Physics. — *On the buckling of a thin-walled circular tube loaaed by pure bending.* II. By C. B. BIEZENO and J. J. KOCH.

(Communicated at the meeting of September 28, 1940.)

6. *Determination of the smallest total characteristic number μ by iteration, the number q of the longitudinal waves, occurring in the corresponding total deformation, being presumed as fixed.*

The most direct way to attack our problem, consisting in solving equation (14) of the preceding section, practically seems impossible. Therefore we have recourse to a generalisation of an iterative method which by the second author has been established in his doctor-thesis [1]).

We consider the following system of linear homogeneous equations:

$$d_i = \sum_{j=1}^{n} \mu\, a_{ij}\, d_j \quad (i = 1, 2 \ldots n) \quad \ldots \ldots \quad (1)$$

which only under the condition

$$\left.\begin{vmatrix} a_{11} - \dfrac{1}{\mu} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - \dfrac{1}{\mu} & \ldots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} - \dfrac{1}{\mu} \end{vmatrix} = 0 \right\} \quad \ldots \ldots \quad (2)$$

admits solutions $d_i\,(i = 1 \ldots n)$ different from zero. It be fixed that $a_{ij} = a_{ji}$, so that all roots of (2) are real. Furthermore it may be assumed that all roots $\mu_1 \ldots \mu_n$ be different, and that

$$|\mu_1| < |\mu_2| < |\mu_3| \ldots . < |\mu_n| . \quad \ldots \ldots \quad (3)$$

To every μ_k corresponds a solution d_{ki} $(i = 1, 2 \ldots n)$, which may be normalized by the condition

$$\Sigma\, d_{ki}^2 = 1 \quad \ldots \ldots \ldots \quad (4)$$

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[1]) Comp. J. J. KOCH, Eenige toepassingen van de leer der eigenfuncties op vraagstukken uit de Toegepaste Mechanica, 1929, Waltman, Delft.

Also: C. B. BIEZENO und R. GRAMMEL, Technische Dynamik, Chapter III, 1939. Springer, Berlin.

It can easily be proved, that — for $k \neq l$ —

$$\sum_{i=1}^{n} d_{ki}\, d_{li} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Indeed we find, making use of (1)

$$\sum_{i=1}^{n} d_{ki}\, d_{li} = \mu_k \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, d_{kj}\, d_{li} = \mu_l \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, d_{lj}\, d_{ki}$$

$$= \mu_l \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ji}\, d_{li}\, d_{kj} = \mu_l \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, d_{kj}\, d_{li}.$$

Comparison of the second and last member of this linked equation, leads — on account of $\mu_l \neq \mu_k$ — to the value zero for all members and especially to $\Sigma\, d_{ki}\, d_{li} = 0$.

Another statement, we make use of in arguing the method of iteration to be explained a few lines further on, is the following one:

An arbitrary set of values ϑ_{1i} $(i = 1, 2 \ldots n)$ can always be expressed as a linear sum of the n solutions d_{ki} $(i = 1, 2 \ldots n)$ corresponding to the n-roots μ_k, so that ϑ_{1i} may be written as:

$$\vartheta_{1i} = a_1\, d_{1i} + a_2\, d_{2i} + \ldots . + a_n\, d_{ni} \;\; (i = 1, 2 \ldots n) . \quad . \quad . \quad (6)$$

Indeed, if we sum up the equations (6) after having multiplied each equation with the number d_{ki}, whose second suffix corresponds with the number i of the equation, we find

$$\sum_{i=1}^{n} d_{ki}\, \vartheta_{1i} = a_1 \sum_{i=1}^{n} d_{ki}\, d_{1i} + a_2 \sum_{i=1}^{n} d_{ki}\, d_{2i} + \ldots + a_k \sum_{i=1}^{n} d_{ki}^2 +$$

$$+ \ldots + a_n \sum_{i=1}^{n} d_{ki}\, d_{ni}$$

or, with regard to (4) and (5):

$$a_k = \sum_{i=1}^{n} d_{ki}\, \vartheta_{1i} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The iteration, mentioned before, consists in deriving from an arbitrary set of quantities ϑ_{1i} another set ϑ_{2i}, defined by

$$\vartheta_{2i} = \sum_{j=1}^{n} a_{ij}\, \vartheta_{1j} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If we substitute in (8) the expression, which for ϑ_{1j} follows from (6), we find

$$\vartheta_{2i} = \sum_{j=1}^{n} a_{ij}\, (\sum_{k=1}^{n} a_k\, d_{kj}) = \sum_{k=1}^{n} a_k \sum_{j=1}^{n} a_{ij}\, d_{kj} . \quad . \quad . \quad . \quad . \quad (9)$$

Taking into account, that

$$d_{ki} = \sum_{j=1}^{n} \mu_k a_{ij} d_{kj} \quad \text{resp.} \quad \sum_{j=1}^{n} a_{ij} d_{kj} = \frac{d_{ki}}{\mu_k}$$

(comp. (1)), the first iteration ϑ_{2i}, derived from ϑ_{1i} becomes:

$$\vartheta_{2i} = \frac{a_1}{\mu_1} d_{1i} + \frac{a_2}{\mu_2} d_{2i} + \ldots + \frac{a_n}{\mu_n} d_{ni} \quad (i = 1, 2 \ldots n). \quad . \quad . \quad (10)$$

We see at once, that, if the iterative process be repeated s times, we get:

$$\vartheta_{s+1,i} = \frac{a_1}{\mu_1^s} d_{1i} + \frac{a_2}{\mu_2^s} d_{2i} + \ldots + \frac{a_n}{\mu_n^s} d_{ni} \quad (i = 1, 2 \ldots n) \, . \quad . \quad (11)$$

and consequently:

$$\frac{\vartheta_{s,i}}{\vartheta_{s+1,i}} = \frac{d_{1i} + \left(\frac{\mu_1}{\mu_2}\right)^{s-1} \frac{a_2}{a_1} d_{2i} + \ldots . + \left(\frac{\mu_1}{\mu_n}\right)^{s-1} \frac{a_n}{a_1} d_{ni}}{d_{1i} + \left(\frac{\mu_1}{\mu_2}\right)^{s} \frac{a_2}{a_1} d_{2i} + \ldots . + \left(\frac{\mu_1}{\mu_n}\right)^{s} \frac{a_n}{a_1} d_{ni}} \mu_1 \quad . \quad (12)$$

The fractions $\left(\frac{\mu_1}{\mu_i}\right)$ being all < 1 (comp. (3)), the second member of this equation tends, with ever-increasing s, to μ_1, so that two conclusions are to be drawn:

1. two consecutive iterations ϑ_{si} and $\vartheta_{s+1,i}$ $(i = 1, 2 \ldots n)$ tend — with increasing s — to proportionality;

2. the "factor of proportionality" represents the smallest characteristic number μ_1.

Theoretically one has to repeat indefinitely the iteration process to obtain the exact-value of μ_1. For technical purposes, which we have in mind here, a relative small number of iterations will do to obtain a sufficient degree of proportionality and to look on the "average" factor of proportionality $\bar{\mu}_1$ as the approximative value of μ_1. It may be emphasized here, that the numerical work which has to be done to arrive at an acceptible approximative value $\bar{\mu}_1$ can remarkably be reduced by using a formula given by KOCH [1]) which, applied to our problem, runs

$$\bar{\mu}_1 = \frac{\sum_{i=1}^{n} \vartheta_{1i} \vartheta_{2i}}{\sum_{i=1}^{n} \vartheta_{2i}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

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[1]) J. J. KOCH, Bestimmung höherer kritischer Drehzahlen schnellaufender Wellen, Verhandlungen des 2ten internationalen Kongresses für technische Mechanik, Zürich 1926.

If in the right hand member we substitute ϑ_{1i} and ϑ_{2i} from (6) and (8) we find — paying attention to (4) and (5) —

$$\bar{\mu}_1 = \frac{1 + \frac{a_2^2}{a_1^2}\frac{\mu_1}{\mu_2} + \ldots + \frac{a_n^2}{a_1^2}\frac{\mu_1}{\mu_n}}{1 + \frac{a_2^2}{a_1^2}\left(\frac{\mu_1}{\mu_2}\right)^2 + \ldots + \frac{a_n^2}{a_1^2}\left(\frac{\mu_1}{\mu_n}\right)^2}\,\mu_1 \quad . \quad . \quad . \quad . \quad (14)$$

The degree of approximation of μ_1 by $\bar{\mu}_1$, depends evidently on the choice of the initial set of values ϑ_{1i}. If this choice was an unfavourable one, it may occur, that the discrepancy between $\bar{\mu}_1$ and μ_1 is greater than admittable. If, in a particular case, there should be some doubt about the accuracy of $\bar{\mu}_1$, the iteration can be repeated once more and μ_1 is then approximated more closely by

$$\bar{\bar{\mu}}_1 = \frac{\sum_{i=1}^{n} \vartheta_{2i}\,\vartheta_{3i}}{\sum_{i=1}^{n} \vartheta_{3i}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

If $\bar{\bar{\mu}}$ should considerably differ from $\bar{\mu}_1$, there is some reason, to go on with a third iteration. At the first sight it seems, that no data are available, to judge "a priori" whether the first set of ϑ_{1i} is a suitable one or not. And it must be admitted that mathematically spoken, such data do not exist. It may be noticed, however, — without going into details here — that technical problems in most cases possess some intrinsic feature, which indeed enables a suitable choice.

In the following section we intend to apply the foregoing results to equation (14) and the corresponding set of equations (15) of section 5. Two remarks have to be made in this connection. The first one bears upon the fact, that the roots of (5, 14) occur in pairs of same absolute magnitude and opposite sign. This requires a slight modification in our procedure to approximate μ_1.

Returning to equation (8), let us consider the case that $\mu_2 = -\mu_1$, then $\vartheta_{s+1,i}$ becomes:

$$\vartheta_{s+1,i} = \frac{a_1}{\mu_1^s} d_{1i} + (-1)^s \frac{a_2}{\mu_1^s} d_{2i} + \frac{a_3}{\mu_3^s} d_{3i} + \ldots + \frac{a_n}{\mu_n^s} d_{ni} \quad . \quad . \quad (16)$$

Accordingly $\vartheta_{s-1,i}$ is given by

$$\vartheta_{s-1,i} = \frac{a_1}{\mu_1{}^{s-2}} d_{1i} + (-1)^{s-2} \frac{a_2}{\mu_1{}^{s-2}} d_{2i} + \frac{a_3}{\mu_3{}^s} d_{3i} + \ldots + \frac{a_n}{\mu_n{}^s} d_{ni}.$$

Therefore the quotient $\vartheta_{s-1,i} : \vartheta_{s+1,i}$ amounts to:

$$\frac{\vartheta_{s-1,i}}{\vartheta_{s+1,i}} = \frac{\left[d_{1i} + (-1)^s \frac{a_2}{a_1} d_{2i}\right] + \left(\frac{\mu_1}{\mu_3}\right)^{s-2} \frac{a_3}{a_1} d_{3i} + \ldots + \left(\frac{\mu_1}{\mu_n}\right)^{s-2} \frac{a_n}{a_1} d_{ni}}{\left[d_{1i} + (-1)^s \frac{a_2}{a_1} d_{2i}\right] + \left(\frac{\mu_1}{\mu_3}\right)^{s} \frac{a_3}{a_1} d_{3i} + \ldots + \left(\frac{\mu_1}{\mu_n}\right)^{s} \frac{a_n}{a_1} d_{ni}} \mu_1^2 \quad . \quad (17)$$

and it is seen that, with increasing s, the quotient of two ϑ's, whose index-numbers differ by the amount 2, now tend to the *square* of the characteristic number μ of smallest absolute magnitude.

Similarly the approximating formula (13) has to be replaced by

$$\mu_1{}^2 = \frac{\sum\limits_{i=1}^{n} \vartheta_{1i}\, \vartheta_{3i}}{\sum\limits_{i=1}^{n} \vartheta_{3i}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Secondly it must be stated explicitely, that the application of the results obtained in this section, to equations (14) and (15) of section 5, entails a transition to $n = \infty$. We take it for granted, that this transition is admissible and do not enter into discussions, which from a mathematical standpoint are indispensable.

7. *Continuation. Determination of the smallest total characteristic number μ by iteration, the number q of the longitudinal waves, occurring in the corresponding total deformation being presumed as fixed.*

The practical application to our buckling problem of the just described iteration involves the knowledge of the influence-numbers a_{ij}. It will be shown now that the calculation of these influence-numbers, (which, though theoretically possible, would lead to elaborate ciphering work) can be omitted by a mechanical interpretation of the law of iteration. To this end we start again from the arbitrary set of numbers ϑ_{1i}, and define, by the aid of these numbers a deformation D^1, composed of a linear sum of the elementary normal deformations D_i:

$$D^1 = \vartheta_{11}\, D_1 + \vartheta_{12}\, D_2 + \vartheta_{13}\, D_3 + \ldots \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We artificially submit our tube to this deformation and *in this fixed shape*, subject it to the action of two unit moments $\overline{M}$, working at its ends. If then we ask the "would-be" load B^1 formally produced by two moments $\overline{M}$, we find — comp. the definition of the influence-numbers in section 5 —

$$B^1 = \vartheta_1 \sum_i a_{i1} B_i + \vartheta_{12} \sum_i a_{i2} B_i + \vartheta_{13} \sum_i a_{i3} B_i + \ldots = \sum_i \left(\sum_j \vartheta_{1j}\, a_{ij}\right) B_i .$$

If this load should be placed on the *unloaded* and undeformed tube, it would produce a deformation D^2, given by

$$D^2 = \sum_i (\sum_j \vartheta_{1j} \, a_{ij}) \, B_i$$

or, with regard to (6, 8)

$$D^2 = \vartheta_{21} \, D_1 + \vartheta_{22} \, D_2 + \vartheta_{23} \, D_3 + \ldots \quad (2)$$

If therefore, we start from an arbitrary deformation D^1, subject the tube (artificially) to this deformation, calculate (formally) the "would-be" load to which this deformation gives rise if the tube — so to say frozen in his shape — is subjected to the action of two unit-moments $\overline{M}$, derive from this "would-be" load the deformation D^2 produced by it if it acts on the undeformed tube, start from D^2 to obtain in the same manner a deformation D^3 a. s. o., then the sets of coefficients ϑ_{ki} $(k = 2, 3 \ldots)$ in the expansions of D^2, D^3, D^k with respect to the elementary normal functions D_i, play the role of the iterative sets, deduced from the set ϑ_{1i} after the law of iteration, given in section **6**.

It will be observed that this mechanical interpretation of our iterative process is free from mentioning the influence-numbers a_{ij}; the only thing we have to do is to start with an arbitrary deformation D^1 and to deduct from this deformation, in a well defined manner, other deformations D^2, D^3 a. s. o. This initial deformation however, can be considered either as a linear sum of the elementary normal deformations D_i, or as a sum of the special functions (2, 6). We will adopt the latter conception, because it facilitates the practical handling of our mechanical iteration process. Accordingly we define from this moment the initial deformation D^1, by [1])

$$\left.\begin{aligned} u^1 &= \sin \lambda \frac{z}{a} \sum_{p=0}^{\infty} u_p^1 \cos p\varphi \\ v^1 &= \sin \lambda \frac{z}{a} \sum_{p=0}^{\infty} v_p^1 \sin p\varphi \end{aligned}\right\} \quad (3)$$

Written in this form, it is an easy matter to derive from it the first "would-be" load B^1, for it can be calculated, (using (5, 1)), that for $M = \overline{M}$ (= unit moment) the partial load corresponding to

$$u_p^1 \cos p\varphi \sin \lambda \frac{z}{a}, \; v_p^1 \sin p\varphi \sin \lambda \frac{z}{a} \quad (4)$$

[1]) It may be remembered once more, that all deductions of the previous and the present section refer to the possibility of buckling, corresponding to a *fixed* value of q. Therefore all suffixes q can be suppressed.

amounts to:

$$\left.\begin{aligned}R &= -\frac{1}{\pi a^2}\frac{\partial^2 u_p^0 \cos p\varphi \sin \lambda \frac{z}{a}}{\partial z^2}\cos\varphi = \\ &= \frac{\lambda^2 \overline{M}}{2\pi a^4} u_p^0 \left[\cos(p-1)\varphi + \cos(p+1)\varphi\right]\sin\lambda\frac{z}{a} \\ \Phi &= -\frac{1}{\pi a^2}\frac{\partial^2 v_p^0 \sin p\varphi \sin \lambda \frac{z}{a}}{\partial z^2}\cos\varphi = \\ &= \frac{\lambda^2 \overline{M}}{2\pi a^4} v_p^0 \left[\sin(p-1)\varphi + \sin(p+1)\varphi\right]\sin\lambda\frac{z}{a}\end{aligned}\right\} \quad . \; . \quad (5)$$

The first iterated deformation, produced by such a load can be found with the aid of formulae (2, 4), (2, 8) and (2, 9), expressing that a load

$$\left.\begin{aligned}R_p &= a_p \cos p\varphi \sin\lambda\frac{z}{a} \\ \Phi_p &= b_p \sin p\varphi \sin\lambda\frac{z}{a}\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

gives rise to a deformation $u_p \cos p\varphi \sin\lambda\frac{z}{a}$, $v_p \sin p\varphi \sin\lambda\frac{z}{a}$, u_p and v_p being related with a_p and b_p by

$$\left.\begin{aligned}u_p &= \alpha_p a_p + \beta_p b_p \\ v_p &= \beta_p a_p + \gamma_p b_p\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

Herewith the data are at our disposal to develop a scheme of practical iteration. It follows from (5), that a deformation D^1, characterized by the numbers

$$u_0^0, u_1^0, v_1^0, u_2^0, v_2^0 \ldots, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

(where v_0^0 is suppressed in consequence of the fact, that the corresponding deformation $\sin 0.\varphi \sin\frac{\lambda z}{a}$ is identically zero), involves — except for the factor of multiplication $\frac{\lambda^2 \overline{M}}{2a}$ — a "would-be" load B^1, — characterized by the coefficients a_p and b_p, — the magnitude of which can be read from the last line of schedule (9)

first row	u_0^0	u_1^0	v_1^0	u_2^0	v_2^0	u_3^0	v_3^0
second row	u_1^0	u_2^0	v_2^0	u_3^0	v_3^0	u_4^0	v_4^0
third row	—	$2\,u_0^0$	—	u_1^0	v_1^0	u_2^0	v_2^0 (9)
sum up	u_1^0	$2\,u_0^0 + u_2^0$	v_2^0	$u_1^0 + u_3^0$	$v_1^0 + v_3^0$	$u_2^0 + u_4^0$	$v_2^0 + v_4^0$
fourth row	$\equiv a_0$	$\equiv a_1$	$\equiv b_1$	$\equiv a_2$	$\equiv b_2$	$\equiv a_3$	$\equiv b_3$

Apart from a single irregularity at the beginning, it is seen that the second and third row of this table are obtained, by shifting the first row to places to the left, resp. to the right. It can readily be seen from schedule (10), — in which the first row, denoted as fourth row is identical with the fourth row from schedule (9) — how the first iteration

	α_0	α_1	γ_1	α_2	γ_2	α_3	γ_3
		β_1		β_2		β_3	
fourth row	u_1^0	$2\,u_0^0 + u_2^0$	v_2^0	$u_1^0 + u_3^0$	$v_1^0 + v_3^0$	$u_2^0 + u_4^0$	$v_2^0 + v_4^0$
fifth row		$\times\alpha_1$	$\times\gamma_1$	$\times\alpha_2$	$\times\gamma_2$	$\times\alpha_3$	$\times\gamma_3$ (10)
sixth row		$\beta_1\times$	$\times\beta_1$	$\beta_2\times$	$\times\beta_2$	$\beta_3\times$	$\times\beta_3$
seventh row	u_0^1	u_1^1	v_1^1	u_2^1	v_2^1	u_3^1	v_3^1

$u_0^1, u_1^1, v_1^1, \ldots$, except for a factor $\frac{a^2}{B}$, tabularly can be calculated from $a_0, a_1, b_1, \ldots$ In practice, the two schedules (9) and (10) naturally will be united in a single one; the multiplication-factors α, β, γ from schedule (10) then being placed at the top of the whole table. It will be remarked, that the only numbers, which in behalf of the iteration have to be calculated beforehand, are the coefficients $\alpha_p, \beta_p, \gamma_p$, given by (**3**, 9) and (**3**, 10). From the set (u_i^1, v_i^1) a new set (u_i^2, v_i^2) is derived in the same way a. s. o., and it is evident that with an increasing number (n) of iterations, proportionality will be reached between (u_i^n, v_i^n) and (u_i^{n+2}, v_i^{n+2}), both sets being characteristic for the deformations D^n and D^{n+2}, the proportionality of which — for ever-increasing n — has been proved in section **6**.

One must hold in mind that, $u_0^1, u_1^1, v_1^1 \dots$ represent the first iteration but for a factor $\dfrac{\lambda^2 \bar{M}}{2\pi a^2 B}$.

Therefore if the process is to be carried on to the n^{th} and $(n+2)^{\text{th}}$ iteration to obtain sufficient proportionality between u_i^n and u_i^{n+2}, then the (approximate) factor of proportionality

$$\mu^* = \frac{u_i^n}{u_i^{n+2}} \quad \dots \dots \dots \dots \quad (11)$$

has to be multiplied by $\left(\dfrac{\lambda^2 \bar{M}}{2\pi a^2 B}\right)^{-2}$ to represent (approximatively) the squared value μ_1.

Consequently the required *smallest* buckling moment, corresponding to the assumed value q (or λ) is given by

$$M = \mu \bar{M} = \frac{2\pi a^2 B \sqrt{\mu^*}}{\lambda^2} \quad \dots \dots \dots \quad (12)$$

8. *The critical buckling moment.* In the heading of the two preceding sections, as well as in the foregoing sentence it has been explicitely expressed, that the iteration process relates to a fixed value of the parameter q (or λ), i. e. to a fixed number of longitudinal waves in the buckling-deformation. Therefore if the *minimum* or critical buckling moment is desired, the iteration process has to be repeated for a great number of values λ. It cannot be denied that this involves a rather tiresome business, as, above all, the calculation of the coefficients $\alpha_p, \beta_p, \gamma_p$ from (3, 9) and (3, 10) requires a great amount of ciphering.

Therefore it is of valuable help, that for large values of λ a prediction can be made about the relation between M and λ^2.

From examples, calculated for technical purposes, the authors have learnt, that for thin-walled and long tubes and for large values of λ^2 the coefficients β and γ are so small, that they scarcely influence the numerical end-result of the iteration.

Moreover, under the same conditions, α_p may be approximated by

$$\alpha_p = \frac{(p^2 + \lambda^2)^2}{(1-\nu^2)\,\lambda^4 + k\,\lambda^8} \quad \dots \dots \dots \quad (1)$$

A rough estimate makes it acceptable that $\sqrt{\mu^*}$ will be influenced only by the iterative numbers connected with low index-numbers p, and furthermore that $\sqrt{\mu^*}$ itself may be roughly estimated by

$$\sqrt{\mu^*} \backsim \frac{1}{2\,\alpha_p} \quad \dots \dots \dots \dots \quad (2)$$

so that for values of λ, high with respect to p, one may write approximately:

$$\sqrt{\mu^*} = \frac{1-\nu^2}{2} + \frac{k}{2}\lambda^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and consequently (comp. 7, (12))

$$M = 2\pi a^2 B \left\{ \frac{1-\nu^2}{\lambda^2} + k\lambda^2 \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This approximation — rough as it is — is of considerable importance. If for a practical case, the iteration has to be carried out for a number of values λ, and if one starts with small, but gradually increasing values of λ^2, one will find that M seems to be a monotonously *decreasing* function of λ^2. It is, of course, quite certain, that a minimum of M exists, but it is rather tiresome, that no data are at hand to estimate for which value of λ^2 this minimum is to be expected. Now it can be seen from (4), that a minimum of M exists, say for $\lambda^2 = \lambda_c^2$, and especially that for $\lambda^2 > \lambda_c^2$, M increases very rapidly, so that the character of the (M, λ^2)-curve is a hyperbolic one.

With this fact in view the iterative work can be considerably abbreviated by approximating the (M, λ^2)-curve by the formula

$$M = \frac{\alpha}{\lambda^2} + \beta\lambda^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Having roughly estimated λ_c^2 by using (4) as a first approximation, one shall take care to iterate M for two values of λ^2, one of which lies considerably *under* the value λ_c^2, the other lying *above* this limit, but in its neighbourhood. The two sets of corresponding values (M_1, λ_1^2) and (M_2, λ_2^2) enable the determination of the coefficients α and β in (5). The minimum value of M, then to be obtained from this formula, represents with great accuracy the critical value of M.

9. *The skew-symmetrical buckling.* Up to now only such buckling-deformations have been considered, which are symmetrical with respect to the plane in which the moments M act. However, still another type of deformations exists, represented by

$$\left.\begin{aligned} u &= \sin\lambda\frac{z}{a}\sum_{p=1}^{\infty} u_p \sin p\varphi \\ v &= \sin\lambda\frac{z}{a}\sum_{p=0}^{\infty} -v_p \cos p\varphi \\ w &= \cos\lambda\frac{z}{a}\sum_{p=1}^{\infty} w_p \sin p\varphi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It can be investigated along the same line as the symmetrical type, and therefore we restrain ourselves here to an abbreviated treatment.

Firstly we substitute the expressions (1) in the equations (4, 2), taking account of (4, 1), and obtain:

$$\left.\begin{aligned}
&\sum_{p=0}^{\infty}\Big\{u_p[1+k(\lambda^4+2\lambda^2p^2+p^4-2p^2+1)]+v_p\left[p+k\frac{3-\nu}{2}\lambda^2 p\right]-\\
&\quad -w_p\left[\nu\lambda+k\left(\lambda^3-\frac{1-\nu}{2}\lambda p^2\right)\right]\Big\}\sin p\varphi=\frac{M\lambda^2}{\pi a^2 B}\sum_{p=0}^{\infty}u_p\sin p\varphi\cos\varphi\\
&\sum_{p=0}^{\infty}\Big\{+u_p\left[p+k\frac{3-\nu}{2}\lambda^2 p\right]+v_p\left[p^2+\frac{1-\nu}{2}\lambda^2+k\frac{3(1-\nu)}{2}\lambda^2\right]-\\
&\quad -w_p\left[\frac{1+\nu}{2}\lambda p\right]\Big\}\cos p\varphi=\frac{M\lambda^2}{\pi a^2 B}\sum_{p=0}^{\infty}v_p\cos p\varphi\cos\varphi\\
&\sum_{p=0}^{\infty}\Big\{-u_p\left[\mu\lambda+k\left(\lambda^3-\frac{1-\nu}{2}\lambda p^2\right)\right]-v_p\left[\frac{1+\nu}{2}\lambda_p\right]+w_p\Big[\lambda^2+\\
&\quad +\frac{1-\nu}{2}p^2+k\frac{1-\nu}{2}p^2\Big]\Big\}\sin p\varphi=0.
\end{aligned}\right\}\quad . \quad (2)$$

Representing the coefficients of u_p, v_p, w_p in the left members of these equations by a_{11}, a_{12}, a_{13}; $a_{21}, a_{22}, a_{23}\ldots$ and replacing the products of goniometrical functions, occurring in the right members by sums of such functions, we find:

$$\left.\begin{aligned}
&\sum_{p=0}^{\infty}[a_{11}u_p+a_{12}v_p+a_{13}w_p]\sin p\varphi=\\
&\qquad=\frac{M\lambda^2}{\pi a^2 B}\sum_{p=0}^{\infty}u_p[\sin(p+1)\varphi+\sin(p-1)\varphi]\\
&\sum_{p=0}^{\infty}[a_{21}u_p+a_{22}v_p+a_{23}w_p]\cos p\varphi=\\
&\qquad=\frac{M\lambda^2}{\pi a^2 B}\sum_{p=0}^{\infty}v_p[\cos(p+1)\varphi+\cos(p-1)\varphi]\\
&\sum_{p=0}^{\infty}[a_{31}u_p+a_{32}v_p+a_{33}w_p]\sin p\varphi=0.
\end{aligned}\right\}\quad . \quad . \quad (3)$$

If then these equations are ordered with respect to *sin pφ*, and *cos pφ*, the following system of equations for u_p, v_p, w_p arises if all coefficients of these functions are put equal to zero:

$$\left.\begin{array}{l}
p=0 \quad a_{22}^{0}\, v_0 = \dfrac{M\lambda^2}{2\pi a^2 B}\, v_1 \\[2ex]
p=1 \left\{\begin{array}{l}
a_{11}^{1}\, u_1 + a_{12}^{1}\, v_1 + a_{13}^{1}\, w_1 = \dfrac{M\lambda^2}{2\pi a^2 B}\, u_2 \\[2ex]
a_{21}^{1}\, u_1 + a_{22}^{1}\, v_1 + a_{23}^{1}\, w_1 = \dfrac{M\lambda^2}{2\pi a^2 B}\, (2 v_0 + v_2) \\[2ex]
a_{31}^{1}\, u_1 + a_{32}^{1}\, v_1 + a_{33}^{1}\, w_1 = 0
\end{array}\right. \\[2ex]
\text{and, in general:} \\[2ex]
\begin{array}{l}\text{for arbitrary}\\ \text{integer } p\end{array} \left\{\begin{array}{l}
a_{11}^{p}\, u_p + a_{12}^{p}\, v_p + a_{13}^{p}\, w_p = \dfrac{M\lambda^2}{2\pi a^2 B}\, (u_{p-1} + u_{p+1}) \\[2ex]
a_{21}^{p}\, u_p + a_{22}^{p}\, v_p + a_{23}^{p}\, w_p = \dfrac{M\lambda^2}{2\pi a^2 B}\, (v_{p-1} + v_{p+1}) \\[2ex]
a_{31}^{p}\, u_p + a_{32}^{p}\, v_p + a_{33}^{p}\, w_p = 0
\end{array}\right.
\end{array}\right\} . \quad (4)$$

Putting

$$\triangle^p \equiv \begin{vmatrix} a_{11}^{p} & a_{12}^{p} & a_{13}^{p} \\ a_{21}^{p} & a_{22}^{p} & a_{23}^{p} \\ a_{31}^{p} & a_{32}^{p} & a_{33}^{p} \end{vmatrix} \quad \text{and} \quad \left.\begin{array}{r} a^p = \dfrac{A_{11}^{p}}{\triangle^p} \\[2ex] b^p = \dfrac{A_{21}^{p}}{\triangle^p} \\[2ex] c^p = \dfrac{A_{22}^{p}}{\triangle^p} \end{array}\right\} \quad . \quad . \quad . \quad . \quad (5)$$

(A_{ij}^{p} designing the minor determinant of the element a_{ij} in $\triangle^p$) we find

$$\left.\begin{array}{l}
u_0 = 0 \\[1ex]
v_0 = \dfrac{M\lambda^2}{2\pi a^2 B}\, \dfrac{v_1}{a_{2n}^{0}} = \dfrac{M\lambda^2}{2\pi a^2 B}\, \dfrac{v_1}{\dfrac{1-\nu}{2}\, \lambda^2 (1+3k)} \\[3ex]
u_1 = \dfrac{M\lambda^2}{2\pi a^2 B}\, \dfrac{[u_2 A_{11}^{1} + (2 v_0 + v_2) A_{21}^{1}]}{\triangle^1} \\[3ex]
v_1 = \dfrac{M\lambda^2}{2\pi a^2 B}\, \dfrac{[u_2 A_{12}^{1} + (2 v_0 + v_2) A_{22}^{1}]}{\triangle^1} \\[3ex]
u_p = \dfrac{M\lambda^2}{2\pi a^2 B}\, [a^p (u_{p-1} + u_{p+1}) + b^p (v_{p-1} + v_{p+1})] \\[3ex]
v_p = \dfrac{M\lambda^2}{2\pi a^2 B}\, [b^p (u_{p-1} + u_{p+1}) + c^p (v_{p-1} + v_{p+1})]
\end{array}\right\} \quad . \quad . \quad . \quad (6)$$

This again is a system of homogeneous linear equations, which only for special values of the bending moment $M = \mu \bar{M}$ admits a solution u_p, v_p.

Obviously from now the determination of the total characteristic numbers μ is identical with that, treated in the previous sections, and therefore we restrict ourselves in reproducing the scheme of iteration:

c^0	a^1	c^1	a^2	c^2	a^3	c^3
		b^1		b^2		b^3
v_0^0	u_1^0	v_1^0	u_2^0	v_2^0	u_3^0	v_3^0
v_1^0	u_2^0	v_2^0	u_3^0	v_3^0	u_4^0	v_4^0
—	—	$2v_0^0$	u_1^0	v_1^0	u_2^0	v_2^0
v_1^0	u_2^0	$2v_0^0+v_2^0$	$u_1^0+u_3^0$	$v_1^0+v_3^0$	$u_2^0+u_4^0$	$v_2^0+v_4^0$
$c^0 v_1^0$	$a^1 u_2^0$	$c^1(2v_0^0+v_2^0)$	$a^2(u_1^0+u_3^0)$	$c^2(v_1^0+v_3^0)$	$a^3(u_2^0+u_4^0)$	$c^3(v_2^0+v_4^0)$
	$b^1(2v_0^0+v_2^0)$	$b^1 u_2^0$	$b^2(v_1^0+v_3^0)$	$b^2(u_1^0+u_3^0)$	$b^3(v_2^0+v_4^0)$	$b^3(u_2^0+u_4^0)$
v_0^1	u_1^1	v_1^1	u_2^1	v_2^1	u_3^1	v_3^1

(7)

Calculations, made in connection with a technical problem, proved that the critical value of M for the anti-symmetrical case exceeds that of the buckling moment, which produces symmetrical deformation.

It may be taken for granted that this statement has general validity.