Mathematics. — *On* LAMBERT's *proof for the irrationality of* π. By J. POPKEN. (Communicated by Prof. J. G. VAN DER CORPUT).

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In their beautiful book "An introduction to the theory of numbers" HARDY and WRIGHT state, there is no simple proof for the irrationality of π [1]). Yet, if LAMBERT's classical proof [2]) is freed from the continued fraction algorithm, it takes a surprisingly easy form [3]). We only require the following lemma:

Lemma: Let $x \neq 0$. For $h = 0, 1, 2, \ldots$ there are polynominals $p_h(x^{-1})$ and $q_h(x^{-1})$ in x^{-1} with integral coefficients and of degree at most $2h$, so that

$$\left.\begin{aligned} & p_h(x^{-1}) \sin x + q_h(x^{-1}) \cos x = \\ & = (-2)^h \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)\ldots(n+h)}{(2n+2h+1)!} x^{2n+1} \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

Proof: a. $h = 0$. Take $p_0(x^{-1}) \equiv 1$, $q_0(x^{-1}) \equiv 0$, then (1) is identical with

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots .$$

b. Supposing (1) to be true for h, then we prove, that (1) also holds for $h+1$ instead of h.

In fact, if (1) is true, then

$$x^{-1} p_h(x^{-1}) \sin x + x^{-1} q_h(x^{-1}) \cos x =$$

$$= (-2)^h \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)\ldots(n+h)}{(2n+2h+1)!} x^{2n}$$

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1) p. 39 and p. 47.

2) J. H. LAMBERT, Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmétiques. Histoire Acad. roy. des sciences et belles lettres. Berlin. Année 1761 (1768), p. 265—322.

3) Nearly the same idea has been applied by HERMITE: Sur quelques approximations algébriques, Oeuvres III, p. 146—149, but HERMITE used an integral.

Differentiation gives

$$\left.\begin{aligned}&\left[\frac{d}{dx}\{x^{-1}p_h(x^{-1})\}-x^{-1}q_h(x^{-1})\right]\sin x+\\&\qquad+\left[x^{-1}p_h(x^{-1})+\frac{d}{dx}\{x^{-1}q_h(x^{-1})\}\right]\cos x\\&=(-2)^h\sum_{n=1}^{\infty}(-1)^n\frac{2n(n+1)(n+2)\ldots(n+h)}{(2n+2h+1)!}x^{2n-1}=\\&=(-2)^{h+1}\sum_{\nu=0}^{\infty}(-1)^\nu\frac{(\nu+1)(\nu+2)\ldots(\nu+h+1)}{(2\nu+2h+3)!}x^{2\nu+1}\end{aligned}\right\}\quad\ldots\quad(2)$$

Putting

$$\left.\begin{aligned}&\frac{d}{dx}\{x^{-1}p_h(x^{-1})\}-x^{-1}q_h(x^{-1})=p_{h+1}(x^{-1}),\\&x^{-1}p_h(x^{-1})+\frac{d}{dx}\{x^{-1}q_h(x^{-1})\}=q_{h+1}(x^{-1}),\end{aligned}\right\}\quad\ldots\ldots\quad(3)$$

it is clear, that $p_{h+1}(x^{-1})$ and $q_{h+1}(x^{-1})$ are polynomials in x^{-1} with integral coefficients and of degree at most $2h+2$, since $p_h(x^{-1})$ and $q_h(x^{-1})$ are polynomials, whose coefficients are integers and are of degree $2h$ at most. By (2) and (3)

$$p_{h+1}(x^{-1})\sin x+q_{h+1}(x^{-1})\cos x=$$

$$=(-2)^{h+1}\sum_{n=0}^{\infty}(-1)^n\frac{(n+1)(n+2)\ldots(n+h+1)}{(2n+2h+3)!}x^{2n+1}.$$

Hence part *b.* of our proof is evident.

Proof of LAMBERT's *theorem:* If π is rational, then $\frac{\pi}{4}=\frac{a}{b}$, where a and b are positive integers. Applying the preceding lemma with $x=\frac{\pi}{4}=\frac{a}{b}$, we derive for any integer $h\geqq 0$

$$\left\{p_h\left(\frac{b}{a}\right)+q_h\left(\frac{b}{a}\right)\right\}\tfrac{1}{2}\sqrt{2}=$$

$$=(-2)^h\left(\frac{h!}{(2h+1)!}\frac{\pi}{4}-\frac{2.3\ldots(h+1)}{(2h+3)!}\left(\frac{\pi}{4}\right)^3+\ldots\right).$$

In the alternating series between brackets the terms decrease in absolute value and tend to zero. Hence

$$2^h\left(\frac{h!}{(2h+1)!}\frac{\pi}{4}-\frac{2.3\ldots(h+1)}{(2h+3)!}\left(\frac{\pi}{4}\right)^3\right)<$$

$$<\tfrac{1}{2}\sqrt{2}\left|p_h\left(\frac{b}{a}\right)+q_h\left(\frac{b}{a}\right)\right|<2^h\frac{h!}{(2h+1)!}\frac{\pi}{4}<\frac{2^h}{h!},$$

$$0<\left|a^{2h}p_h\left(\frac{b}{a}\right)+a^{2h}q_h\left(\frac{b}{a}\right)\right|<\sqrt{2}\,\frac{2^h\,|a|^{2h}}{h!}\ldots \quad (4)$$

Since $p_h(x^{-1})$ is a polynomial in x^{-1} with integers as coefficients and of degree say g the number $p_h\left(\frac{b}{a}\right)$ can be written as a fraction with denominator a^g. But $g \leqq 2h$, therefore $a^{2h}p_h\left(\frac{b}{a}\right)$ is an integer and the same reasoning shows, that $a^{2h}q_h\left(\frac{b}{a}\right)$ also is an integer.

Hence for integral $h \geqq 0$

$$\left|a^{2h}p_h\left(\frac{b}{a}\right)+a^{2h}q_h\left(\frac{b}{a}\right)\right|$$

is a positive integer. But now (4) leads to a contradiction, since

$$\lim_{h\to\infty}\sqrt{2}\,\frac{2^h\,|a|^{2h}}{h!}=0.$$