Hydrodynamics. — *Some considerations on the development of boundary layers in the case of flows having a rotational component.* By J. M. BURGERS. (Mededeeling N⁰. 40 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hoogeschool te Delft.)

(Communicated at the meeting of December 28, 1940.)

1. The object of the following note is to consider some problems which arise in the study of the boundary layers formed in various parts of rotating pumps or ventilators. In particular we wish to give attention to the influence of the centrifugal forces upon the flow in such boundary layers [1]).

We will begin by deriving the equations for the flow in the boundary layer along a rotating wall in a general form. Special forms of these equations which can be obtained by introducing appropriate simplifications, then may be applied to the cases to be considered more in detail.

It is assumed throughout the following lines that the motion of the fluid is stationary with respect to a system of coordinates rotating with the angular velocity ω about an axis fixed in space. When a righthanded rectangular system x, y, z is used, the hydrodynamical equations can be put into the form:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+2\,(\omega_y\,w-\omega_z\,v)-\omega^2\,r\frac{\partial r}{\partial x}=-\frac{1}{\varrho}\frac{\partial p}{\partial x}+f_x \quad . \quad . \quad (1a)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+2\,(\omega_z\,u-\omega_x\,w)-\omega^2\,r\frac{\partial r}{\partial y}=-\frac{1}{\varrho}\frac{\partial p}{\partial y}+f_y \quad . \quad . \quad (1b)$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+2\,(\omega_x\,v-\omega_y\,u)-\omega^2\,r\frac{\partial r}{\partial z}=-\frac{1}{\varrho}\frac{\partial p}{\partial z}+f_z \quad . \quad (1c)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Here u, v, w are the components of the velocity of the fluid, measured with respect to the rotating system of coordinates; p is the pressure; f_x, f_y, f_z represent the frictional forces per unit of mass; $\omega_x, \omega_y, \omega_z$ are the components of the angular velocity ω along the coordinate axes; finally r is the distance from a point to the axis of rotation. As will be seen the

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[1]) For a general treatment of the theory of boundary layer flow the reader is referred to the chapters on this subject in S. GOLDSTEIN, Modern developments in fluid dynamics I, II (Oxford 1938), where an extensive summary is given of the work of a great number of authors.

terms depending upon the angular velocity in these equations respectively represent the compound centripetal acceleration, according to CORIOLIS' theorem, and the ordinary centripetal acceleration.

In applying these equations to the flow in a boundary layer we shall take the y-axis normal to the wall. Within the boundary layer v at most will be of the same order of magnitude as the boundary layer thickness δ; the same will apply to f_y. It follows that we may write equation (1b):

$$\frac{1}{\varrho}\frac{\partial p}{\partial y} \simeq \omega^2 r \frac{\partial r}{\partial y} - 2(\omega_z u - \omega_x w) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which shows that within the boundary layer p can vary at most with an amount of the order δ. It is customary to neglect this variation, and to consider p as independent of y within the boundary layer. The value of p to be used in the remaining equations (1a) and (1c) then is determined by the flow outside of the boundary layer, which by hypothesis is not influenced by frictional forces.

We will assume that the motion outside of the boundary layer, when considered with respect to *non* rotating coordinates, is free from vorticity. The components U, V, W of this motion, defined with reference to the rotating system, consequently must satisfy the relations:

$$\frac{\partial W}{\partial y} - \frac{\partial V}{\partial z} = -2\,\omega_x, \text{ etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It follows that the pressure p is determined by the equation:

$$p = const. - \tfrac{1}{2}\varrho\,(U^2 + V^2 + W^2) + \tfrac{1}{2}\varrho\,\omega^2 r^2 \quad . \quad . \quad . \quad . \quad (5)$$

The equations for the motion in the interior of the boundary layer now can be written:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\varrho}\frac{\partial p}{\partial x} + \omega^2 r\frac{\partial r}{\partial x} - 2\,\omega_y w + f_x \quad . \quad . \quad (6a)$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\varrho}\frac{\partial p}{\partial z} + \omega^2 r\frac{\partial r}{\partial z} + 2\,\omega_y u + f_z \quad . \quad . \quad (6b)$$

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In these equations p should be considered as a known function of x and z.

2. In sections **2.—6.** we shall be concerned with the flow in the boundary layer developing along a surface of revolution, and it will be assumed that the motion is wholly symmetrical with respect to the axis of rotation.

In applying equations (6) — (7) the origin of the coordinate system will be taken in a point of the surface along which the boundary layer develops (compare fig. 1); the x-axis shall be tangential to the intersection

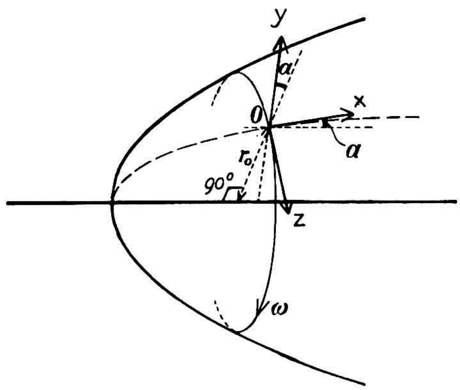

Fig. 1.

of the surface with a meridian plane; the z-axis shall be perpendicular to the meridian plane, in such a way that the system shall be righthanded. The angle between the x-axis and the axis of rotation will be denoted by a; the positive direction of rotation is the one indicated in the diagram.

Denoting the distance from O to the axis of rotation by r_0, we have: $r^2 = (r_0 + x \sin a + y \cos a)^2 + z^2$, from which, for $z = 0$:

$$\partial r/\partial x = \sin a \; ; \; \partial r/\partial y = \cos a \; ; \; \partial r/\partial z = 0.$$

Likewise for $z = 0$ we have:

$$\partial u/\partial z = - w \sin a/r \; ; \; \partial v/\partial z = - w \cos a/r \; ; \; \partial w/\partial z = + u \sin a/r.$$

As, moreover:

$$\omega_x = + \omega \cos a \; ; \; \omega_y = - \omega \sin a \; ; \; \omega_z = 0,$$

the equations valid outside of the boundary layer:

$$\partial U/\partial z - \partial W/\partial x = - 2\omega_y \; ; \; \partial V/\partial z - \partial W/\partial y = + 2\omega_x,$$

lead to the relation:

$$\partial W/\partial r + W/r = -2\omega,$$

both for $x = const.$ and for $y = const.$ It follows that:

$$W = C/r - \omega r. \qquad (8)$$

Here C is a constant, and C/r represents the tangential component of the flow measured with respect to fixed axes (the absolute tangential component). Hence eq. (5) simplifies to:

$$p = const. - \tfrac{1}{2}\varrho\,(U^2 + V^2 + C^2/r^2) \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

It is to be observed that we may put $V = 0$ at the exterior limit of the boundary layer, so long as we remain sufficiently close to the origin of the coordinate system.

Equations (6*a*), (6*b*), (7) consequently take the forms:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{\partial U}{\partial x} + \left\{\frac{(w + \omega r)^2}{r} - \frac{C^2}{r^3}\right\} \sin a + f_x \quad . \qquad (10a)$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} = \qquad -\left(2\omega + \frac{w}{r}\right) u \sin a + f_z \quad . \quad . \quad . \qquad (10b)$$

$$\frac{\partial v}{\partial y} = -\frac{1}{r}\,\frac{\partial\,(u\,r)}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

3. We first consider the case where $\omega = 0$ (no rotation) and $w = 0$, $C = 0$, so that there is no tangential velocity and the motion is confined to the meridian planes. Equations (10)—(11) then reduce to:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{\partial U}{\partial x} + f_x \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

$$\frac{\partial v}{\partial y} = -\frac{1}{r}\,\frac{\partial\,(ur)}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

We will assume that for a given form of the surface the boundary layer flow determined by these equations is wholly known, in particular also when the motion of the fluid in the boundary layer is turbulent. As is well known, in the immediate neighbourhood of the wall the left hand side of eq. (12), representing the effect of the inertia, vanishes, so that here the motion practically is determined by the "exterior driving force" $U(\partial U/\partial x)$ and the frictional force f_x. The frictional force is dependent upon the distribution of the velocity in the boundary layer, which distribution must satisfy the conditions that $u = 0$ at $y = 0$ and $u = U$ at $y = \delta$. In those cases where $\partial U/\partial x$ is negative, the "exterior" force tends to drive the fluid in the negative direction; when the boundary layer has become of sufficient thickness in order that the frictional force will have become rather small, a flow in the negative direction actually sets in close to the surface. The appearance of such a "counter flow" or back-flow brings about a separation from the surface of the original positive boundary layer flow, which separation soon becomes of such an amount that the concept of a boundary layer is no longer applicable [2]).

[2]) S. GOLDSTEIN, *l.c.* Vol. I, p. 56 and fig. 22.

It will be supposed that the course of this process is known for the boundary layer flow described by eqs. (12)—(13); our purpose will be to discuss the effects which must be expected to appear when the tangential velocity w and the rotation of the surface are superposed upon this flow.

4. We now re-introduce the tangential velocity w, keeping to the case of a non rotating surface, so that $\omega = 0$. The tangential velocity either can be produced by the action of the rotating blade system of the pump, or it may have been produced by a system of fixed guiding blades, placed ahead of the surface along which we will consider the boundary layer. In order that the flow outside of the boundary layer shall be irrotational, so that here $W = C/r$, the circulation around the blades which have produced the tangential velocity must be the same for all distances from the axis.

In this case (13) remains unchanged, whereas (12) is replaced by the two equations:

$$\left\{\begin{aligned} & u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{\partial U}{\partial x} + \left(\frac{w^2}{r} - \frac{C^2}{r^3}\right)\sin\alpha + f_x \quad . \quad . \quad (14\,a) \\ & u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} = -\frac{uw}{r}\sin\alpha + f_z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14\,b) \end{aligned}\right.$$

The case to be investigated can be compared to one in which $w = W = 0$, the surfaces limiting the field of flow remaining the same. So long as the boundary layer is sufficiently thin we may assume that the values of U and of $\partial U/\partial x$ are the same in both cases, which consequently differ only by the presence of the tangential velocity in one of them, and by the corresponding term in eq. (9) for the pressure. It will be seen that the term depending upon C in this equation adds to p an amount which increases with r.

The flow in the boundary layer now possesses a tangential component along with the component u. The distribution of the first one is governed by eq. (14b), which does not contain a term independent of w (as f_z is a function of w). Associated with eq. (14b) are the boundary conditions:

$$w = 0 \text{ at } y = 0; \; w = W = C/r \text{ at } y = \delta.$$

As there is no tangential pressure gradient we may expect that eq. (14b), so long as u is everywhere positive (which will be the case so long as the boundary layer does not separate from the surface), will determine a monotonous course of the function $w(y)$; at the same time it will give information about the variation of δ along a meridian section of the surface.

It is more important to consider eq. (14a). Here the quantity $(w^2/r - C^2/r^3)$ will be negative in the boundary layer, in particular close to the surface, where w vanishes. Hence to the "exterior force" $U(\partial U/\partial x)$ already considered before in **3.** there is added another "impressed force", the sign of which depends upon that of $\sin\alpha$. If r_0 increases when we move downstream along a meridian, $\sin\alpha$ will be positive, and this new force is a retarding one. Hence when $U(\partial U/\partial x)$ should be negative, the already existing tendency for separation will be increased; when $\partial U/\partial x$ is positive the stability of the flow at any rate is impaired and with a sufficiently large value of C a possibility for the occurrence of separation even may arise in this case.

We must be careful, however, in making conclusions of this kind, as there are other effects which also influence the behaviour of the boundary layer. In the first place as the velocity outside of the boundary layer has increased from U to the value $\sqrt{U^2 + W^2}$, the motion in the boundary layer which perhaps might have been laminar with $W = 0$, may have become turbulent. The transition to turbulence generally makes the boundary layer flow of a more stable nature and defers the tendency towards separation. This effect naturally loses its importance when the flow in the boundary layer should have been turbulent already from the beginning. — Another effect is connected with the *curvature* of the motion around the surface: in turbulent motion there is a tendency towards a decrease of turbulent intermixing and turbulent friction if it takes place along a convex wall; while the opposite tendency makes itself felt along a concave wall. Now a decrease of the turbulent frictional forces appears to increase the possibility for the occurrence of separation.

Hence there are several effects which must be taken into account, and it often may be difficult to determine their combined result. By way of example we may consider the case indicated in fig. 2, where it is assumed

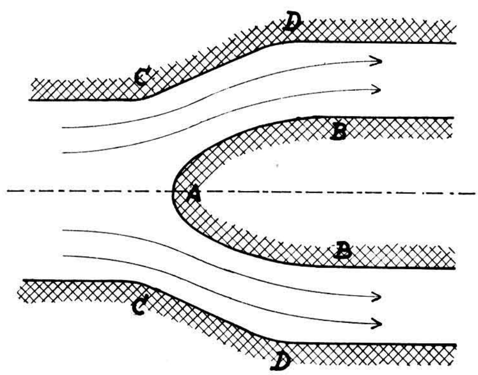

Fig. 2.

that U decreases when we pass from CC to DD, so that $U(\partial U/\partial x)$ is negative. When the motion in the boundary layers is turbulent already with $W = 0$, it is probable that the introduction of a tangential component will increase the chance for the occurrence of separation along the interior surface AB, as here we have both the influence of the negative term $(w^2/r - C^2/r^3) \sin \alpha$ and the decrease of turbulent intermixing due to the flow along a convex surface. — At the exterior surface CD the tendencies discussed oppose each other. Experimental evidence points to a preponderant influence of the increased turbulent intermixing and to a decrease of the chance for separation [3]).

In the case sketched in fig. 3 we have $\sin \alpha < 0$ along the part EF of the

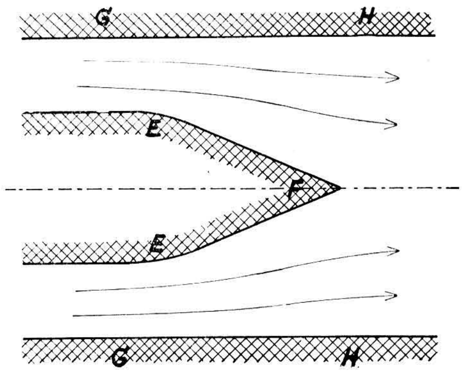

Fig. 3.

interior surface; hence the additional "impressed force" in eq. (14a) now drives forward the fluid, and somewhat should diminish the chance for the occurrence of separation which is caused by the negative value of $\partial U/\partial x$. The flow along the convex surface, however, brings a decrease of turbulent intermixing, and it is possible that this effect is preponderant and increases the tendency towards separation. It must be remarked, moreover, that although in practice there are sometimes found cases in which the diameter of the interior boundary surface decreases over a short distance, it is customary to remove as far as possible the rotational velocity of the fluid before it enters a space in which the interior surface retracts to zero radius (as at F in fig. 3), as the increase of the tangential velocity, which otherwise would be obtained near the axis, would bring about a loss of energy.

5. Next we turn to the case where the surface itself is rotating, with the angular velocity ω, while in the motion outside of the boundary layer there is no absolute tangential velocity, so that $C = 0$. Instead of eqs. (14a)—(14b) we obtain:

$$\left\{ \begin{aligned} & u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{\partial U}{\partial x} + \frac{(w + \omega r)^2}{r} \sin \alpha + f_x \quad . \quad . \quad (15\,a) \\ & u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} = - \frac{u}{r} (w + 2\,\omega r) \sin \alpha + f_z \quad . \quad . \quad . \quad (15\,b) \end{aligned} \right.$$

[3]) It is known that the efficiency of a diffuser (a pipe of increasing cross section, used for obtaining an increase of pressure through a decrease of the velocity) increases when a rotation is set up in the flow entering the diffuser. Compare *e.g.* H. PETERS, Energieumsetzung in Querschnittserweiterungen bei verschiedenen Zulaufsbedingungen, Ingenieur-Archiv 2, p. 92, 1931.

If we introduce the absolute tangential velocity:

$$w_1 = w + \omega r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

eq. (15*b*) can be transformed into:

$$u \frac{\partial w_1}{\partial x} + v \frac{\partial w_1}{\partial y} = - \frac{u w_1}{r} \sin a + f_z \quad . \quad . \quad . \quad . \quad . \quad (17)$$

As f_z just as well may be considered as a function of w_1 (within the boundary layer the derivative $\partial r/\partial y$ is negligible in comparison with $\partial w/\partial y$) this equation is identical in form with eq. (14*b*). The boundary conditions associated with it are:

$$w_1 = \omega r \text{ at } y = 0; \; w_1 = 0 \text{ at } y = \delta.$$

We may expect that again there will be found a monotonous course of w_1 between these values.

Now considering eq. (15*a*) it will be seen that the additional "impressed force" $(w + \omega r)^2/r \, . \sin a$ is of the opposite sign as the one appearing in eq. (14*a*). Hence if we return to a case as was sketched in fig. 2, where $\sin a > 0$, there now is obtained a forward driving force, which consequently diminishes the chance for the occurrence of separation. It will be evident that this forward driving force is due to the centrifugal effect of the fluid in the rotating boundary layer; hence it will always drive this fluid towards the section of the largest diameter. The presence of such an effect can be useful in cases as that of fig. 2, where owing to the enlargement of the section of the channel $\partial U/\partial x$ has a negative value.

6. It may be of interest to make an estimate of the magnitude of the rotational velocity necessary to balance the retarding influence due to a negative value of $\partial U/\partial x$. For this purpose it is convenient to transform eq. (15*a*) by integrating it with respect to y over the thickness of the boundary layer. Passing over the details we obtain:

$$\frac{d}{dx} \int_0^\delta dy \, r u^2 - U \frac{d}{dx} \int_0^\delta dy \, r u = \delta r U \frac{\partial U}{\partial x} + \sin a \int_0^\delta w_1^2 \, dy - \frac{r F_x}{\varrho} \quad (18)$$

where F_x denotes the frictional force per unit area exerted on the wall. We assume the following approximations, which in most cases of turbulent flow in the boundary layer may be sufficiently accurate for deriving an estimate:

$$u \cong U (y/\delta)^{1/7}; \quad w_1 \cong \omega r - \omega r (y/\delta)^{1/7}.$$

Equation (18) then can be transformed into:

$$\frac{7}{72} U^2 \frac{d}{dx} (r \delta) = \frac{r F_x}{\varrho} - r \delta \left\{ \frac{23}{72} U \frac{\partial U}{\partial x} + \frac{\sin a}{36} \omega^2 r \right\} \quad . \quad . \quad (19)$$

Hence in order that the retarding effect of a negative $\partial U/\partial x$ may be balanced, the value of the circumferential velocity ωr should satisfy the inequality:

$$(\omega r)^2 \sin \alpha \geqq -\frac{23}{2} r U \frac{\partial U}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

A similar calculation can be made for the more general case to which refers eq. (10a), where the absolute tangential velocity W outside of the boundary layer is not zero. In that case the second term on the right hand side of eq. (18) must be replaced by:

$$\sin \alpha \left\{ \int_0^\delta w_1{}^2 \, dy - \delta \frac{C^2}{r^2} \right\}.$$

For w_1 we must take the approximate formula:

$$w_1 \cong \omega r + (C/r - \omega r)(y/\delta)^{1/7},$$

and the second term of the right hand member of eq. (19) becomes

$$-r\delta \left\{ \frac{23}{72} U \frac{\partial U}{\partial x} + \frac{\sin \alpha}{36\, r} \left(\omega r - \frac{C}{r} \right) \left(\omega r + \frac{8\, C}{r} \right) \right\}.$$

When the absolute tangential velocity W is in the same direction as the angular velocity ω, it must be counted as positive (see fig. 1), so that also C is positive. If we write $C/r = k \, . \, \omega r$, it is found that condition (20) is replaced by:

$$(1 - k)(1 + 8\, k)(\omega r)^2 \sin \alpha \geqq -\frac{23}{2} r U \frac{\partial U}{\partial x}.$$

The left hand member has a maximum for $k = 7/16 \cong 0{,}44$; with this value we find:

$$\frac{81}{32} (\omega r)^2 \sin \alpha \geqq -\frac{23}{2} r U \frac{\partial U}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Experiments on the influence of rotation on the behaviour of the boundary layer along a sphere have been carried out by LUTHANDER and RYDBERG [4]). As the authors state the effects of the centrifugal force of the rotating boundary layer, tending to drive the fluid towards the equator of the sphere, clearly come out in many details of their results, but the whole

[4]) S. LUTHANDER und A. RYDBERG, Experimentelle Untersuchung über den Luftwiderstand bei einer um eine mit der Windrichtung parallele Achse rotierenden Kugel, Physik. Zeitschr. 36, p. 552, 1935.

phenomenon appears to be complicated very much by the influence of the rotational velocity upon the state of turbulence in the boundary layer. A comparison with eq. (20) cannot be made, as in the case of the flow along a sphere $\partial U/\partial x$ and *sin* α both are positive at the front side, and negative at the back side.

7. We now leave aside the consideration of boundary layers along surfaces of revolution, and turn to those parts of hydrodynamic or aerodynamic machinery, where the field, owing to the presence of the blades, no longer possesses perfect rotational symmetry. In this case the simplified equations (10*a*), (10*b*), (11) cannot be used, and we must revert to eqs. (6*a*), (6*b*), (7). The conclusions that can be drawn with regard to the flow in the boundary layers reduce to a discussion of the influence of those terms in the right hand members of (6*a*) and (6*b*) which are due to the rotation.

By way of example we may consider a highly simplified type of axial pump, as indicated in fig. 4. It has been supposed that the four blades are

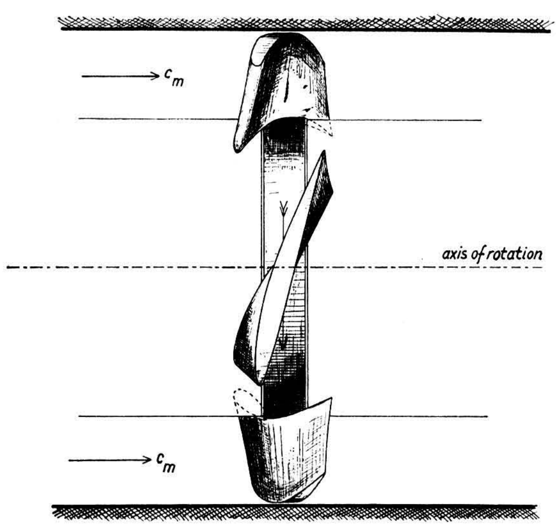

Fig. 4.

mounted on a disk (shaded in the diagram), which in the direction indicated by the arrow rotates between two fixed cylindrical surfaces. In the diagram the case has been imagined where the parts in the neighbourhood of the leading edges and of the trailing edges of the blades move over the non rotating surfaces.

We first consider the flow in the boundary layer along the cylindrical surfaces limiting the field. With reference to a rotating system of axes (introduced in order that the flow may appear stationary), the *y*-axis, as before, being normal to the surface, the motion of the fluid is governed by the equations:

$$\left.\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}&=-\frac{1}{\varrho}\frac{\partial p}{\partial x}+f_x \\ u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}&=-\frac{1}{\varrho}\frac{\partial p}{\partial z}+f_z \\ \frac{\partial v}{\partial y}&=-\frac{\partial u}{\partial x}-\frac{\partial w}{\partial z} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (22)$$

In these equations the angular velocity does not appear. Hence in the boundary layer along the rotating part of the interior surface (*i.e.* along the cylindrical surface of the disc or hub which carries the blades), where the boundary condition for the relative tangential velocity is $w = 0$, no direct influence of the rotation is to be perceived.

At the non rotating cylindrical surfaces the boundary condition for w becomes: $w = -\omega r$. Consequently along these surfaces there exists a tangential flow (in the negative direction with reference to the rotating coordinate system). In so far as such a flow is found in the spaces between the blades it will strike upon the under sides of the blades (which in the diagram have been taken as practically flat). The blades, so to speak, "scrape off" the fluid from the outer cylindrical boundary. and from those parts of the inner boundary which they overlap. Considering with reference to the cylindrical surfaces we may expect that this arrangement has the effect of a forward driving force upon the fluid in the boundary layer. On the other hand, considering with reference to the blades, we see that a certain amount of fluid is heaped up before the under sides of the blades, while fluid is drawn away from the upper sides (the convex sides in the diagram). These effects may have a certain influence upon the boundary layer flow along the blades themselves [5]).

In the boundary layers along the blades the centrifugal force will tend to produce a motion in the radial direction. The effect of this centrifugal force, however, cannot be discussed without giving attention at the same time to the pressure distribution generated by the motion of the blades. An approximate picture can be obtained by means of the reasoning indicated in the next section.

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[5]) Experiments on the influence of "scraping off" the boundary layer have been carried out by H. P. J. VERBEEK, Bepaling van den luchtweerstand van een bol met draaiende wieken, De Ingenieur 48, p. W 157, 1933. The rotating vanes moving over the back side of the sphere used in these experiments were straight, so that they did not exert a forward driving force upon the fluid in the boundary layer. It was found that at those REYNOLDS numbers, where the flow in the boundary layer still is laminar and separation normally occurs before the equator, a decrease of resistance, pointing to a delay of the separation, could be obtained by scraping off the boundary layer. A satisfactory interpretation of all the results found in these experiments is difficult, however, and perhaps more exact investigations will be necessary.

8. The blades of the pump can be considered as vortex sheets; in the case of blades bearing a constant circulation the vortex lines from the interior cylindrical surface run to the exterior one, often in a nearly radial direction, so that they are all cut by any cylindrical surface imagined between the interior and the exterior boundary of the field. We shall write γ for the strength of the vortex sheet per unit length, measured along the intersection of the blade with such a cylindrical surface; the integral of γ along the whole length of the line of intersection is equal to the circulation Γ around the blade. Usually γ has a maximum somewhere in the neighbourhood of the leading edge of the blade; in certain cases the maximum may be found actually on the leading edge, but with suitable forms of the blade it can be displaced more towards the centre for the normal working condition of the pump.

Now let W_{rel} be the effective relative velocity of the fluid in the neighbourhood of the blade, which quantity is given by:

$$W_{\text{rel}} = \sqrt{c_m^2 + (\omega r - \tfrac{1}{2} \triangle c_u)^2} \quad . \; . \; . \; . \; . \; . \; . \quad (23)$$

Here it has been assumed that upstream of the blades the absolute tangential velocity is zero, so that $C = 0$, there being only an axial velocity of constant magnitude c_m; $\frac{1}{2} \triangle c_u$ represents a correction due to the presence of the other blades, which with sufficient approximation is given by: $N\Gamma/4\pi r$ (N being the number of blades). It may be remarked that in a pump of the simple type sketched in fig. 4, the flow outside of the boundary layer practically is confined to cylindrical surfaces, so that any radial component may be neglected.

The pressure difference between the two sides of the blades is given by:

$$p_1 - p_2 = \varrho \, W_{\text{rel}} \, \gamma \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (24)$$

and when the blades are not too thick the excess of pressure at a point of the under side will have the value:

$$p_1 = \tfrac{1}{2} \varrho \, W_{\text{rel}} \, \gamma - \tfrac{1}{8} \varrho \, \gamma^2 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (25a)$$

and the defect of pressure at a point of the upper side:

$$p_2 = - \tfrac{1}{2} \varrho \, W_{\text{rel}} \, \gamma - \tfrac{1}{8} \varrho \, \gamma^2 \quad . \; . \; . \; . \; . \; . \; . \quad (25b)$$

Combining these amounts with the potential of the centrifugal forces we have, at the under side:

$$p_1 - \tfrac{1}{2} \varrho \, \omega^2 r^2 = \tfrac{1}{2} \varrho \, W_{\text{rel}} \, \gamma - \tfrac{1}{8} \varrho \, \gamma^2 - \tfrac{1}{2} \varrho \, \omega^2 r^2 \quad . \; . \; . \quad (26a)$$

and at the upper side:

$$p_2 - \tfrac{1}{2} \varrho \, \omega^2 r^2 = - \tfrac{1}{2} \varrho \, W_{\text{rel}} \, \gamma - \tfrac{1}{8} \varrho \, \gamma^2 - \tfrac{1}{2} \varrho \, \omega^2 r^2. \quad . \; . \quad (26b)$$

For a pump with simple blades in the normal working condition the value of γ, although being a function of the distance from the leading edge, does not change very much with r. Hence, as W_{rel}, according to (23), increases with r, it will be seen that at the upper side of the blade the fluid in the boundary layer is acted upon by a centrifugal field of force, and that the same effect may be found at the under side provided the value of γ is not too high. In the neighbourhood of the leading edge, however, locally rather high values of γ may occur (unless the blades are specially designed so as to avoid such values, which can be done for a particular working condition only); in that case the pressure excess can amount to $\frac{1}{2} \varrho W_{rel}^2$. Nevertheless the quantity $p_1 - \frac{1}{2} \varrho \omega^2 r^2$ will not increase with increasing values of r, provided the situation is not of such kind, that the high value of γ is to be found only at the outer part of the blade.

We may conclude that in general there will be a tendency towards a centrifugal flow of the fluid in the boundary layers along the blades, a marked one at the upper sides and a less significant one at the under sides.

9. Thus far we have given attention to motion in the radial direction only. When the distribution of the pressure over the blade surface should be known in detail, it will be possible to find out whether there may be marked tendencies for the appearance of a boundary layer flow in a direction opposite to that of the flow outside of the boundary layer. Such a tendency may be found at the upper side, from the trailing edge towards the region of minimum pressure; and in certain cases at the under side from the region of maximum pressure towards the leading edge. As is well known separation of the boundary layer flow may occur at the upper side when the pressure distribution is of an unsuitable type, as will occur when the angle of incidence of the effective relative velocity of the fluid upon the blade is too high. (In designing the blades care is taken that such a situation will not occur in the normal working range of the pump.)

It is further known that with a well rounded off leading edge there is no danger for separation at the under side of the blade when the region of high pressure is near to the leading edge. In those cases where the blade is designed so as to have the region of maximum pressure more towards the centre, the maximum pressure will not be so high as to cause serious danger for the occurrence of separation.

It must be noted finally that the compound centrifugal acceleration of CORIOLIS will produce a curvature of the streamlines in the boundary layer. In the case pictured in fig. 4 the flow along the under side of the blades *e.g.* continually will suffer a deviation to the right.