§ 8.

Wenn wir in

$$Q = 0_{13,23}$$

nach (1) das Paar 13 durch 22 ausdrücken und 23 durch die Ableitung $\frac{1}{2}(22)_1$ ersetzen, so lässt sich die Invariante Q so schreiben:

$$Q = -\tfrac{3}{4}\, 0_{22,23} = \tfrac{3}{8}\, 0_{(22)_1}\, 0_{22} = \tfrac{3}{8}\, (H)_{(22)_1}. \quad . \quad . \quad . \quad . \quad (72)$$

Auf Grund dieser Darstellung lässt sich $Q(\varphi)$ einfach berechnen, wobei die Gerade φ_{ik} durch (60) gegeben ist. Es wird

$$Q(\varphi) = \tfrac{3}{8}\, [H(\varphi)]_{[M'_{22}(\varphi)]_1},$$

also nach (70) und wenn wir wieder die rechte Seite von (67) mit $(w' x)$ bezeichnen:

$$Q(\varphi) = \tfrac{3}{8}\, \lambda\, (\lambda - 8\,Q)^2 \,.\, (H w'_1).$$

Da $(H w') = 0$ ist, folgt durch Differentiation

$$(H_1 w') + (H w'_1) = 0 \;,\; (H w'_1) = -(H_1 w'),$$

also, da wir oben schon $(H_1 w')$ berechnet haben,

$$(H w'_1) = -\tfrac{8}{3}\, Q\, \lambda\, (8\,Q - \lambda), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

Somit wird

$$Q(\varphi) = -Q \,.\, \lambda^2\, (8\,Q - \lambda)^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

Für $\lambda = 4Q$ gibt dies für die Heftgerade α_{ik}:

$$Q(\alpha) = -4^5 \,.\, Q^6 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

Weiters berechnen wir die zu φ_{ik} gehörige Heftgerade $\alpha(\varphi)$. Nach (58) ist

$$\alpha(\varphi) = \tfrac{3}{2}\, (H(\varphi)\, H_1(\varphi))_{ik} + 4\, Q(\varphi) \,.\, \varphi_{ik}.$$

Aus (70) finden wir

$$(H(\varphi)\, H_1(\varphi))_{ik} = \lambda^2\, (8\,Q - \lambda)^4 \,.\, (H H_1)_{ik};$$

also wird nach (74) und (60):

$$\alpha(\varphi) = \tfrac{3}{2}\, \lambda^2 (8\,Q - \lambda)^4 \,.\, (H H_1)_{ik} - 4\,Q\,\lambda^2\, (8\,Q - \lambda)^3 \,.\, [\tfrac{3}{2} (H H_1)_{ik} + \lambda \,.\, 0_{ik}]$$

$$\alpha(\varphi) = \lambda^2 (8\,Q - \lambda)^3 \,.\, [\tfrac{3}{2} (4\,Q - \lambda) \,.\, (H H_1)_{ik} - 4\,Q\,\lambda \,.\, 0_{ik}] \quad . \quad . \quad (76)$$

Setzen wir hier $\lambda = 4Q$, so ergibt sich

$$\alpha_{ik}(\alpha) = (-4\,Q)^7 \,.\, 0_{ik} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

Für $Q \neq 0$ ist also die Heftfläche der Heftfläche mit der ursprünglichen Regelfläche F identisch, was geometrisch zu erwarten war.

Astronomy. — *Spontaneous development of a gaseous disc revolving round the sun into rings and planets.* II. By H. P. BERLAGE Jr., Research Associate Roy. Magn. and Meteorological Observatory, Batavia.

(Communicated at the meeting of March 30, 1940.)

Whether the nebula will really start producing rings depends on the condition that its potential energy decreases during the process. This condition requires that the left side of (42) is positive. Controlling this for the values of x, φ, ψ, γ and δ already found, we have to be extremely careful. We know that these values have to be slightly corrected in order to satisfy the 3 equations (40) (41) (42) in their unapproximated form. The application of this correction, however insignificant, has now become essential, because otherwise we should perhaps overlook that the left side of (42) which is to be positive, is identically $= 0$ when we take account in the variations of small quantities of the first order of magnitude only.

Indeed, as the nebula is supposed to start on its new course from a state of instable equilibrium the variation applied leaves $\triangle U$ and $\triangle V$ both $= 0$ up to first order small quantities. The variation of U and V in second order small quantities will decide whether U and V are decreasing or increasing.

The only way out of insurmountable complications is then to leave the general track and to start the calculation with a value of x which has by trial been found to agree with the actual state of the planetary system. We are then able to show that the nebula was inclined to follow the suggested course.

A very good approximation to the actual conditions is obtained for

$$a = 7.4 \times 10^{-7},\; b = 14.8 \times 10^{-7},\; x = 0.5 \quad . \quad . \quad . \quad . \quad (48)$$

We find

$$\frac{\gamma}{\delta} = -0.441 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Let us take as the one independent variable the important "amplitude" $\varkappa$ of the density fluctuation, or

$$\varkappa^2 = \gamma^2 + \delta^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

We then get successively

$$\gamma = -0.404\,\varkappa,\; \delta = 0.916\,\varkappa,\; \psi = -0.00533\,\varkappa\, a,\; \varphi = -0.0408\,\varkappa. \quad (51)$$

How is the structure of the planetary system that would be created in this way? The density maxima are along circles satisfying (21) or

$$\sqrt{r_n} - \sqrt{r_{n-1}} = \frac{2\pi}{b} = 4.24 \times 10^6 \quad . \quad . \quad . \quad . \quad . \quad (52)$$

If r_1 is the radius of the smallest ring

$$tg\,\frac{7.4 \times 10^{-7} \times r_1^{\frac{1}{2}}}{0.5} = 0.441 \quad \ldots\ldots \quad (53)$$

or

$$r_1^{\frac{1}{2}} = 0.28 \times 10^6 \ldots\ldots \quad (54)$$

Hence

$$r_n^{\frac{1}{2}} = (0.28 + 4.24\,n) \times 10^6. \ldots\ldots \quad (55)$$

and

$$r_n = (0.28 + 4.24\,n)^2 \times 10^{12}\ \text{cm} \ldots\ldots \quad (56)$$

When writing this series out we obtain

0.078   20.5   77.2   169   297   462   660   889 etc. $\times 10^{12}$ cm

or in astronomical units

0.005   1.37   5.15   11.3   19.8   30.8   43.9   60.0 etc.

Numbering the planets accordingly

I   II   III   IV   V   VI   VII   VIII

suitable distances from the sun are obtained for

	actual mean distance from the sun in A.U.
III = Jupiter	5.2
IV = Saturn	9.5
V = Uranus	19.2
VI = Neptune	30.4
VII = Pluto	40

Is it by chance that r_1 nearly corresponds with the sun's radius $r_0 = 7 \times 10^{10}$ cm? If not, we could have found x if we had put in (47) for r_m the solar radius r_0. Is it not very likely that the only actually existing "boundary condition", a density optimum along the equator of the solar globe has fixed the value of x? If this be so, with given a and ϱ_0 the course of evolution could have been predicted quantitatively.

Now, let us calculate the masses of the planets. This can be done with a planimeter, when we have drawn the curve

$$r^{\frac{3}{2}}\, e^{-7.4 \times 10^{-7}\, r^{\frac{1}{2}}} \ldots\ldots \quad (57)$$

by measuring the successive surfaces between the axis of abscissae, the curve and ordinates through the m i n i m a of

$$-\gamma \sin b\, r^{\frac{1}{2}} + \delta \cos b\, r^{\frac{1}{2}} \ldots\ldots \quad (58)$$

If then we make the sum total of the calculated masses of Jupiter, Saturn, Uranus, Neptune and Pluto equal to the sum total of the actual masses, we get

TABLE I.

Planet	Distance in A.U.	Mass		
		in grams $\times 10^{29}$	earth = 1	actual
I	0.005	0.09	1.5	
II	1.37	7.19	122	
Jupiter	5.15	15 99	271	318
Saturn	11.3	7.83	133	95
Uranus	19.8	2.09	35.4	14.6
Neptune	30.8	0.34	5.9	17.3
Pluto	43.9	0.05	0.8	0.8
Total		33.58	569.6	

The agreement between the computed and actual masses is promising, but it is difficult to avoid the conclusion t h a t t h e p l a n e t a r y s y s t e m w a s n o t c r e a t e d i n o n e a c t. Ring I will have been united with the sun and ring II surely never gave birth to that big planet with a mass equal to 122 times the mass of the Earth. In its place appeared Mercury, Venus, Earth, Mars and the planetoids, most probably in a second act of creation. We will come back to this later.

We concluded that a second approximation of $\triangle U$ and $\triangle V$ only could decide which of both is positive or negative. Hence we have to develop (15) into

$$\left.\begin{aligned} \varrho_e + \triangle\varrho_e = \varrho_0\, e^{-a r^{\frac{1}{2}}} (1+\varphi) \left(1 - \psi r^{\frac{1}{2}} + \frac{\psi^2}{2} r\right) \\ \{1 - 2\gamma \sin b r^{\frac{1}{2}} + 2\delta \cos b r^{\frac{1}{2}} + (\gamma^2 + \delta^2) + (\delta^2 - \gamma^2) \cos 2 b r^{\frac{1}{2}} + 2\gamma\delta \sin 2 b r^{\frac{1}{2}}\} \end{aligned}\right\} \quad (59)$$

or

$$\left.\begin{aligned} \frac{\triangle\varrho_e}{\varrho_e} = \varphi + (\gamma^2 + \delta^2) - \psi(1+\varphi) r^{\frac{1}{2}} + (1+\varphi)(-2\gamma \sin b r^{\frac{1}{2}} + 2\delta \cos b r^{\frac{1}{2}}) + \\ + \frac{\psi^2}{2} r - \psi r^{\frac{1}{2}} (-2\gamma \sin b r^{\frac{1}{2}} + 2\delta \cos b r^{\frac{1}{2}}) + \{(\delta^2 - \gamma^2) \cos 2 b r^{\frac{1}{2}} + 2\gamma\delta \sin 2 b r^{\frac{1}{2}}\} \end{aligned}\right\} \quad (60)$$

Changing in our previous equations (40) (41) (42)

φ into $\varphi + (\gamma^2 + \delta^2)$
ψ $\quad \psi(1+\varphi)$
γ $\quad \gamma(1+\varphi)$
δ $\quad \delta(1+\varphi)$

we get the new equations as far as first order small quantities. Evidently, up to these quantities the solution can be obtained in the same way. We shall get again $\triangle U = 0$ and $\triangle V = 0$, but now with values of φ, ψ, γ, δ slightly different from the former solution. The second order quantities decide about the sign of $\triangle U$ and $\triangle V$. The natural course is indicated by

$$\triangle U > 0$$

and now this changes into the condition

$$\left.\begin{array}{r}\frac{4!}{\frac{1}{2}a^5}\left[15\left(\frac{\psi}{a}\right)^2-12\left(\frac{\psi}{a}\right)x^6(x^2+1)^{-6}[-\gamma\{-6x^5+21x^3-6x\}+\right.\\ \left.+\delta\{x^6-15x^4+15x^2-1\}]+\frac{1}{32}(\tfrac{1}{2}x)^5(\tfrac{1}{4}x^2+1)^{-5}\right.\\ \left.[-\gamma\delta\{-20x^4+160x^2-64\}+(\delta^2-\gamma^2)\{x^5-40x^3+80x\}]\right]>0\end{array}\right\} . \quad (62)$$

Substituting $x=0.5$ it reduces to

$$0.002015\,\delta^2>0 \quad \ldots\ldots\ldots \quad (63)$$

or

$$0.00168\,\varkappa^2>0. \quad \ldots\ldots\ldots \quad (64)$$

The left side being positive the condition is satisfied and we have proved that *the gaseous disc will show spontaneously a tendency to concentration into concentric rings.*

If U_0 is the original kinetic energy of the system

$$\triangle U=0.00168\,\varkappa^2\,U_0 \quad \ldots\ldots\ldots \quad (65)$$

We are now in the position to calculate

$$\theta_0 \text{ and } \theta_e, \quad U_0 \text{ and } U_e$$

the moment of momentum and kinetic energy of the planetary system at the beginning and at the end of its evolution from the disc.

If m be the mass of a planet and r its final distance from the sun, we get the following table.

TABLE II.

Planet	m	$r^{\frac{1}{2}}$	r	$m\,r^{\frac{1}{2}}$	$\frac{m}{r}$
I	0.09×10^{29}	0.28×10^{6}	0.0784×10^{12}	0.0×10^{35}	1.15×10^{16}
II	7.19	4.52	20.5	32.5	3.51
Jupiter	15.99	8.76	77.2	140	2.08
Saturn	7.83	13.00	169	102	0.46
Uranus	2.09	17.24	297	36.2	0.07
Neptune	0.34	21.48	462	7.3	0.01
Pluto	0.05	25.72	660	1.3	0.00
Σ	33.58×10^{29}			319.3×10^{35}	7.28×10^{16}

But

$$\theta_e=(fM)^{\frac{1}{2}}\,\Sigma\,m\,r^{\frac{1}{2}} \quad \ldots\ldots\ldots \quad (66)$$

or with $f=6.67\times10^{-8}$ and $M=2.00\times10^{33}$

$$\theta_e=3.69\times10^{50} \quad \text{c. g. s.} \quad \ldots\ldots\ldots \quad (67)$$

whereas

$$U_e=\tfrac{1}{2}fM\,\Sigma\frac{m}{r} \quad \ldots\ldots\ldots \quad (68)$$

or

$$U_e=4.84\times10^{42} \quad \text{c. g. s.} \quad \ldots\ldots\ldots \quad (69)$$

The total mass, total moment of momentum and kinetic energy of the disc are found by

$$m_0=(2\pi)^{\frac{3}{2}}\left(\frac{RT}{fM}\right)^{\frac{1}{2}}\varrho_0\frac{6!}{\frac{1}{2}a^7} \quad \ldots\ldots\ldots \quad (70)$$

$$\theta_0=(2\pi)^{\frac{3}{2}}(RT)^{\frac{1}{2}}\varrho_0\frac{7!}{\frac{1}{2}a^8} \quad \ldots\ldots\ldots \quad (71)$$

$$U_0=\tfrac{1}{2}(2\pi)^{\frac{3}{2}}(fMRT)^{\frac{1}{2}}\varrho_0\frac{4!}{\frac{1}{2}a^5} \quad \ldots\ldots\ldots \quad (72)$$

with quite sufficient approximation. Substituting provisionally

$$(RT)^{\frac{1}{2}}\varrho_0=y \quad \ldots\ldots\ldots \quad (73)$$

we obtain

$$(2\pi)^{\frac{3}{2}}\frac{y}{(fM)^{\frac{1}{2}}}\frac{6!}{\frac{1}{2}a^7}=335.9\times10^{28} \quad \ldots\ldots\ldots \quad (74)$$

or

$$y=2.08\times10^{-4} \quad \ldots\ldots\ldots \quad (75)$$

We then get

$$\theta_0=3.68\times10^{50} \quad \text{c. g. s.} \quad \ldots\ldots\ldots \quad (76)$$

and

$$U_0=4.10\times10^{42} \quad \text{c. g. s.} \quad \ldots\ldots\ldots \quad (77)$$

Comparing (67) and (76) we can confirm that the moment of momentum of the system has not changed from the origin, whereas (69) shows that the kinetic energy has increased. The total increase is represented by

$$\triangle U=0.18\,U_0 \quad \ldots\ldots\ldots \quad (78)$$

It is of interest to compare

$$\triangle U=0.00168\,\varkappa^2\,U_0 \quad \ldots\ldots\ldots \quad (65)$$

Although the approximations used in our calculations allowed it to be carried through only in cases when $\varkappa \ll 1$, we are now permitted to conclude that $\varkappa$ might at most have reached a value comparable with 10.

The value of $\varkappa$ actually reached depended on the limit of stability of the rings. In previous papers I have shown that a critical limit in the motion of our rotating nebula is reached, when

$$\frac{fM}{RT}+\frac{d}{dr}\left(r^3\frac{d\,lg\,\varrho_e}{dr}\right)=0\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (79)$$

This limit is fixed by the possibility of adiabatic radial displacements of masselements within the nebula. It is the limit where laminar motion is replaced by turbulent motion. The left side of (79) can never be negative.

Introducing in (79)

$$\varrho_e=\varrho_0\, e^{-ar^{\frac{1}{2}}-2\gamma\sin br^{\frac{1}{2}}+2\delta\cos br^{\frac{1}{2}}}$$

we get the following condition of stability

$$\frac{fM}{RT}-\frac{5}{2}br^{\frac{3}{2}}\left[\frac{a}{2b}+\delta\sin br^{\frac{1}{2}}+\gamma\cos br^{\frac{1}{2}}\right]-\frac{b^2}{2}r^2[-\gamma\sin br^{\frac{1}{2}}+\delta\cos br^{\frac{1}{2}}]>0 \quad (80)$$

Trying it out quantitatively, let us substitute

$$a=7.4\times10^{-7} \text{ and } b=14.8\times10^{-7}.$$

As we have to try out values of $\varkappa$ many times greater than 1, we can as well neglect the term $\frac{a}{2b}=\frac{1}{4}$ between the first brackets. After having done this, both terms between brackets reach the absolute value of $\varkappa$ at their maximum. Hence the second and third terms on the left side are comparable for

$$r^{\frac{1}{2}}=\frac{5}{b} \text{ or } r=11.4\times10^{12}\,\text{cm.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (81)$$

Beyond this distance, which is smaller than one A.U. the third term is the more dangerous one. We may therefore conclude that, roughly, a ring becomes instable, when $\varkappa$ has increased as far as to satisfy the equation

$$\frac{fM}{RT}-\frac{\varkappa}{2}b^2r^2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (82)$$

or when

$$\varkappa=\frac{12.1\times10^{37}}{RT\,r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (83)$$

When this critical value of $\varkappa$ is reached it allows locally small parts of the mass to unite freely by their own attraction. Concentration to small solid particles will start, the formation of meteoric bodies is stimulated. However, the final concentration of these bodies or of a ring, which has remained gaseous, into one or more big bodies, depends on the mean density which the ring has attained. This second critical limit which has to be crossed is ROCHE's limit.

It is not the author's intention to deal in this paper with the formation

of satellites more fully. But here we have to mention it in passing. The rings of Saturn, rotating within ROCHE's limit prove that condensation to meteorites was permitted, whereas it is known that the mean density of some of the inner satellites of Saturn is so low that they must be clouds of solid particles like comets heads.

It may be that every planet passed through the planetesimal state, but certainly those planets did, which in their evolution crossed the first mentioned limit of $\varkappa$ before ROCHE's limit was crossed.

This second limit follows from the condition that the density of the rings is sufficiently high to ascertain their condensation into individual bigger globes. Applying ROCHE's theory this condition is satisfied, when, roughly, the mean density of a ring has reached 14 times the mean density of a central body with the ring at its circumference. To start the concentration, at least the maximum density in any ring has to reach this value. Hence, a ring becomes instable when $\varkappa$ has increased as far as to satisfy the equation

$$\varrho_0\, e^{-ar^{\frac{1}{2}}+2\varkappa}=14\times\frac{M}{\frac{4}{3}\pi r^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (84)$$

or

$$0.434\,(-ar^{\frac{1}{2}}+2\varkappa)={}^{10}log\left(\frac{14}{\varrho_0}\times\frac{M}{\frac{4}{3}\pi r^3}\right) \quad . \quad . \quad . \quad . \qquad (85)$$

With

$$\varrho_0=2.08\times10^{-4}\,(RT)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (86)$$

we obtain

$$\varkappa=1.15\,(3.21\times10^{-7}\,r^{\frac{1}{2}}+37.0+\tfrac{1}{2}\,log\,RT-3\,log\,r) \quad . \quad . \qquad (87)$$

To fix the ideas, let us assume that in our tenuous slightly absorbing original nebula, we have at earthdistance from the sun

$R=2\times10^6$ (a gas with $\mu=40$), $T=100°$ (blackbody-temperature$=280°$)

hence, with our previous assumption (4), anywhere

$$RT=2\times10^8 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (88)$$

We then get for the density of the nebula at the centre, that is in contact with the sun

$$\varrho_0=1.48\times10^{-8}\ \text{gr cm}^{-3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (89)$$

and for the two critical limits of $\varkappa$

$$\varkappa_1=6.05\times10^{29}\,r^{-2},\quad \varkappa_2=1.15\,(3.21\times10^{-7}\,r^{\frac{1}{2}}+41.15-3\,log\,r) \quad (90)$$

Writing these values out, the following table is obtained

TABEL III.

	K_1	K_2
II	1440	3.08
Jupiter	101	2.63
Saturn	21.2	3.03
Uranus	6.84	3.76
Neptune	2.83	4.66
Pluto	1.40	5.68

I dare say that this table contains the clue to many problems. However, how the breaking up of the rings and the formation of the embryos of the planets takes place, we can only guess. Nor are we sure of the absolute magnitude of the figures in the table, in consequence of the uncertainty in (88). Taking the values in the table as they are, and remembering that the lower of the two limits of $\varkappa$ is decisive, we have to assume that the generation of the planets proceeded in the following surprising order of succession

Pluto	$\varkappa_1 = 1.40$
Jupiter	$\varkappa_2 = 2.63$
Neptune	$\varkappa_1 = 2.83$
Saturn	$\varkappa_2 = 3.03$
II	$\varkappa_2 = 3.08$
Uranus	$\varkappa_2 = 3.76$

It is of importance to remark that $\varkappa$ remains inferior to 10, a value which, as we have seen, could hardly have been surpassed. Pluto and Neptune reached $\varkappa_1$ before $\varkappa_2$. They were in danger to remain in the form of planetoids. Neptune has evidently escaped this danger, Pluto perhaps not. Its excentric behaviour suggests that there may be other "plutoids". Moreover the next planet beyond Pluto, revolving at a mean distance of 60 A.U. would still possess, according to our theory, appreciable mass, say 0.1 of the mass of the Earth. Hence we have to leave open the possible existence of a peripheral ring of meteorites.

It completes our picture, when we imagine that with $\varkappa = 3$ at the moment of the final breaking up of the rings, the densities reached at the crests are e^{12} or 10^5 times superior to the densities in the near valleys.

Looking away from Pluto, Neptune and Jupiter started condensation. This might perhaps be the reason, why Jupiter and Neptune are heavier and the other planets lighter than they ought to be, according to table I. When trying to grasp what will happen, it appears probable that when a ring starts condensation into one or more separate bodies the other rings will start supplying matter to the ring in progress of disintegration in order to restore the violated state of equilibrium in the nebula. So Jupiter grew partly at the cost of II and Saturn, Neptune at the cost of Uranus. And it becomes more and more probable that at last all these planets came into

existence with the exception of II which lost almost all its matter before succeeding in its own concentration. However, the big planets once created, there came a moment when the inconsiderable rest of the gas inside the orbit of Jupiter, not more than 2 times the earth's mass, was once more reconstructed in discform and rearranged in a series of concentric rings, giving birth to a new planetary system much smaller than the first, but of a comparable character

	mass (earth = 1)
Mercury	0.06
Venus	0.82
Earth	1
Mars	0.11
Planetoids	very small

The author has convinced himself that new values a, b and ϱ_0, suiting reasonably well the masses and distances of these planets can be found. However, he assumes that the solid base now obtained allows him to be short and to leave this point as well as the formation of the satellite systems to future investigation.

It is of interest to get a better idea of the structure of the rings before they condensed to planets. Developing the density ϱ in the neighbourhood of a maximum at radius r_m, we get

$$\varrho = \varrho_0 (1 + \varrho) \, Exp \left[-(\alpha + \psi) r^{\frac{1}{2}} - \frac{fM}{2RT} \frac{h^2}{r^3} + 2\varkappa \left\{ 1 - \tfrac{1}{8} b^2 \frac{(r - r_m)^2}{r} \right\} \right] \quad (91)$$

Putting

$$r - r_m = k \quad \ldots \ldots \ldots \quad (92)$$

the equation of the curves of equal density in a meridional section through a ring is

$$\frac{fM}{2RT} \frac{h^2}{r^3} + \tfrac{1}{4} \varkappa b^2 \frac{k^2}{r} = \text{constant} \quad \ldots \ldots \quad (93)$$

For the smaller values of h and k these curves are ellipses. If the major and minor axes are 2α and 2β we have

$$\frac{\beta}{\alpha} = \left(\frac{\varkappa RT}{2fM} \right)^{\frac{1}{2}} br \quad \ldots \ldots \quad (94)$$

This ratio increases with r. Inserting known values we get

$$\frac{\beta}{\alpha} = \varkappa^{\frac{1}{2}} \times 1.28 \times 10^{-15} r \quad \ldots \ldots \quad (95)$$

To fix the ideas, let us put $\varkappa = 3$ at the moment, when the rings loose their stability. Then

$$\frac{\beta}{\alpha} = 2.22 \times 10^{-15} r \quad \ldots \ldots \quad (96)$$

This ratio is for

Jupiter	0.172
Saturn	0.374
Uranus	0.655
Neptune	1.02
Plato	1.46

The rings near the sun were flat. The smaller the ring the flatter it is. The rings of the smaller planets within the orbit of Jupiter, although the values $\varkappa$ and b have changed, all have been flat. Only those of Uranus, Neptune and Pluto attained a toruslike structure.

This, of course, will aid us to explain some features of these planets with their satellite systems. And here again we meet the 2 flat rings of Saturn, close round their primary, as living proofs of a structure, now clearly understood in its genesis by theory.

It is a stimulating aspect of the account which we have tried to give that the solar system in successive stages of evolution, could be identified with a CARTESIAN whirl, with KANT's disc and the rings of LAPLACE. Even CHAMBERLIN and MOULTON's planetesimals were met at a certain stage, whereas for JEANS' foreign star there might have been work to do at the moment of conception. It might have started the evolution, leaving the rest to be done spontaneously. At any rate, this beautifully ordered structure, which is our planetary system, is essentially self-made.

January 1940.

Anthropology. — *On the Increase of Stature in the Netherlands and the Possibility of its Explanation by Genetic Changes.* By W. A. MIJSBERG (Batavia).

(Communicated at the meeting of March 30, 1940.)

Since the middle of the preceding century the stature of the population of the Netherlands has considerably increased. This fact has been established by studying the measurements taken by the Boards which yearly carry out medical examinations on the male subjects that have reached the age of 19 years. As these examinations have to decide as to the fulfilment of military service they are compulsory; consequently the stature of each male of the said age is taken.

The extensive material on stature brought together in this way has been studied by BRUINSMA [1]), BOLK [2]) and VAN DEN BROEK [3]). As a result of their investigations it can be stated that in the period 1863—1925 the mean stature has continually increased, the total increase during that period being no less than 65 mm (VAN DEN BROEK).

A long time ago already it was established that at the age of 19 about 99 % of the adult stature is attained. Therefore the increase cannot be explained by assuming that nowadays the total stature is reached at an earlier time of life than was the case formerly. One is forced to admit that at present the adults are also taller. According to VAN DEN BROEK in 1921—1925 the mean stature of the conscripts was 170.77 cm. From the figures published by BENDERS (1938) [4]) it appears that since the latter period the increase has not been arrested. There is no reason to suppose that the increase should have been limited to the male part of the population.

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