Physics. — *On the buckling of a thin-walled circular tube loaded by pure bending*. I. By C. B. BIEZENO and J. J. KOCH.

(Communicated at the meeting of June 29, 1940.)

1. *Introduction*. If a thin-walled circular tube in his end-sections is loaded by two equal and opposite bending moments M, it may be stated that its cross section alters its circular shape into an oval one, owing to the fact, that, apart from the normal bending stresses in the cross section of the tube, there arise tangential bending-stresses in its meridional sections. A closer examination of the fact learns, that the curvature $\frac{1}{\varrho}$ of the "axis" of the tube does not increase proportionally with the loading moment M.

If "a" be the radius of the tube, h its thickness, E the elasticity-modulus of the material and ν the reciprocal value of POISSON's coefficient, the following relation between M and ϱ exists:

$$M = \pi a^3 h \left[\frac{1}{\varrho} - \frac{3}{2}(1-\nu^2)\frac{a^4}{\varrho^3 h^2}\right] \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the graphical representation of which is shown qualitatively in fig. 1.

It is seen from this figure that there exists a critical value M_{crit} of M, characterized by the fact, that no increase of M occurs, if $1/\varrho$ increases. Hence a break-down of the tube is to be expected.

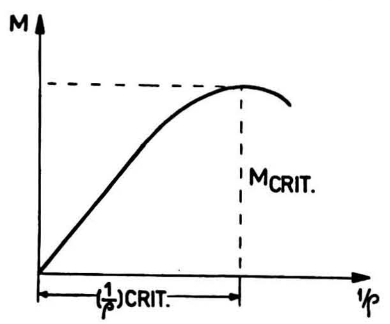

Fig. 1.

The phenomenon here described has been studied at great length by BRAZIER [1]).

In the present paper quite another phenomenon is studied, which occurs for certain critical values of M, and is characterized by the simultaneous appearance of longitudinal and circumferential waves in the cylindrical shape of the tube. We assume, that, — if unloaded —, the tube possesses such initial curvature, that under the action of the buckling moment it is straight, and — in cross-section — circular and of constant thickness h. Hereby our buckling problem relates to

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[1]) Comp. BRAZIER, Aeronautical Research Committee, Reports and Memoranda No. 1081, M 49.

a *straight* circular tube, loaded at his ends by linear changing bending stresses.

Preliminary we shall have to solve some detail-questions (sections 2—6). In section 2 the formulae for the displacements and stresses of a cylindrical tube, submitted to prescribed radial, tangential and axial stresses, R, Φ and Z are reproduced. In section 3 a system of particular loads B is defined and calculated, which plays a fundamental role in our proper buckling problem. The loads B will be called *"elementary normal loads"*, the corresponding deformations, "elementary normal deformations". Section 4 deals with the differential equations of the buckling problem; more particularly it is shown that the differential equations for the displacements obtained in section 2 may be looked upon as the required buckling-equations, provided that R, Φ and Z be replaced by adequate "would-be" forces. The so-obtained differential equations are homogeneous and linear in the displacements and therefore only admit solutions (different from zero) for special values of the loading moment $M = \mu \overline{M}$ ($\overline{M}$ = unit of bending moment). The values μ, for which buckling of the tube is possible, will be called the "total characteristic values" of our problem, the corresponding deformations T of the tube "total normal deformations". Section 5 bears on the development of the "total normal deformations" into a series of the "elementary normal deformations". Finally it is shown in section 6, how by iteration the smallest characteristic value μ can approximately be calculated.

2. *The cylindrical circular tube of constant thickness under a prescribed load-system.* As shown in fig. 2 the position of an arbitrary point of

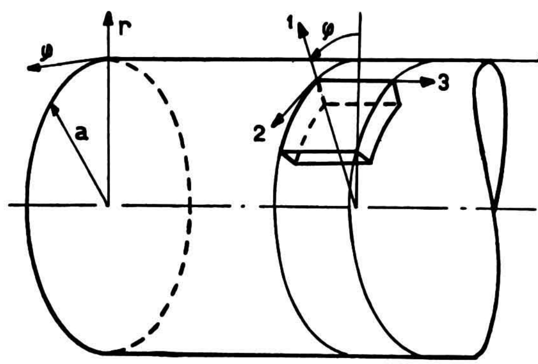

Fig. 2.

the middle-surface of the tube is given by its "cylinder"-coordinates a, φ, and z. The radial, tangential and axial displacements of such a point are called u, v, w; the components of the external load of the tube with reference to the unit of surface, and taken in radial, tangential and axial direction are designed by R, Φ, and Z. Fig. 3 shows the nomenclature of the so-called "internal" forces and moments, all of which refer to the unit of length of their corresponding section.

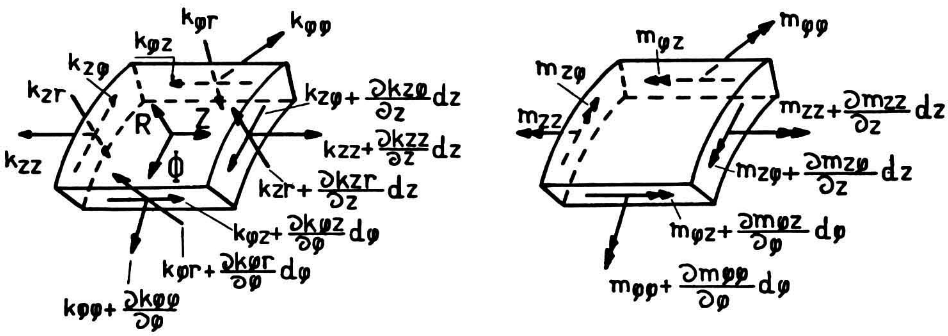

Fig. 3.

If, for abbreviation, we put

$$A^* = \frac{Eh^3}{12\,(1-\nu^2)}, \quad B = \frac{Eh}{1-\nu^2} \quad \left(\nu = \frac{1}{m}\right) \quad . \quad . \quad . \quad . \quad (1)$$

and for later purposes

$$\frac{A^*}{Ba^2} = \frac{h^2}{12a} = k, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1^*)$$

the equations for u, v, w may be written as follows:

$$\left.\begin{aligned}
&\frac{1}{a}\left(\frac{u}{a} + \frac{1}{a}\frac{\partial v}{\partial \varphi} + \nu\frac{\partial w}{\partial z}\right) + \frac{A^*}{B}\left(\frac{1}{a^4}\frac{\partial^4 u}{\partial \varphi^4} + \frac{2}{a^2}\frac{\partial^4 u}{\partial \varphi^2\,\partial z^2} + \frac{\partial^4 u}{\partial z^4} + \right.\\
&\left. + \frac{2}{a^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{u}{a^4} - \frac{3-\nu}{2a^2}\frac{\partial^3 v}{\partial \varphi\,\partial z^2} + \frac{1-\nu}{2a^3}\frac{\partial^3 w}{\partial \varphi^2\,\partial z} - \frac{1}{a}\frac{\partial^3 w}{\partial z^3}\right) - \frac{R}{B} = 0\\
&\frac{1}{a^2}\frac{\partial u}{\partial \varphi} + \frac{1}{a^2}\frac{\partial^2 v}{\partial \varphi^2} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial z^2} + \frac{1+\nu}{2a}\frac{\partial^2 w}{\partial \varphi\,\partial z} + \frac{A^*}{B}\left(-\frac{3-\nu}{2a^2}\frac{\partial^3 u}{\partial \varphi\,\partial z^2} + \right.\\
&\left. + 3\,\frac{1-\nu}{2a^2}\frac{\partial^2 v}{\partial z^2}\right) + \frac{\Phi}{B} = 0\\
&\frac{\nu}{a}\frac{\partial u}{\partial z} + \frac{\nu+1}{2a}\frac{\partial^2 v}{\partial \varphi\,\partial z} + \frac{1-\nu}{2a^2}\frac{\partial^2 w}{\partial \varphi^2} + \frac{\partial^2 w}{\partial z^2} + \frac{A^*}{B}\left(\frac{1-\nu}{2a^3}\frac{\partial^3 u}{\partial \varphi^2\,\partial z} - \right.\\
&\left. - \frac{1}{a}\frac{\partial^3 u}{\partial z^3} + \frac{1-\nu}{2a^4}\frac{\partial^2 w}{\partial \varphi^2}\right) + \frac{Z}{B} = 0.
\end{aligned}\right\} \quad . \quad (2)$$

The internal forces and moments can be calculated from u, v, w, by the relations (3):

$$
\left.\begin{aligned}
k_{\phi\phi} &= \frac{A^*}{a^3}\left(\frac{\partial^2 u}{\partial \varphi^2} + u\right) + B\left(\frac{u}{a} + \frac{1}{a}\frac{\partial v}{\partial \varphi} + \nu\frac{\partial w}{\partial z}\right) \\
k_{\phi z} &= \frac{(1-\nu)\,A^*}{2a^2}\left(\frac{\partial^2 u}{\partial \varphi\,\partial z} + \frac{1}{a}\frac{\partial w}{\partial \varphi}\right) + \frac{(1-\nu)\,B}{2}\left(\frac{\partial v}{\partial z} + \frac{1}{a}\frac{\partial w}{\partial \varphi}\right) \\
m_{\phi\phi} &= \frac{(1-\nu)\,A^*}{2a}\left(2\frac{\partial^2 u}{\partial \varphi\,\partial z} - \frac{\partial v}{\partial z} + \frac{1}{a}\frac{\partial w}{\partial \varphi}\right) \\
k_{z\phi} &= \frac{(1-\nu)\,A^*}{2a^2}\left(-\frac{\partial^2 u}{\partial \varphi\,\partial z} + \frac{\partial v}{\partial z}\right) + \frac{(1-\nu)\,B}{2}\left(\frac{\partial v}{\partial z} + \frac{1}{a}\frac{\partial w}{\partial \varphi}\right) \\
k_{zz} &= -\frac{A^*}{a}\frac{\partial^2 u}{\partial z^2} + B\left(\nu\frac{u}{a} + \frac{\nu}{a}\frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial z}\right) \\
m_{z\phi} &= A^*\left(\frac{\nu}{a^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\nu}{a^2}\frac{\partial v}{\partial \varphi} - \frac{1}{a}\frac{\partial w}{\partial z}\right) \\
m_{zz} &= \frac{(1-\nu)\,A^*}{a}\left(-\frac{\partial^2 u}{\partial \varphi\,\partial z} + \frac{\partial v}{\partial z}\right)
\end{aligned}\right\} \quad . \quad (3)
$$

We see at once, that for the special loads

$$
\left.\begin{aligned}
R &= a_{pq}\cos p\varphi \sin \lambda\frac{z}{a} \\
\Phi &= b_{pq}\sin p\varphi \sin \lambda\frac{z}{a} \\
Z &= 0
\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (4)
$$

— where p and q design arbitrary positive, integer numbers and λ stands for

$$
\lambda = \frac{\pi q a}{l} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)
$$

(l being the length of the cylinder) — the equations (2) admit solutions of the following type

$$
\left.\begin{aligned}
u &= u_{pq}\cos p\varphi \sin \lambda\frac{z}{a} \\
v &= v_{pq}\sin p\varphi \sin \lambda\frac{z}{a} \\
w &= w_{pq}\cos p\varphi \cos \lambda\frac{z}{a}
\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (6)
$$

u_{pq}, v_{pq} and w_{pq} representing functions of p and q, which are to be calculated from the equations (7)

$$\left.\begin{aligned}
&[1+k(p^4+2p^2\lambda^2+\lambda^4-2p^2+1)]\,u_{pq}+\left[p+\frac{3-\nu}{2}kp\lambda^2\right]v_{pq}+\\
&\quad+\left[-\nu\lambda+k\left(\frac{1-\nu}{2}p^2\lambda-\lambda^3\right)\right]w_{pq}=\frac{a^2}{B}a_{pq}\\
&\left[p+\frac{3-\nu}{2}kp\lambda^2\right]u_{pq}+\left[p^2+\frac{1-\nu}{2}(1+3k)\lambda^2\right]v_{pq}-\\
&\quad-\frac{1+\nu}{2}p\lambda w_{pq}=\frac{a^2}{B}b_{pq}\\
&\left[-\nu\lambda+k\left(\frac{1-\nu}{2}p^2\lambda-\lambda^3\right)\right]u_{pq}-\frac{1+\nu}{2}p\lambda v_{pq}+\\
&\quad+\left[\lambda^2+\frac{1-\nu}{2}(1+k)p^2\right]w_{pq}=0
\end{aligned}\right\} . \quad (7)$$

(It may be noted, that for all solutions (7), the displacements u and v are zero for $z=0$ and $z=l$, whereas w is different from zero at the ends of the cylinder). We restrict ourselves to solving u_{pq} and v_{pq} and find, under the essential condition, that the thickness h of the tube be small enough to neglect all terms which contain the parameter k (comp. 1*) in higher than the first degree

$$\left.\begin{aligned}
u_{pq}&=\alpha_{pq}\,a_{pq}+\beta_{pq}\,b_{pq}\\
v_{pq}&=\beta_{pq}\,a_{pq}+\gamma_{pq}\,b_{pq}
\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

α_{pq}, β_{pq} and γ_{pq} standing for:

$$\alpha_{pq}=\left(\frac{T_1}{N}\right)_{pq}\frac{a^2}{B};\; \beta_{pq}\equiv-\left(\frac{T_2}{N}\right)_{pq}\frac{a^2}{B};\; \gamma_{pq}=\left(\frac{T_3}{N}\right)_{pq}\frac{a^2}{B} \quad . \; . \quad (9)$$

T_1, T_2, T_3, N themselves standing for:

$$\left.\begin{aligned}
T_1&=(1+k)p^4+[2\lambda^2+2(1-\nu)\lambda^2k]\,p^2+(1+3k)\lambda^4\\
T_2&=p\,[[1+k(2\lambda^2+1)]\,p^2+(\nu+2)\lambda^2+2k\lambda^4]\\
T_3&=kp^2(p^2-1)^2+k\frac{2(2-\nu)}{1-\nu}\lambda^2p^4+\left[1+\frac{5-\nu}{1-\nu}k\lambda^4+\right.\\
&\quad\left.+\frac{2(\nu-\nu^2-2)}{1-\nu}k\lambda^2+k\right]p^2+2(1+\nu)\lambda^2+\frac{2}{1-\nu}k(\lambda^6-2\nu\lambda^4)\\
N&=kp^8+k[4\lambda^2-2]\,p^6+k[6\lambda^4-2(4-\nu)\lambda^2+1]\,p^4+k[4\lambda^6-\\
&\quad-6\lambda^4+2(2-\nu)\lambda^2]\,p^2+(1-\nu^2)\lambda^4+k\lambda^8-2\nu\lambda^6k+(4-3\nu^2)\lambda^4k.
\end{aligned}\right\} . \quad (10)$$

3. *The "elementary normal loads" B, and the corresponding "elementary normal deformations" D.* As "elementary normal loads" B and

corresponding "elementary normal deformations" D we define such loads R_{pq}, Φ_{pq} (comp. 2, 4) and such corresponding displacements u_{pq}, v_{pq}, for which

$$\frac{R_{pq}}{u_{pq}}=\frac{\Phi_{pq}}{v_{pq}}=\omega\,. \quad \ldots \ldots \ldots \quad (1)$$

Condition (1) is identical with the requirement, that the equations

$$\left.\begin{aligned} a_{pq} &= \omega\,(\alpha_{pq}\,a_{pq}+\beta_{pq}\,b_{pq}) \\ b_{pq} &= \omega\,(\beta_{pq}\,a_{pq}+\gamma_{pq}\,b_{pq}) \end{aligned}\right\} \text{ or } \left.\begin{aligned} (\omega\,\alpha_{pq}-1)\,a_{pq}+\omega\,\beta_{pq}\,b_{pq} &= 0 \\ \omega\,\beta_{pq}\,a_{pq}+(\omega\,\gamma_{pq}-1)\,b_{pq} &= 0 \end{aligned}\right\}. \quad (2)$$

admit solutions a_{pq}, b_{pq} different from zero. (Comp. 2, 8 and 2, 4).

Therefore the equation

$$\begin{vmatrix} \omega\,\alpha_{pq}-1 & \omega\,\beta_{pq} \\ \omega\,\beta_{pq} & \omega\,\gamma_{pq}-1 \end{vmatrix}=0 \quad \ldots \ldots \ldots \quad (3)$$

has to be satisfied by ω. The two (real) roots of this equation are:

$$\begin{matrix} 1/\omega^{*}_{pq} \\ \text{resp.} \\ 1/\omega^{**}_{pq} \end{matrix} = \frac{\alpha_{pq}+\gamma_{pq}}{2} \pm \sqrt{\left(\frac{\alpha_{pq}-\gamma_{pq}}{2}\right)^2+\beta_{pq}{}^2} \quad \ldots \ldots \quad (4)$$

The corresponding loads and displacements — which of course are definite except for a factor of proportionality $\varkappa$ — may readily be calculated from the equations (3).

With:

$$\left.\begin{aligned} E^{*}_{pq} &\equiv -\frac{\alpha_{pq}-\gamma_{pq}}{2\beta_{pq}}+\sqrt{\left(\frac{\alpha_{pq}-\gamma_{pq}}{2\beta_{pq}}\right)^2+1}\,;\; E^{**}_{pq}\equiv -\frac{\alpha_{pq}-\gamma_{pq}}{2\beta_{pq}}-\sqrt{\left(\frac{\alpha_{pq}-\gamma_{pq}}{2\beta_{pq}}\right)^2+1} \\ F^{*}_{pq} &\equiv \frac{\alpha_{pq}+\gamma_{pq}}{2}+\sqrt{\left(\frac{\alpha_{pq}-\gamma_{pq}}{2}\right)^2+\beta^2_{pq}}\,;\; F^{**}_{pq}\equiv \frac{\alpha_{pq}+\gamma_{pq}}{2}-\sqrt{\left(\frac{\alpha_{pq}-\gamma_{pq}}{2}\right)^2+\beta^2_{pq}} \\ G^{*}_{pq} &= E^{*}_{pq}\,F^{*}_{pq}\,; \qquad\qquad G^{**}_{pq}=E^{**}_{pq}\,F^{**}_{pq} \end{aligned}\right\} (5)$$

they can be written as follows:

$$\left.\begin{aligned} R^{*} &= \varkappa^{*}\cos p\varphi\,\sin\lambda\frac{z}{a} & R^{**} &= \varkappa^{**}\cos p\varphi\,\sin\lambda\frac{z}{a} \\ \Phi^{*} &= \varkappa^{*}\,E^{*}_{pq}\sin p\varphi\,\sin\lambda\frac{z}{a} & \Phi^{**} &= \varkappa^{**}\,E^{**}_{pq}\sin p\varphi\,\sin\lambda\frac{z}{a} \\ u^{*} &= \varkappa^{*}\,F^{*}_{pq}\cos p\varphi\,\sin\lambda\frac{z}{a} & u^{**} &= \varkappa^{**}\,F^{**}_{pq}\cos p\varphi\,\sin\lambda\frac{z}{a} \\ v^{*} &= \varkappa^{*}\,G^{*}_{pq}\sin p\varphi\,\sin\lambda\frac{z}{a} & v^{**} &= \varkappa^{**}\,G^{**}_{pq}\sin p\varphi\,\sin\lambda\frac{z}{a} \end{aligned}\right\}. \quad (6)$$

the first solution obviously belonging to ω^*, the second one to ω^{**}. To get rid of the undeterminate factors $\varkappa^*$ and $\varkappa^{**}$ we standardize our solutions by the condition of standardization

$$\int_0^l \int_0^{2\pi} (u^* R^* + v^* \Phi^*)\, a\, d\varphi\, dz = \frac{\pi al}{2} \text{ resp. } \int_0^l \int_0^{2\pi} (u^{**} R^{**} + v^{**} \Phi^{**})\, a\, d\varphi\, dz = \frac{\pi al}{2} \quad (7)$$

and find hereby:

$$\varkappa^* = \frac{1}{\sqrt{F^*_{pq} + E^*_{pq}\, G^*_{pq}}} \text{ resp. } \varkappa^{**} = \frac{1}{\sqrt{F^{**}_{pq} + E^{**}_{pq}\, G^{**}_{pq}}} \quad . \quad . \quad . \quad (8)$$

Henceforth the expression "elementary normal function" will be used in the same sense as "standardized elementary normal function", so that from now the elementary normal loads and deformations will be represented by the eqs. (6) and (8).

In the following sections we shall have to deal with systems of elementary normal functions, belonging to a fixed value of the second affix q. In such case this affix will be suppressed. It can easily be proved, that, under this understanding, the following so-called "orthogonality"-relations exist between the functions (6):

$$\left.\begin{array}{l}
\left.\begin{array}{l}
\displaystyle\int_0^l \int_0^{2\pi} [R^*_k u^*_l + \Phi^*_k v^*_l]\, a d\varphi dz = 0 \\
\displaystyle\int_0^l \int_0^{2\pi} [R^{**}_k u^{**}_l + \Phi^{**}_k v^{**}_l]\, a d\varphi dz = 0
\end{array}\right\} k \neq l \\
\left.\begin{array}{l}
\displaystyle\int_0^l \int_0^{2\pi} [R^*_k u^{**}_l + \Phi^*_k v^{**}_l]\, a d\varphi dz = 0 \\
\displaystyle\int_0^l \int_0^{2\pi} [R^{**}_k u^*_l + \Phi^{**}_k v^*_l]\, a d\varphi dz = 0
\end{array}\right\} \text{for } all \text{ values of } k \text{ and } l.
\end{array}\right\} \begin{array}{l} k \text{ and } l \text{ denoting values of the} \\ \text{first affix } p; \\ \text{the second affix} \\ q \text{ being fixed.} \end{array} \quad . \quad . \quad (9)$$

4. *The differential-equations of the buckling tube.* If only the load components R, Φ and Z and the displacements u, v, w be adequately interpreted, the equations (2, 2) may be looked upon as the differential-equations of the buckling tube, loaded by axial (and linearly changing) bending stresses at its endsections. As stated in section 1, we assume that, — thanks to its initial curvature — the tube, under influence of its buckling moments $M = \mu \overline{M}$, can be regarded as straight and of constant thickness. The displacements u, v, w of any point of the middle-surface, *from this initial state of stress and strain* (II) replace from this moment the quantities, designed in the same way in equation 2, 2, denoting originally the displacements from the *unstressed state* (I).

One could be inclined, in applying the equations (2, 2) to the buckling problem, to put R, Φ and Z equal to zero, due to the fact that these quantities evidently now stand for the *supplementary* loads, which eventually arise with the transition from state II to the (indefinitely) neighbouring buckling-configuration (II′). The following fact, however, has to be observed. An arbitrary element of the tube, as represented in fig. 4, being in state II, finds itself in equilibrium, under the axial forces $Q\,ad\varphi$, Q representing the external force pro unit of circumferential length exerted on the end-sections of the tube in the points $(z=0, \varphi)$ and $(z=l, \varphi)$.

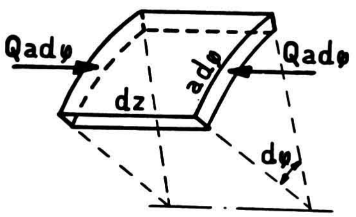

Fig. 4.

If now the transition takes place from state II to state II′ these forces retain their magnitude (because the external load of the tube does not alter at all), but they change in direction in accordance with the change in shape and curvature of the surface-elements on which they act. Therefore these forces (though unchanged in magnitude) produce components in the directions r, φ and z, which, divided by the surface $ad\varphi\, dz$, play the role of R, Φ and Z in the equations (2, 2).

It can be shown [1]), that in our case R, Φ, Z amount to

$$R=-Q\frac{\partial^2 u}{\partial z^2},\quad \Phi=-Q\frac{\partial^2 v}{\partial z^2},\quad Z=0 \quad . \quad . \quad . \quad . \quad (1)$$

and therefore the differential-equations of our buckling problem run as follows:

$$\left.\begin{aligned}
&\frac{1}{a}\left(\frac{u}{a}+\frac{1}{a}\frac{\partial v}{\partial\varphi}+\nu\frac{\partial w}{\partial z}\right)+\frac{A^*}{B}\left(\frac{1}{a^4}\frac{\partial^4 u}{\partial\varphi^4}+\frac{2}{a^2}\frac{\partial^4 u}{\partial\varphi^2\partial z^2}+\frac{\partial^4 u}{\partial z^4}+\right.\\
&\qquad\left.+\frac{2}{a^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{u}{a^4}-\frac{3-\nu}{2a^2}\frac{\partial^3 v}{\partial\varphi\partial z^2}+\frac{1-\nu}{2a^3}\frac{\partial^3 w}{\partial\varphi^2\partial z}-\frac{1}{a}\frac{\partial^3 w}{\partial z^3}\right)+\\
&\qquad+\frac{Q}{B}\frac{\partial^2 u}{\partial z^2}=0\\
&\frac{1}{a^2}\frac{\partial u}{\partial\varphi}+\frac{1}{a^2}\frac{\partial^2 v}{\partial\varphi^2}+\frac{1-\nu}{2}\frac{\partial^2 v}{\partial z^2}+\frac{1+\nu}{2a}\frac{\partial^2 w}{\partial\varphi\partial z}+\frac{A^*}{B}\left(-\frac{3-\nu}{2a^2}\frac{\partial^3 u}{\partial\varphi\partial z^2}+\right.\\
&\qquad\left.+3\frac{1-\nu}{2a^2}\frac{\partial^2 v}{\partial z^2}\right)-\frac{Q}{B}\frac{\partial^2 v}{\partial z^2}=0\\
&\frac{\nu}{a}\frac{\partial u}{\partial z}+\frac{1+\nu}{2a}\frac{\partial^2 v}{\partial\varphi\partial z}+\frac{1-\nu}{2a^2}\frac{\partial^2 w}{\partial\varphi^2}+\frac{\partial^2 w}{\partial z^2}+\frac{A^*}{B}\left(\frac{1-\nu}{2a^3}\frac{\partial^3 u}{\partial\varphi^2\partial z}-\right.\\
&\qquad\left.-\frac{1}{a}\frac{\partial^3 u}{\partial z^3}\right)+\frac{1-\nu}{2a^4}\frac{\partial^2 w}{\partial\varphi^2}=0
\end{aligned}\right\} \quad . \quad (2)$$

[1]) Comp. W. FLÜGGE, Statik und Dynamik der Schalen; Springer, Berlin 1934. C. B. BIEZENO and R. GRAMMEL, Technische Dynamik; Springer, Berlin 1939.

There only remains to express $\frac{Q}{B}$ in terms of the loading moment $M = \mu\overline{M}$. We find (comp. fig. 5 and 2, 1):

$$\frac{Q}{B} = \frac{M\,a\cos\varphi}{\pi\,a^3\,h}\,h.\,\frac{(1-\nu^2)}{E\,h} = \frac{\mu\,\overline{M}\,(1-\nu^2)}{\pi\,a^2\,E\,h}\cos\varphi \quad . \quad . \quad . \quad . \quad (3)$$

It has already been stated in section 1, that the eqs. (2) are homogeneous in u, v, w and their derivatives, and therefore only have solutions, different from $u = v = w = 0$, for distinct values of the parameter μ, respectively for distinct values of the loading moment $M = \mu\overline{M}$. Those values will be called the "total characteristic numbers" respectively the "total characteristic moments" of our problem; the corresponding deformations (T) the "total characteristic deformations". From now it is our subject to calculate the *smallest* characteristic moment.

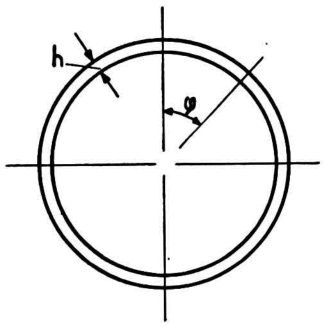

Fig. 5.

5. *The expansion of a "total characteristic deformation" T into a series of elementary normal deformations D. The influence numbers a_{ij}.* The main result of the preceding section lays in the fact, that in case of buckling the external loading moments M of the tube give rise to a "would-be" surface load R, Φ, Z given by

$$R = -Q\frac{\partial^2 u}{\partial z^2} = -\frac{M}{\pi a^2}\frac{\partial^2 u}{\partial z^2}\cos\varphi,\ \ \Phi = -Q\frac{\partial^2 v}{\partial z^2} = -\frac{M}{\pi a^2}\frac{\partial^2 v}{\partial z^2}\cos\varphi,\ \ Z = 0 \quad (1)$$

the magnitude of which depends upon the total characteristic deformation u, v, w, that corresponds with M. We learn from (1) that — if this total deformation be decomposed in a set of other deformations $u_1, v_1, w_1; u_2, v_2, w_2 \ldots$ —, such that $u = u_1 + u_2 + \ldots$, $v = v_1 + v_2 + \ldots$, $w = w_1 + w_2 + \ldots$, the loadsystem (1) may be decomposed in a set of other loadsystems, each of which is calculable by (1) from u_1, v_1, w_1, u_2, v_2, w_2 etc.

If therefore the total characteristic deformation T be expanded in a (infinite) series of "elementary normal deformations" D:

$$T = \sum_{i=1}^{\infty} d_i\,D_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad , \quad (2)$$

and if the loadsystem, derived from D_i with the aid of (1) be called $\overline{B}_i$ then the "would-be" loadsystem belonging to the total characteristic deformation under consideration, may be written as:

$$\Sigma\, d_i\,\overline{B}_i.$$

Each system $\overline{B}_i$, on its part, can be expanded in a (finite or infinite) series of elementary normal loadsystems B, so that, at the end the would-be loadsystem (1), produced by the total characteristic deformation under consideration, can be regarded upon as the sum of an infinite number of groups of a (finite or infinite) number of elementary normal loads B.

Two remarks of some importance are here to be made. Firstly it has been silently assumed, that all deformations and load systems, introduced in this section belong to the *same* parameter λ (comp. (2, 5)), which is characteristic for the total deformation T, so that all deformations and loads, considered here, have the same number of longitudinal waves. In accordance with a remark, already made in section 3, we therefore denote the functions T, B and D by one single suffix p, relating to the number of circumferential waves.

Secondly, we could — of course — have developed *directly* the load system (1) into a series of the elementary normal loads. We did not, however, proceed in this way in view of the introduction of a system of so-called influence-numbers a_{ij}, which will now be defined.

If (artificially) the tube be given the elementary normal deformation D, then we can — by the aid of (1) — formally calculate the "would-be" loadsystems roused by two unit bending moments $\overline{M}$, acting at the ends of the tube. As stated before, this loadsystem can be developed in a (finite or infinite) series of elementary, normal functions B. The coefficient a_{ij}, which in this expansion belongs to the elementary normal function B_i is called "the influence number *of* the elementary normal deformation D_j *with respect to* the elementary normal load B_i.

This formal definition provides us with an expedient to obtain system of homogeneous linear equations for the coefficients d_i in the expression (2). Indeed, if a deformation D_j provokes a "would-be" load, which contains a_{ij} times B_i, (assumed that the tube be charged by unit bending moments $\overline{M}$), then a deformation $d_j D_j$ provokes a would-be load, which contains $\mu d_j a_{ij}$ times B_i, assumed that the tube be charged by two bending moments $M = \mu \overline{M}$.

The total characteristic deformation $T = \sum_{i=1}^{\infty} d_i D_i$ therefore provokes a would-be load, which contains B_i

$$\sum_{j=1}^{\infty} \mu\, d_j\, a_{ij} \text{ times.}$$

On the other hand it was stated in section 2, that the deformation $T = \sum_{i=1}^{\infty} d_i D_i$ only can be maintained by the load $\sum_{i=1}^{\infty} d_i B_i$, and therefore the system of equations

$$d_i = \sum_{j=1} \mu\, d_j\, a_{ij} \quad i = 1, 2 \ldots \qquad (3)$$

must hold.

It only has solutions different from $d_i = 0$ $(i = 1, 2 \ldots)$ if $\frac{1}{\mu}$ satisfies the equation:

$$\begin{vmatrix} a_{11} - \frac{1}{\mu} & a_{12} & a_{13} \cdots \cdots \\ a_{21} & a_{22} - \frac{1}{\mu} & a_{23} \cdots \cdots \\ a_{31} & a_{32} & a_{33} - \frac{1}{\mu} \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix} = 0 \quad . \quad . \quad . \quad (4)$$

which formally can be considered as representing our problem.

Now we proceed by proving, that — if only the elementary normal functions be suitably numbered — the reciprocal relation $a_{ij} = a_{ji}$ holds, so that equation (4) is a secular one, possessing only real roots. To this end we calculate by using (1) the would-be loads R, Φ — to be denoted by $\overline{R}^*$, $\overline{\Phi}^*$, resp. $\overline{R}^{**}$, $\overline{\Phi}^{**}$ — belonging to the elementary normal deformations u^*, v^*, resp. u^{**}, v^{**} defined by (3, 6) and (3, 8). They are — if the "order" of the underlying elementary deformation be indicated by p —

$$\left.\begin{aligned} \overline{R}^*_p &= \frac{\lambda^2 F^*_p}{2\pi a^2 \sqrt{F^*_p + E^*_p\, G^*_p}} \left[\cos(p+1)\,\varphi + \cos(p-1)\,\varphi\right] \sin \frac{\lambda z}{a} \\ \overline{\Phi}^*_p &= \frac{\lambda^2 G^*_p}{2\pi a^2 \sqrt{F^*_p + E^*_p\, G^*_p}} \left[\sin(p+1)\,\varphi + \sin(p-1)\,\varphi\right] \sin \frac{\lambda z}{a} \\ \overline{R}^{**}_p &= \frac{\lambda^2 F^{**}_p}{2\pi a^2 \sqrt{F^{**}_p + E^{**}_p\, G^{**}_p}} \left[\cos(p+1)\,\varphi + \cos(p-1)\,\varphi\right] \sin \frac{\lambda z}{a} \\ \overline{\Phi}^{**}_p &= \frac{\lambda^2 G^{**}_p}{2\pi a^2 \sqrt{F^{**}_p + E^{**}_p\, G^{**}_p}} \left[\sin(p+1)\,\varphi + \sin(p-1)\,\varphi\right] \sin \frac{\lambda z}{a} \end{aligned}\right\} \quad . \quad . \quad (5)$$

Both load-systems $\overline{R}^*_p$, $\overline{\Phi}^*_p$, and $\overline{R}^{**}_p$, $\overline{\Phi}^{**}_p$ can be linearly expressed in the elementary load-systems

$$B^*_{p-1} = (R^*_{p-1}, \Phi^*_{p-1}); \quad B^*_{p+1} = (R^*_{p+1}, \Phi^*_{p+1}); \quad B^{**}_{p-1} = (R^{**}_{p-1}, \Phi^{**}_{p-1});$$
$$B^{**}_{p+1} = (R^{**}_{p+1}, \Phi^{**}_{p+1}).$$

If we restrict ourselves to the system $\overline{R}^*_p, \overline{\Phi}^*_p$, and if we put

$$\left.\begin{aligned} \overline{R}^*_p &= \delta^1_p\, R^*_{p-1} + \delta^2_p\, R^{**}_{p-1} + \delta^3_p\, R^*_{p+1} + \delta^4_p\, R^{**}_{p+1} \\ \overline{\Phi}^*_p &= \delta^1_p\, \Phi^*_{p-1} + \delta^2_p\, \Phi^{**}_{p-1} + \delta^3_p\, \Phi^*_{p+1} + \delta^4_p\, \Phi^{**}_{p+1} \end{aligned}\right\} \quad . \; . \; . \quad (6)$$

the coefficients δ^i_p $(i = 1, 2 \ldots 4)$ can be found by using the relations

$$\left.\begin{aligned} \frac{\pi a l}{2}\,\delta^1_p &= \int\limits_0^l \int\limits_0^{2\pi} [\overline{R}^*_p\, u^*_{p-1} + \overline{\Phi}^*_p\, v^*_{p-1}]\, a d\varphi\, dz \\ \frac{\pi a l}{2}\,\delta^2_p &= \int\limits_0^l \int\limits_0^{2\pi} [\overline{R}^*_p\, u^{**}_{p-1} + \overline{\Phi}^*_p\, v^{**}_{p-1}]\, a d\varphi\, dz \\ \frac{\pi a l}{2}\,\delta^3_p &= \int\limits_0^l \int\limits_0^{2\pi} [\overline{R}^*_p\, u^*_{p+1} + \overline{\Phi}^*_p\, v^*_{p+1}]\, a d\varphi\, dz \\ \frac{\pi a l}{2}\,\delta^4_p &= \int\limits_0^l \int\limits_0^{2\pi} [\overline{R}^*_p\, u^{**}_{p+1} + \overline{\Phi}^*_p\, v^{**}_{p+1}]\, a d\varphi\, dz \end{aligned}\right\} \quad . \; . \; . \quad (7)$$

which can be verified by substituting the expressions (6) in the right-hand members and by taking into account the relations of orthogonality established in section 3 (comp. 3, 9). Substitution of the explicite (5) of $\overline{R}^*_p, \overline{\Phi}^*_p, \overline{R}^{**}_p, \overline{\Phi}^{**}_p$ into (7) gives:

$$\left.\begin{aligned} \delta^1_p &= \frac{\lambda^2}{2\pi a^2}\left[\frac{F^*_p F^*_{p-1} + G^*_p G^*_{p-1}}{\sqrt{F^*_p + E^*_p G^*_p}\,\sqrt{F^*_{p-1} + E^*_{p-1} G^*_{p-1}}}\right] \\ \delta^2_p &= \frac{\lambda^2}{2\pi a^2}\left[\frac{F^*_p F^{**}_{p-1} + G^*_p G^{**}_{p-1}}{\sqrt{F^*_p + E^*_p G^*_p}\,\sqrt{F^{**}_{p-1} + E^{**}_{p-1} G^{**}_{p-1}}}\right] \\ \delta^3_p &= \frac{\lambda^2}{2\pi a^2}\left[\frac{F^*_p F^*_{p+1} + G^*_p G^*_{p+1}}{\sqrt{F^*_p + E^*_p G^*_p}\,\sqrt{F^*_{p+1} + E^*_{p+1} G^*_{p+1}}}\right] \\ \delta^4_p &= \frac{\lambda^2}{2\pi a^2}\left[\frac{F_p F_{p+1} + G_p G_{p+1}}{\sqrt{F^*_p + E^*_p G^*_p}\,\sqrt{F^{**}_{p+1} + E^{**}_{p+1} G^{**}_{p+1}}}\right] \end{aligned}\right\} \quad . \; . \quad (8)$$

If again we put

$$\left.\begin{aligned} \overline{R}^{**}_p &= \varepsilon^1_p\, R^*_{p-1} + \varepsilon^2_p\, R^{**}_{p-1} + \varepsilon^3_p\, R^*_{p+1} + \varepsilon^4_p\, R^{**}_{p+1} \\ \overline{\Phi}^{**}_p &= \varepsilon^1_p\, \Phi^*_{p-1} + \varepsilon^2_p\, \Phi^{**}_{p-1} + \varepsilon^3_p\, \Phi^*_{p+1} + \varepsilon^4_p\, \Phi^{**}_{p+1} \end{aligned}\right\} \quad . \; . \; . \quad (9)$$

we find in an analogous way

$$\left.\begin{aligned}
\varepsilon_p^1 &= \frac{\lambda^2}{2\pi a^2}\left[\frac{F_p^{**}F_{p-1}^{*}+G_p^{**}G_{p-1}^{*}}{\sqrt{F_p^{**}+E_p^{**}G_p^{**}}\sqrt{F_{p-1}^{*}+E_{p-1}^{*}G_{p-1}^{*}}}\right]\\
\varepsilon_p^2 &= \frac{\lambda^2}{2\pi a^2}\left[\frac{F_p^{**}F_{p-1}^{**}+G_p^{**}G_{p-1}^{**}}{\sqrt{F_p^{**}+E_p^{**}G_p^{**}}\sqrt{F_{p-1}^{**}+E_{p-1}^{**}G_{p-1}^{**}}}\right]\\
\varepsilon_p^3 &= \frac{\lambda^2}{2\pi a^2}\left[\frac{F_p^{**}F_{p+1}^{*}+G_p^{**}G_{p+1}^{*}}{\sqrt{F_p^{**}+E_p^{**}G_p^{**}}\sqrt{F_{p+1}^{*}+E_{p+1}^{*}G_{p+1}^{*}}}\right]\\
\varepsilon_p^4 &= \frac{\lambda^2}{2\pi a^2}\left[\frac{F_p^{**}F_{p+1}^{**}+G_p^{**}G_{p+1}^{**}}{\sqrt{F_p^{**}+E_p^{**}G_p^{**}}\sqrt{F_{p+1}^{**}+E_{p+1}^{**}G_{p+1}^{**}}}\right]
\end{aligned}\right\} \quad . \quad . \quad (10)$$

It goes without saying that each of the coefficients δ_p^i and ε_p^i $(i=1,2\ldots4)$ represents an influence-number a_{ij} in the previously defined sense. Furthermore it is evident, that no other influence-coefficients exist except those represented by (8) and (10).

The question, which suffixes must be ascribed to the coefficient a, to let it represent a given δ or ε, depends upon the way in which the elementary normal deformations $D_i \equiv (\overset{*}{u}_i, \overset{*}{v}_i)$ and $D_i^{**} \equiv (\overset{**}{u}_i, \overset{**}{v}_i)$ are arranged. We fix, that

$D_0^* \equiv 0$ will be indicated by D_i $(i=0)$ and in consequence B_0^* by B_i $(i=0)$
D_0^{**} „ „ „ „ D_i $(i=1)$ „ „ „ B_0^{**} by B_i $(i=1)$
D_1^* „ „ „ „ D_i $(i=2)$ „ „ „ B_1^* by B_i $(i=2)$
D_1^{**} „ „ „ „ D_i $(i=3)$ „ „ „ B_1^{**} by B_i $(i=3)$
a. s. o.

Bearing in mind that δ_p^1 represents the influence-coefficient of the elementary normal load B_{p-1}^* with respect to the elementary normal deformation D_p^*, and that B_{p-1}^* and D_p^* in the just defined sequences of normal loads B_i and normal deformations D_i have the numbers $i=2(p-1)$, resp. $i=2p$, then it is obvious that δ_p^1 has to be called $a_{2(p-1),2p}$, and that the significance of the coefficients δ_p^i, ε_p^i in general can be derived from the following scheme

$$\left.\begin{aligned}
\delta_p^1 &\equiv a_{2(p-1),2p} & \varepsilon_p^1 &= a_{2(p-1),2p+1}\\
\delta_p^2 &\equiv a_{2(p-1)+1,2p} & \varepsilon_p^2 &= a_{2(p-1)+1,2p+1}\\
\delta_p^3 &\equiv a_{2(p+1),2p} & \varepsilon_p^3 &= a_{2(p+1),2p+1}\\
\delta_p^4 &\equiv a_{2(p+1)+1,2p} & \varepsilon_p^4 &= a_{2(p+1)+1,2p+1}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

From the equations (8) and (10) we deduce

$$\delta_p^1 = \delta_{p-1}^3\ ,\ \varepsilon_p^1 = \delta_{p-1}^4\ ,\ \delta_p^2 = \varepsilon_{p-1}^3\ ,\ \varepsilon_p^2 = \varepsilon_{p-1}^4,$$

and consequently from equations (11)

$$\left.\begin{array}{l} a_{2(p-1),2p} = a_{2p,2(p-1)} \\ a_{2(p-1),2p+1} = a_{2p+1,2(p-1)} \\ a_{2(p-1)+1,2p} = a_{2p,2(p-1)+1} \\ a_{2(p-1)+1,2p+1} = a_{2p+1,2(p-1)+1} \end{array}\right\} \quad \ldots\ldots \quad (12)$$

These equations obviously can be contracted in the single relation of reciprocity:

$$a_{ij} = a_{ji}. \quad \ldots\ldots\ldots \quad (13)$$

The numbering of normal functions, introduced in this section, therefore make the left-hand side of (4) to a symmetrical determinant. If furthermore we give attention to the fact, that in our new nomenclature influence-coefficients of the type a_{ii} do not occur, and that a great number of the other coefficients is zero, equation (4) can be replaced by:

$$\begin{vmatrix} \frac{1}{\mu} & a_{12} & a_{13} & 0 & 0 & 0 & 0\ldots \\ a_{21} & -\frac{1}{\mu} & 0 & a_{24} & a_{25} & 0 & 0\ldots \\ a_{31} & 0 & -\frac{1}{\mu} & a_{34} & a_{35} & 0 & 0\ldots \\ 0 & a_{42} & a_{43} & -\frac{1}{\mu} & 0 & a_{46} & a_{47}\,.\,. \\ 0 & a_{52} & a_{53} & 0 & -\frac{1}{\mu} & a_{56} & a_{57}\,.\,. \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix} = 0 \quad (14)$$

with $a_{ij} = a_{ji}$.

After a well-known theorem the roots of this equation are one and all real. If the columns 2, 3, 6, 7, 10, 11,... and the rows 1, 4, 5, 8, 9,... of the determinant are multiplied by -1, no alteration takes place in its general shape, except that all terms $\frac{1}{\mu}$ change their sign. Therefore it can be stated beforehand, that all roots of equation (14) occur in pairs of equal magnitude and opposite sign.

Every root μ_k of (13) corresponds to a total characteristic deformation T_k

$$T_k = \sum_{i=1}^{\infty} d_{ki}\, D_i,$$

the coefficients d_i satisfying the equations:

$$d_{ki} = \sum_{j=1}^{n} \mu_k\, a_{ij}\, d_{kj}\ \ (i = 1, 2\ldots). \quad \ldots\ldots \quad (15)$$