Anatomy. — *On the shape of froth chambers.* By S. T. BOK. (Communicated by Prof. M. W. WOERDEMAN.)

(Communicated at the meeting of October 26, 1940.)

In the tissues of the human body there is a great variety of froth structures and possibly the spatial networks formed in those tissues by protoplasmic and other rods are related to froth structures in so far that these rods may be found at the place of earlier froth edges. The quantitative investigation of these tissues is ever hampered by the lack of an adequate geometrical analysis of froth structures. We even find the most diverging communications in the literature about the shape of the froth chambers and hence a calculation of the most elementary properties of the shape of a froth — such as for instance the total edgelength in a given volume — is not even possible by approximation. Therefore we have investigated the shape of froth, the general results of which investigation are described in the following pages.

The properties of shape of froth are greatly different, according as the chambers are practically of the same size or not. When the difference in size is great the chambers are all more or less globular: in the spaces left between the larger globes (even when they nearly touch) the smaller globes can find sufficient room without materially affecting the shape of the larger ones. In froths the chambers of which are of the same size, on the other hand, the shape of the chambers approaches that of polyhedrons, i.e. bodies bounded by flat faces. For in that case the chambers are separated by thin, practically flat walls, meeting in approximately straight edges. Only froths with chambers of the same order of size will be discussed here.

To such a froth the two well known laws apply, viz.

1. three walls always meet in one edge (under approximately equal angles) and

2. four edges always meet in one point of junction (also under approximately equal angles).

From this it follows that the interfacial angle between two adjacent chamber walls is approximately 120° (i.e. about $^1/_3 \times 360°$) and that the four edges meeting in one point will spatially behave approximately like the four lines connecting the centre of a regular tetrahedron with its four vertices, so that the angle between two intersecting edges will be approximately 109° 28′ 16,......″ (being the angle between the lines in the tetrahedron mentioned, that is an angle with $cos = -^1/_3$).

The two rules mentioned follow from the conditions for equilibrium in a froth. A froth consists of two substances which do not mix, either two liquids, or a liquid and a gas, of which the one filling the chambers is disperse, i.e. divided into a number of non-coherent parts, separated by the second, coherent, so continuous, substance, which forms the walls. When the disperse substance fills only a small part of the space we use the term emulsion, otherwise it is a froth. From this it follows that the spatial structure of an emulsion gradually passes into that of a froth, as the relative volume of the dispersed substance (emulsion globules or contents of the froth chambers) increases. The greater the relative volume of the froth chambers, the thinner the walls and the more apparent the froth character. Very thin walls, however, can only exist when the surface tension is great as compared with the specific gravities. The structure of a "fine" froth therefore mainly depends on the surface tension and this is approximately the same in all froth walls. For the surface tension depends on the nature of the two adjacent substances and of the curve radius of the surface. In a froth with a thin wall whose chambers are practically the same size, the greater part by far of the surface is very little curved, only a narrow strip along the edges showing a more marked curvature. So the potential energy, accumulated in the partition planes depends principally upon the nature of the two substances (which nature is the same everywhere), and on the total size of those partition planes. The attempt to establish the equilibrium, which therefore, is mainly an attempt to occupy the smallest possible area, results in a tangential tension in the froth walls, which is approximately the same everywhere. The three walls meeting in one edge exert approximately the same degree of tension, and the three powers (approximately equally great) will only establish an equilibrium if they form approximately equal angles, so that together the three walls must form approximately equal interfacial angles. And where a following froth plane intersects the edge, the interfacial angle between this plane and two of the former being again ca. 120°, the four intersecting lines, the edges, again intersect under approximately equal angles.

In the literature various theories have been put forward concerning certain polyhedrons, which are supposed approximately to have the shape of froth chambers and which, when a great number of them is piled up, are supposed more or less to fulfil the conditions for equilibrium mentioned.

BUFFIN's theory, that this is the rhombic dodecahedron, is the least felicitous, although many after him supported his theory, the last of his followers being SEIFRIZ in 1930. We may derive this rhombic dodecahedron from the situation, found in regularly arranged globes. The regular arrangement (one of the two so-called densest globe cumulations) is characterized by the fact that any 12 globes touching a "centre" globe (all globes having the same size), have their centres in the centres of the edges of a cube whose centre of gravity coincides with the centre of the central globe (see fig. 1, left top corner). If we imagine such regularly arranged globes equally increasing in volume while their centres remain in the same place, they will be flattened against each other in such a way that each plane of contact is found in the place of the tangent plane which, in the original configuration, may be imagined through the point of contact between the two globes. In the case of maximal growth, therefore, they become polyhedrons, which may also be construed in the regular arrangement by construing the mutual tangent plane of two globes through each point of contact. The lines of intersection of those tangent planes are the edges of the polyhedron obtained and it appears that this polyhedron is a rhombic dodecahedron (see fig. 1, right top corner). BUFFIN verified this construction by tying peas in a bag and swelling them in water. The flattened peas frequently had the shape of a rhombic dodecahedron. That this, however, does not correspond to the formation of a froth, is in my opinion

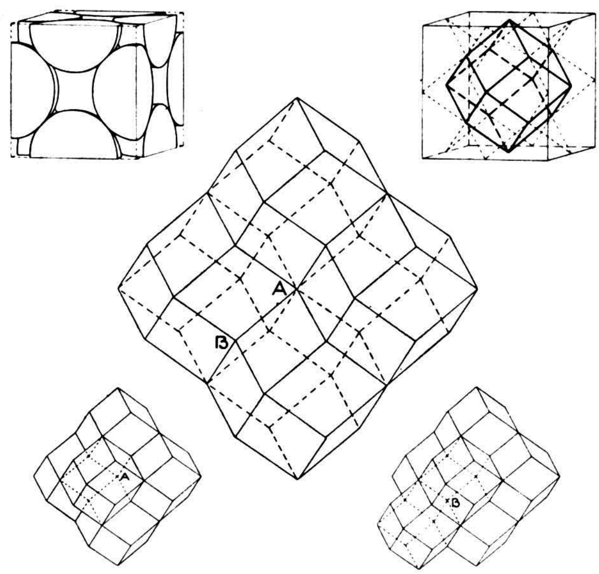

Fig. 1. Regularly piled up globes (see left top corner), when swollen form perfectly fitting rhombic dodecahedrons (see right top corner). In half the points of contact of the edges formed thus 8 edges meet (point A in the central figure), in the other points their number is 4 (point B). Continued piling up, namely, does not add edges to points A and B, as may be seen from the bottom figures.

clear from the fact that in this configuration of planes 4 edges meet in only half of all the points of junction. In the remaining points 8 edges meet (two of which always being in line, see A in fig. 1, centre). So this configuration does not fulfil the conditions of the second main rule of a froth (4 edges in every point of junction). And all the froths I examined fulfil this rule. Moreover, in froths I only found very few tetragons as walls and these resembled squares far more than the rhombs of a rhombic dodecahedron, which have angles of 109° 28′ 16,...″ and 70° 31′ 43,......″. Finally the size of one of these angles varies very much from the ideal value of 109° 28′ 16......″. So BUFFIN's thought construction and his experiment do not apply to froths, and this may be understood from the fact that the surface tension did not act as shaping power.

The same error is found in the starting point of Lord KELVIN (1887), who endeavoured to find a body which was to satisfy three requirements, 1. in cumulation the space should be entirely filled, 2. in every vertex 4 edges should meet and 3. the interfacial angles between the planes should be about 120°. The orthic tetrakaidecahedron mentioned by him fulfils these requirements. The two interfacial angles occurring in it are 109° 28′

16,......" and 125° 15′ 25,......". In this train of thought the surface tension does not play a part either and it follows that the result is not applicable to froth. This polyhedron is namely bounded by 8 regular hexagons and 6 squares, and in the froths some hexagons and squares do indeed occur, but they form a small minority. This solution does not fulfil the requirements either that the angles between the edges should be approximately 109° 28′ 16,......": in the orthic tetrakaidecahedron the angles are 90° and 120°.

Both (and similar) reasonings follow a faulty trend of thought: in searching for a polyhedron agreeing as much as possible with the rules of a froth, which on cumulation will completely fill the space. On nearer consideration it is this last requirement which appears to be contradictory to the fact that the shape of a froth depends on surface tensions.

The requirement of complete filling of space namely, implies the idea that the froth walls are flat. In such flat walls the surface tension must everywhere be exactly the same, the curvature being the same (see above) and in order to establish an equilibrium the interfacial angles between the planes must all be identical like the angles between the edges, which ought to be exactly 109° 28′ 16,......". But it is impossible for bodies bounded by flat faces to have only these angles, for the total of the angles of a polygon in a flat plane is always a complete multiple of 180°, and the angle mentioned of $109\frac{1}{2}$° and that of 180° have no common divisor. *So a froth with purely flat walls cannot exist.* Only curving of the faces can solve the problem, on the one hand such a curving being attended with differences in the surface tension on account of which the interfacial angles and the angles between the edges may be somewhat unequal and on the other hand, owing to the curvature, the total of the angles of one face not needing to be exactly a full multiple of 180°. As a matter of fact I found the walls in all the froths examined to be slightly curved, as appeared from the division of light on the glistening froth walls. In figures 3 and 4, two photographs of froth of soapsuds, the edges are seen to be slightly curved. The shape of the froth chambers, therefore differs from a polyhedron with ideal angles, not only in that the angles between its edges as well as its interfacial angles deviate somewhat from the ideal values, but also in that the walls are slightly curved. And that means that the polyhedron does not completely fill the space on cumulation: the interfacial angles of this polyhedron need not be exactly 120° and the three polyhedrons meeting in one edge need not fill the space completely. The only requirements we may make of the approximate polyhedron are those of the two main rules and it is exclusively these requirements which are based on the conditions for equilibrium.

From the properties of a froth we can deduce yet another indication of this polyhedron, although this is not a conditio sine qua non. This indication follows from the consideration that all the chambers adjacent to one specific chamber will in principle have the same relation to that one chamber, there is at least no a priori reason why one should behave differently in principle from its neighbours. This suggests that in the froth structure there is a certain tendency to make the chamber shape approach to a regular polyhedron.

In my opinion there are four arguments in favour of this being the regular dodecahedron.

In the first place both the interfacial angles and the angles between the edges of the regular dodecahedron have values approaching pretty near to the values mentioned. In the other regular polyhedrons the difference of those angles and interfacial angles is considerably greater, as may be seen from the following table.

	Interfacial angle	Angle between the edges
regular tetrahedron	70° 31′ 44″	60°
„ hexahedron	90°	90°
„ octahedron	109° 28′ 16″	60°
„ dodecahedron	**116° 33′ 54″**	**108°**
„ icosahedron	138° 11′ 23″	60°
in froth approximately	**120°**	**109° 28′ 16″**

So the interfacial angle of the regular dodecahedron differs not quite $3\frac{1}{2}°$ and the angle of the edges not quite $1\frac{1}{2}°$ from the angles which in flat walls would establish the equilibrium. In all the other regular polyhedrons the differences are considerably greater.

The second argument is that it is only in accumulation of the regular dodecahedron (after modification in the above sense) that 3 planes meet in one edge and 4 edges in one vertex. When regular dodecahedrons are piled up not all the walls can coalesce, a.o. because the three interfacial angles (of over $116\frac{1}{2}°$) which can meet in one edge, leave an opening of ca. $360° - 3 \times 116\frac{1}{2}° = 10\frac{1}{2}°$. But round one regular dodecahedron 12 other nearly regular dodecahedrons can be placed, exactly fitting into the space if some of its interfacial angles are taken a few degrees greater. It appears then that along each edge 3 planes meet and 4 edges in each vertex (compare fig. 2), which corresponds with the figures in the froth structure.

A pile of regular hexahedrons (cubes) will fit perfectly without any modification. But when the vertices coalesce 4 planes will meet in each edge and 6 edges in each vertex, so we do not get the numbers 3 and 4 which invariably are found in froth structure.

When piling up regular tetrahedrons, narrow clefts are always left: the interfacial angle is just over $70\frac{1}{2}°$ and so the five meeting in one edge, together fill $352\frac{1}{2}^\circ$ which is $7\frac{1}{2}^\circ$ short of the completion of 360°. When we imagine them slightly modified in shape (a.o. by enlarging the interfacial angles to averagely 72°), 5 walls will meet in each edge and 12 edges in each point of junction, which numbers deviate still more from the numbers of the froth structure.

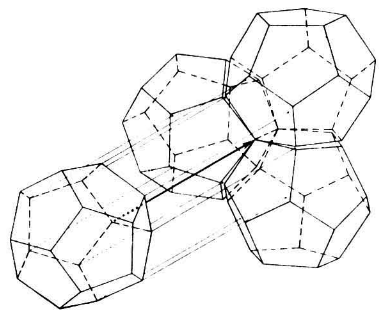

Fig. 2. Four regular dodecahedrons may be piled up nearly fitting round one point.

Regular octahedrons of equal size when piled up also leave clefts (interfacial angle $109\frac{1}{2}^\circ$, so too large for four and too small for five such octahedrons in one edge). Perfectly fitting piling may be obtained by reducing their interfacial angle from just over 109° to 90°. Six such bodies together form a rhombic dodecahedron [1]) and the space may be filled by piling up rhombic dodecahedrons. It is then seen that 4 walls meet along each edge and 8 edges in each vertex, which is again contradictory to the main rules of a froth.

Perfectly fitting accumulation of bodies approaching the shape of a regular icosahedron is not possible. Moreover, in a point of junction of a regular icosahedron 5 edges meet already, which is too great a number to fulfil the condition of the second main rule of froth structure. In accumulation this number would become greater still, and the discrepancy would be even more pronounced.

So the regular dodecahedron is the only regular polyhedron to which the shape of the froth chamber might approach under the condition that 3 planes must meet in one edge and 4 edges in one vertex.

The third argument is given by the consideration that on accumulation of globes of the same size 12 globes can always be in contact with one. On mutual flattening therefore bodies with 12 faces will be formed and if these bodies should approach the regular polyhedron this ought to be the regular polyhedron showing 12 planes, that is the regular dodecahedron.

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[1]) An octahedron has 6 vertices. Through 4 vertices a diagonal plane may be brought, bounded by four edges and having the shape of a square. The 6 octahedrons mentioned may now be placed so that a diagonal face of each coalesces with the wall of a cube. Of the planes within the cube 2 coalesce each time, of the planes outside the cube two are always in line. Together each two planes named secondly form one plane of the rhombic dodecahedron.

Fourth argument: by far the majority of the walls in a froth are pentagonal and the regular dodecahedron only consists of pentagons, the other regular polyhedrons are bounded by triangles (viz. the tetra-, octa- and icosahedron) or by squares (the hexahedron, the cube).

That most of the walls in a froth are pentagonal is a fact found in each of the froths investigated for this purpose (with chambers of about the same size). Meanwhile a superficial observation of such a froth may easily lead the investigator astray, as the surface chambers have a different shape. When, for instance, we examine a froth obtained by blowing air through a blowpipe into a layer of soapsuds in a cylinder glass (which froth entirely may fill the space of the glass above the liquid) we are at first struck by the lines along which the froth walls attach themselves to the glass wall and nearly all these lines form hexagons of a fairly regular shape. (The same is true of the lines connecting the walls between the upper chambers with the walls separating the chambers from the open air. If looked for, the same picture may be observed on the surface of the liquid at the bottom of the glass). In these surface chambers the conditions for equilibrium are different owing to the fact that on the outer surface of the froth there are only walls running in, not running out. This is the cause that walls between the surface chambers, especially in their most superficial parts, can place themselves pretty truly in such a position that all the interfacial angles are 120°, owing to which a section perpendicular to their edges must show a regular hexagon. And owing to the fact that these walls are perpendicular to the surface, this surface shows such a section.

The influence of the surface being less felt in those chambers that are situated farther away from the surface, we see the general froth structures discussed in this paper, only in the centre of the froth, at least when the cylinder glass is wide enough. In order to make them clearly visible I photographed a froth of soapsuds (made in a cylinder glass) by means of a lens with a comparatively small focus distance (5 cm) and a large opening (*F*: 3.5), owing to which only the interior gave a sharp picture. It is evident from the photographs that most of the walls are pentagonal (figures 3 and 4).

The preponderance of pentagons I found not only in all the froths consisting of a liquid and a gas, but also in froths of two liquids. Such a froth structure occurs in the yolk of an egg, for instance of a hen's egg. In it the yellow substance lies as a liquid in separate parts of about the same size, separated by walls of an uncouloured liquid. By boiling the yellow liquid coagulates: when carefully pushed or stirred with a small glass stick, a piece of hard boiled yolk, placed in a drop of water on an object glass will separate into a number of coagulated froth chambers. The photographs in fig. 5 demonstrate the fact that these bodies approach the shape of polyhedrons, mainly bounded by pentagons. (I am grateful to Prof. H. G. BUNGENBERG DE JONG for indicating this highly demonstrative example.)

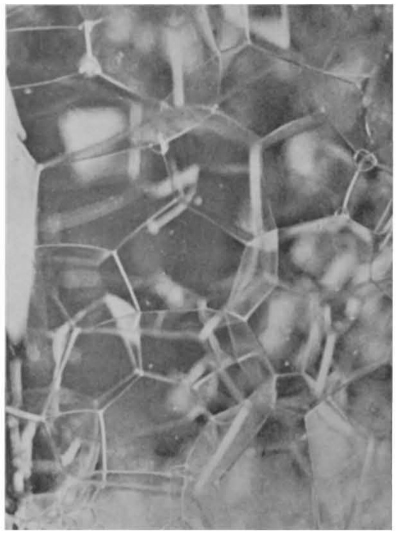

Fig. 3. The majority of walls in a froth of soapsuds are slightly curved pentagons.

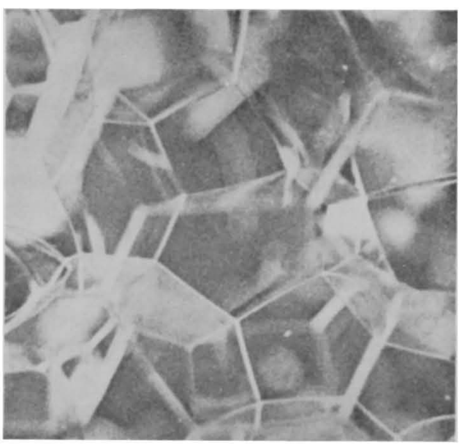

Fig. 4. A few walls in a froth of soapsuds are hexagons, fewer still are tetragons (one of each is seen in the photograph).

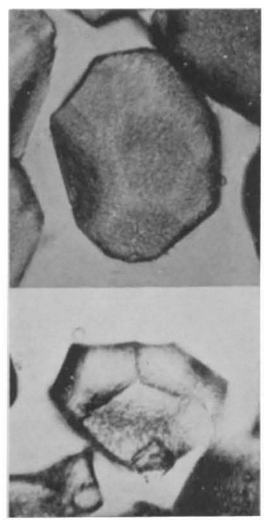

Fig. 5. Coagulated yolk of egg with pentagonal faces.

Meanwhile, when bodies approaching in shape the regular dodecahedron are piled up, a phenomenon occurs, in consequence of which not all the walls can be pentagonal, but in which some hexagons and fewer tetragons must occur, which is also seen to be the case in the froth photographs.

Starting from a regular dodecahedron a rind of 12 other dodecahedrons may be built round the first, which slightly deviate from the regular dodecahedron in the above sense. By each time fixing new edges to each vertex, so that 4 edges meet in each vertex and the edges each time enclose pentagons, we can for an unlimited space of time continue building new rinds round the former ones, all consisting of pentagon dodecahedrons. But the shape of these dodecahedrons will deviate the more from the original, central dodecahedron as they are further removed from it. For it appears that their tangential dimensions (their dimensions in a direction perpendicular to the line connecting their centre with that of the central dodecahedron) ever become greater. The cause of this lies in the discrepancy between the growth of the tangential surface of the rind (proportionate to the square of its radius) and the increase in the number of dodecahedrons per rind. This number is determined by the number and the position of the uncovered pentagons of the previous rind, and it increases more slowly than the surface, so that one dodecahedron must cover a larger part of the rind as the distance from the central dodecahedron increases. If this construction is carried out (I did it with twisted copper-wire), the space can ever be taken up by additional pentagon dodecahedrons, but in doing so either the outer dodecahedrons must be larger in all directions, which leaves their shape approximately that of regular dodecahedrons but which increases their volume, or their volume may be left almost unchanged, but then their radial dimensions (along the line running through their centre and that of the central dodecahedron), should be taken increasingly smaller. Then they are more and more flattened, finally becoming broad, thin slices, which cannot exist as froth chambers. In no single way can the space over a large extension be occupied by bodies which are everywhere approximations of regular dodecahedrons of the same size.

This difficulty may be met by placing between the dodecahedrons some bodies with a greater number of planes, in which the interfacial angles need only be changed a little. Without entering into details it may be demonstrated from fig. 6, an arrangement of nearly fitting regular dodecahedrons, enclosing a hexagon. On either side of this hexagon a body is bounded, which on completion of the accumulation would be a tetrakaidecahedron, bounded by 12 pentagons and 2 hexagons. In the same way other configurations produce polyhedrons with a few tetragons. Indeed this occurrence of hexagons and tetragons among the great majority of pentagons is found in the froths. Fig. 4 shows a hexagon by the side of a tetragon. As a rule such a tetragon appears to be approximately a square.

Finally we can discuss briefly the relation between the rhombic dodecahedron, which was supposed to be the result of a regular cumulation of globes and the regular dodecahedron which approaches the actual proportion in the froth.

The twelve globes in a regular arrangement touching some central globe, are not arranged equally round that central globe. If of the cube on whose edge centres lie the centres of the peripheral globes, we select a vertex, we see that 3 globes which touch

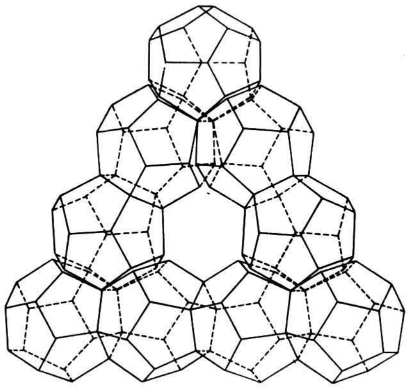

Fig. 6. Regular dodecahedrons can form nearly fitting hexagons.

each other are arranged round it (fig. 7 left). Round the centre of a face of that cube, on the other hand, there are 4 globes, between which a greater space is left open than between the three globes mentioned first. Hence the peripheral globes can be divided in a natural way into 4 groups, each consisting of 3 globes touching each other. Owing to the fact that these three globes also touch the central globe, their three centres and the centre of the central globe lie in the vertices of a regular tetrahedron, being the most stable configuration of 4 globes. When we imagine lines drawn through the centre of the central globe and through each of the centres of these tetrahedrons, these lines also intersect 4 non-adjacent vertices of the cube (see fig. 7 right top corner), owing to which they form mutually equal angles of 109° 28′ 16, ...″. They may therefore be conceived as the four trigonal polar axes of the regular system. The three peripheral globes of one group are equidistant from their axis and may be revolved round that axis. They then leave the regular arrangement, to recover it after a rotation of 120°.

In the regular arrangement it is seen that any peripheral globe always touches 4 other peripheral globes. The tangent plane between the peripheral globe selected and the central globe is therefore intersected by the four tangent planes between the central globe and the other peripheral globes mentioned, i.e. the tangent plane becomes a tetragon. The four peripheral globes touching the peripheral globe selected are not arranged regularly round it: they touch two by two, leaving two openings of the same size, situated diametrically opposite each other. The tetragon deducted above is therefore a rhomb. Hence the development of the rhombic dodecahedron. After the revolution round the polar axis described we find other proportions between the peripheral globes. It appears then that any peripheral globe has come to lie between 5 others: so in the case of growth in this configuration the globes will develop into dodecahedrons bounded by pentagons. These are no regular pentagons owing to the peripheral globes being dispersed irregularly round the selected one. But in this configuration the number of contacts between the peripheral globes is considerably smaller than in the regular one (viz. 12 instead of 36), for now only the three globes out of each group touch and there is no contact between globes of different groups. Therefore this configuration, which I propose to call tetratoid (see fig. 7 right bottom corner), leaves room enough for each peripheral globe to relinquish its contact with the other peripheral globes (a peripheral globe is no longer hemmed in by four other peripheral globes touching it). This opens the possibility that the 12 peripheral globes distribute themselves perfectly equally round the central globe (none of the peripheral globes touching another) and this configuration causes the regular dodecahedron. But the peripheral globes can also distribute themselves *approximately*

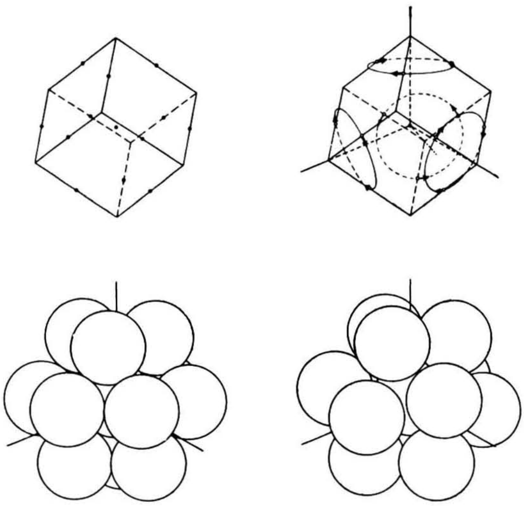

Fig. 7. The regular and the tetratoid arrangement of globes. The figures on the left demonstrate the regular, those on the right the tetratoid arrangement. The top figures show the position of the globe centres with regard to a cube. In the right figure they have revolved from the regular position round the four pictured trigonal polar axes by the length of the arrows (that is 15° in a positive sense).

regularly round the central one, and then the result is the polyhedron which is an *approximation* of the regular dodecahedron with which they have in common that they are both bounded by 12 pentagons.

From this systematic description it follows that the regular arrangement represents only one specific arrangement of the peripheral globes, and one with a maximum of mutual points of contact, which implies a minimum of freedom, and that from this, through rotation on the one hand, and through dispersion of the globes on the other, an infinite number of other positions may develop with fewer points of contact and hence with more freedom. The only other configuration with the maximal number of points of contact is the hexagon (different from the regular one in that out of the groups of three peripheral globes one isolated group has revolved 60° round its axis). Only in these two specific positions the 12 tangent planes are tetragons (in the regular one they are rhombs, in the hexagon they are rhombs and isosceles trapezia), in all the other positions they are pentagons.

The regular and the hexagonal arrangements can be constructed further than the first rind, while the structural plan remains exactly the same. This is not the case with the tetratoid and free arrangements. I have not succeeded in extending them with globes of the same size in such a way that the 12 peripheral globes surrounding any central globe are distributed so that each is surrounded by 5 others. This is in agreement with the fact mentioned above, that the extension with dodecahedrons, bounded by pentagons is not possible without changing the volume or the general shape of those dodecahedrons.

The proposition defended here, that the froth chambers are approximations of regular dodecahedrons has already been put forward by several investigators. Most of them only adduced the argument that among the regular polyhedrons it is only the regular dodecahedron which has interfacial angles and angles between the edges approximately agreeing to the angles which develop when 3 flat planes meet in one line under equal interfacial angles and when 4 straight lines meet in one point under equal angles. They did not account for the deviations from those angles and they did not refute the arguments of the authors who mentioned other polyhedrons as approximations of the shape of the froth chambers, so that the uninitiated reader had no arguments for a correct choice. I think I have provided such theoretical arguments in the above, and moreover to have shown that the regular dodecahedron is more in keeping with the actual proportions in a froth than other shapes mentioned in the literature. As for the latter, it was particularly shown that the majority of walls in a froth are *pentagons* (of the regular polyhedron only the dodecahedrons have pentagonal walls) and that among them there occur a few hexagons and tetragons, which must also occur now and then when approximately regular dodecahedrons are piled up.