{L}}\,\mathrm{d}\mathfrak{B}_h). \quad (45)$$

According to (30) this must vanish for every variation δx^h vanishing at the boundary of U. *The integrand of* (45) *is the quantity which would have to vanish if the operator* δ_L *were replaced by a variation* δ *of the variables* ϑ^h *and* λ^r *alone, without change of the point* x_h (Cf. Ph. Th. (57)).

Hence we have proved:

Theorem 4. *The equations of motion* (17) *are satisfied if and only if*

$$\boxed{\int\limits_0^{\theta_1} d\theta \int\limits_{S_\theta} (\delta_{\mathrm{L}}\,S^{\mathrm{d}V} + \lambda^r\,\delta_{\mathrm{L}}\,N_r^{\mathrm{d}V} + \vartheta^i\,\delta_{\mathrm{L}}\,P_i^{\mathrm{d}V} - \mathfrak{p}\,\vartheta^h\,\delta_{\mathrm{L}}\,\mathrm{d}\mathfrak{B}_h) = 0} \quad (46)$$

for every variation δx^h *vanishing at the boundary of* U.

Evidently the last term in the integrand of (46) only defines $\delta_{\mathrm{L}}\,\mathrm{d}\mathfrak{B}_i$. Deforming $\mathrm{d}\mathfrak{B}_i$ simply by "dragging along", we get $\delta_{\mathrm{L}}\,\mathrm{d}\mathfrak{B}_i = 0$, and this term vanishes.

The right member of (30) becomes

$$\int\limits_B P_i^{\mathrm{d}B}\,\delta x^i = \Big[\int\limits_{S_\theta} P_i^{\mathrm{d}V}\,\delta x^i\Big]_0^{\theta_1} + \int\limits_0^{\theta_1} d\theta \int\limits_{O_\theta} \mathrm{d}\mathfrak{O}_{hj}\,\vartheta^j\,\mathfrak{P}^{h}_{\cdot i}\,\delta x^i, \quad . \quad (47)$$

where $\mathrm{d}\mathfrak{O}_{hj}$ are the components of an element of the boundary O_θ of S_θ with appropriate exterior orientation. Hence, if δx^h vanishes at the boundary of each S_θ (though not necessarily over S_0 and S_{θ_1} themselves) and if $\delta_{\mathrm{L}}\mathrm{d}\mathfrak{B}_i = 0$, we obtain, equating the right members of (45) and (47):

$$\int\limits_0^{\theta_1} d\theta \int\limits_{S_\theta} (\delta_{\mathrm{L}}\,S^{\mathrm{d}V} + \lambda^r\,\delta_{\mathrm{L}}\,N_r^{\mathrm{d}V} + \vartheta^i\,\delta_{\mathrm{L}}\,P_i^{\mathrm{d}V} - \frac{d}{d\theta}\,P_i^{\mathrm{d}V}\,\delta x^i) = 0.$$

Differentiating with respect to θ_1 and replacing θ_1 by θ we get

$$\int\limits_{S_\theta} (\delta\,S^{\mathrm{d}V} + \lambda^r\,\delta\,N_r^{\mathrm{d}V} + \vartheta^i\,\delta\,P_i^{\mathrm{d}V} - \delta x^i\,\frac{d}{d\theta}\,P_i^{\mathrm{d}V}) = 0, \quad . \; . \quad (48)$$

where the invariant LIE-variation δ_{L} has been replaced by the uninvariant variation δ again.

§ 4. *Variational principles of Fermat's type.*

Let us suppose now that the fluid moves in such a way, that a function $\varphi = \varphi(x^h)$ exists, such that

$$\mathfrak{p}_r\,\partial_i\,\lambda^r = -\,\mathfrak{q}\,\partial_i\,\varphi, \qquad \mathfrak{q} =_{\mathrm{df}} \mathfrak{p}_j\,\vartheta^j. \quad \ldots\ldots \quad (49)$$

Then the equations of motion (25) become:

$$\frac{d_{\mathrm{L}}}{d\theta}\,\mathfrak{p}_i = \mathfrak{q}\,\partial_i\,\varphi. \quad \ldots\ldots\ldots\ldots \quad (50)$$

Transvection with ϑ^i shows that the density

$$\mathfrak{a} =_{\mathrm{df}} e^{-\varphi}\,\mathfrak{q}\,. \quad \ldots\ldots\ldots\ldots\ldots \quad (51)$$

is a constant of the motion:

$$\frac{d_{\mathrm{L}}\,\mathfrak{a}}{d\theta}=0. \qquad (52)$$

Moreover variation of the integral

$$I =_{\mathrm{df}} \int_{\theta_0}^{\theta_1} e^{\varphi}\, d\theta = \int_{\theta_0}^{\theta_1} \mathfrak{a}^{-1}\, \mathfrak{p}_i\, dx^i \qquad (53)$$

with $\delta\theta=0$ at the boundaries, gives by (50), (51):

$$\delta I = \int_{\theta_0}^{\theta_1} e^{\varphi}\, \delta\varphi\, d\theta = \int_{\theta_0}^{\theta_1} \delta x^i\, d_{\mathrm{L}}\, \mathfrak{a}^{-1}\, \mathfrak{p}_i. \qquad (54)$$

Hence, if the variation is performed by "dragging along" ϑ^h (which is equivalent with the condition that δx^h is dragged along by the operator d_{L}), i.e. $\delta_{\mathrm{L}}\vartheta^h = -\, d_{\mathrm{L}}\delta x^h/d\theta = 0$, then the integration in the last member of (54) can be carried out by using (52) and we obtain:

Theorem 5[23a]*). If* (49) *is satisfied throughout the fluid, then*

$$\mathfrak{a} =_{\mathrm{df}} e^{-\varphi}\, \mathfrak{p}_i\, \vartheta^i$$

is a constant of the motion and

$$\boxed{\delta \int_{\theta_0}^{\theta_1} e^{\varphi}\, d\theta = \delta \int_{\theta_0}^{\theta_1} \mathfrak{a}^{-1}\, \mathfrak{p}_i\, dx^i = [\mathfrak{a}^{-1}\, \mathfrak{p}_i\, \delta x^i]_{\theta_0}^{\theta_1}} \qquad (55)$$

for any variation δx^h with $\delta_{\mathrm{L}}\vartheta^h = 0$[24]).

Corollary: If the fluid is homogeneous and if

$$\partial_i\, \eta = 0. \qquad (56)$$

throughout the motion, then

$$\delta \int_{\theta_0}^{\theta_1} \pi_i\, dx^i = [\pi_i\, \delta x^i]_{\theta_0}^{\theta_1}. \qquad (57)$$

Proof: By hypothesis $n=1$. We drop the suffix r and write $\mathfrak{p}'$ for $\partial\mathfrak{p}/\partial\lambda$. Then (56) is equivalent with

$$\partial_i\, \lambda = -\, \partial_i\, \pi_j\, \vartheta^j = -\, \pi_j\, \vartheta^j\, \partial_i \log(-\pi_j\, \vartheta^j),$$

[23a]) The theorem remains valid if to the right member of (49) a term of the form $-\gamma\, \mathfrak{p}_i$ is added and if the first statement is replaced by "$\mathfrak{a}\, e^{\int -\gamma\, d\theta}$ is a constant of the motion", $\mathfrak{a}$ being defined by (51). If to the right members of (50) and (52) a term $\gamma\, \mathfrak{p}_i$ and $\gamma\, \mathfrak{a}$ respectively is added, the demonstration remains valid.

[24]) In that case the values of θ can be kept constant during the deformation, so that $\delta\theta$ vanishes identically, and that (54) is valid.

where $\pi_i =_{\mathrm{df}} \mathfrak{p}_i/\mathfrak{p}'$, or

$$\mathfrak{p}_r\, \partial_i\, \lambda^r = \mathfrak{p}'\, \partial_i\, \lambda = -\, \mathfrak{q}\, \partial_i \log(-\pi_j\, \vartheta^j)$$

so that (49) is satisfied with $e^{\varphi} = -\, \pi_j \vartheta^j$. Substitution into (52) leads immediately to the proof of the corollary. Evidently (52) holds with $\mathfrak{a} = -\mathfrak{p}'$, i.e. the condition (56) implies that the equation of continuity (cf. § 5 (68)) is satisfied. The equation of continuity, however, is *not* sufficient in order that (56) be satisfied.

As a particular case let us introduce metric again and assume that an equation of state of the form (14) exists with $n=1$ (homogeneous relativistic fluid) and $f_i=0$. Then (49) becomes

$$\partial_i\, \lambda = \varepsilon\, \vartheta_0\, \partial_i\, \varphi. \qquad (58)$$

with

$$\vartheta_0 = \vartheta^h\, i_h = \sqrt{g_{ij}\, \vartheta^i\, \vartheta^j} = c\,(k\,T_0)^{-1}$$

and

$$\varepsilon = \sqrt{g^{ij}\, \pi_i\, \pi_j}\,,\quad \pi_i = -\,\varepsilon\, i_i.$$

Hence we have also

$$\nabla_i\, \mathfrak{p} = \mathfrak{p}'\, \nabla_i\, \lambda - \mathfrak{p}'\, \varepsilon\, \nabla_i\, \vartheta_0 = \mathfrak{p}'\, \varepsilon\, \vartheta_0\, \partial_i\,(\varphi - \log \vartheta_0).$$

A relation $u\,\partial_i v = \partial_i w$, however, implies that w and u are functions of v alone and that $u = dw/dv$.

As $\mathfrak{p}'$, ε and ϑ_0 are always $\neq 0$ (even >0), $\varphi - \log\vartheta_0$ and then also $\mathfrak{p}'\,\varepsilon\,\vartheta_0 = -\,\mathfrak{p}_i\,\vartheta^i = -\,\mathfrak{q} = \varrho + \mathfrak{p}$ (cf. (3) and (10)), hence also ϱ is a function of $\mathfrak{p}$ alone[25]) and we have

$$\varphi = \log\vartheta_0 + \int \frac{d\mathfrak{p}}{\mathfrak{p}'\,\varepsilon\,\vartheta_0} = \log\vartheta_0 - \int \frac{d\mathfrak{p}}{\mathfrak{p}_i\,\vartheta^i}. \qquad (59)$$

Or also, introducing the "index-function"[26]) $F =_{\mathrm{df}} e^{\varphi}/\vartheta_0$, whence

$$\log F =_{\mathrm{df}} -\int \frac{d\mathfrak{p}}{\mathfrak{p}_i\,\vartheta^i} = \int \frac{d\mathfrak{p}}{\varrho + \mathfrak{p}}, \qquad (60)$$

the integral (53) becomes by (49):

$$I = \int_{\theta_0}^{\theta_1} e^{\varphi}\, d\theta = \int_{\theta_0}^{\theta_1} F\,\vartheta_0\, d\theta = \int_{\theta_0}^{\theta_1} F\, ds. \qquad (61)$$

[25]) This means that the quantities $\mathfrak{q}$ etc. depend upon the coordinates x^h through $\mathfrak{p}$ alone, i.e. that $\delta\mathfrak{q} = \frac{d\mathfrak{p}}{d\mathfrak{q}}\,\delta\mathfrak{p}$ for all variations δ of the form $\delta = \delta x^j\,\partial_j$, but of course not for *arbitrary* variations. Relations of this type with respect to the *spatial* coordinates alone are called by BJERKNES barotropic relations. Cf. V. BJERKNES [1] p. 85.

[26]) Cf. J. L. SYNGE [2] p. 391.

Moreover $\mathfrak{a}^{-1}$ becomes

$$\mathfrak{a}^{-1} = e^{\varphi}\, \mathfrak{q}^{-1} = - F\, \vartheta_0\, (\varrho + \mathfrak{p})^{-1} = - \vartheta_0 \frac{dF}{d\mathfrak{p}}, \quad . \quad . \quad . \quad (62)$$

whence

$$\mathfrak{a}^{-1}\, \mathfrak{p}_i = (\mathfrak{a}\, \vartheta_0)^{-1}\, \mathfrak{p}_j\, \vartheta^j\, i_i = F i_i.$$

Hence we have proved, the argument being reversible:

Theorem 6. *If a homogeneous fluid has an equation of state of the form* (14) *and moves in such a way, that* ϱ *is a function of* $\mathfrak{p}$ *alone,* $k\, T_0\, (dF/d\mathfrak{p})^{-1}$ *is a constant of the motion and*

$$\boxed{\delta \int e^{\varphi}\, d\theta = \delta \int F\, ds = [F_i\, \delta x^i]}, \quad . \quad . \quad . \quad . \quad (63)$$

(*where* F *is given by* (60), *whereas* $F_i =_{\mathrm{df}} F i_i$) *for any variation satisfying the conditions of Theorem* 5.

The variational principle (63), was found in 1924 already by L. P. EISENHART[27]) by a method based on the geometry of paths. It was proved in a different way and discussed further by J. L. SYNGE [2] p. 393. We obtained it here as a metrical specialisation of (54).

Finally we apply theorem 5 to a volume, filled with black radiation, which can be considered as a homogeneous fluid with $\lambda = 0$ and $\mathfrak{q} = \mathfrak{p}_i\, \vartheta^i = -4\mathfrak{p}$ (cf. R.G. § 10). Then (49) is automatically satisfied with $\varphi = 0$, so that I simply becomes

$$I = \int d\theta = \int k T\, dt = \int c^{-1}\, k T_0\, ds,$$

It is of importance to note that the proper mass of a volume filled with radiation (being greater than the sum of the proper masses of the photons, which is zero) is always positive, viz $3\, \mathfrak{p}\, d\mathfrak{V}/c^2$. Therefore the average velocity of black radiation is always $< c$ (e.g. $= 0$ if the radiation is enclosed in a box at rest). Hence $ds \neq 0$, so that i^h, T_0, etc. remain finite. Of course this does not apply to a single beam of light, which is a limiting case with $\mathfrak{p} = \mathfrak{p}_i\, \vartheta^i = 0$. Moreover (14) is satisfied with $\lambda = 0$. Hence $F = \vartheta_0^{-1} = k\, T_0$. Finally we remark that by theorem 5 $\mathfrak{a} = \mathfrak{q}$, hence $\mathfrak{p}$, hence T_0 is a constant of the motion, so that we have proved

Theorem 7: *Black radiation moves always according to the variational principle*

$$\delta \int_{\theta_0}^{\theta_1} d\theta = -\tfrac{1}{4}\, [\mathfrak{p}^{-1}\, \mathfrak{p}_i\, \delta x^i]_{\theta_0}^{\theta_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

27) Cf. L. P. EISENHART [1] p. 214—216.

which becomes by metrical specialisation, T_0 *being constant along the rays,*

$$\delta \int ds = [i_i\, \delta x^i]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

Hence the "stream-lines" of black radiation are geodesics.

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Anatomy. — *The cytology of the cortex in the opossum (Didelphys virginiana) considered in relation to some general problems of cortical evolution.* By W. RIESE and G. E. SMYTH [1]). (Communicated by Prof. C. U. ARIËNS KAPPERS.)

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In the extensive literature devoted to the comparative anatomy of the cerebral cortex it would appear that in the cytotectonic descriptions too little attention is payed to the differentiation of the constituent cells themselves. So the cortical lamination of the opossum has been carefully studied by GRAY (1924); the excitable areas of the cortex of this animal by GRAY and TURNER (1924) and the origin and course of the pyramidal tracts by TURNER (1924). Nevertheless little or no information is available on the structural differentiation of the cells in the several cortical laminae and in representative cortical areas.

In the present paper we shall deal with some cellular differentiations in the cortex of the opossum as an addition to our recent observations on the cellular structure of the thalamus and corpus striatum and other subcortical centers of this primitive mammal [2]).

In consequence of the lack of systematic phylogenetic studies of cellular differentiation in mammals no absolute standard exists whereby the degree of differentiation of nerve cells may be tested. In these observations the criteria adopted are based on ontogenetic studies, in particular on the conclusions arrived at by RIESE (1939) from investigations on the development of the brain of the bear (Ursus arctos). They are in close agreement with some earlier observations on the development of cortical cells in pig and rabbit embryos reported by PATON (1900).

Realizing the fundamental difference between embryonic and adult cell structures, also in lower mammals, the comparisons established in this paper are only meant to indicate certain analogies. With this reservation the criteria of structural differentiation may be summarized as follows:

1. The amount of cytoplasm, particularly in relation to the bulk of the nucleus, and the sharpness of demarcation between cytoplasm and the surrounding substances. The more primitive the cell, the more poorly demarcated seems the cell outline. The bulk of the cytoplasm relative to that of the nucleus tends to increase with the progressive evolution of the cell.

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[1]) From the Laboratoire de Physiologie générale de la Sorbonne et Laboratoire d'Ethologie des Animaux Sauvages au Muséum National d'Histoire Naturelle.

[2]) These observations will be communicated soon.