Hydrodynamics. — *On the distinction between irregular and systematic motion in diffusion problems.* By J. M. BURGERS. (Mededeeling No. 41 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hoogeschool te Delft.)

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1. The term "diffusion" is used to denote the gradual spreading of some form of matter through a medium of different nature, from regions where this matter is present in high concentration to regions from where it is absent, or where the concentration is low. The diffusion is the consequence of irregular movements to which the particles of the diffusing matter are subjected, and which may find their origin either in the molecular motion of the medium in which the particles are dispersed, or in coarser, turbulent motions present in this medium. The "irregularity" of the movements of the diffusing particles expresses itself in the circumstance that these movements do not show any preference for a particular direction: a given particle has the same chance to be driven in a positive as in a negative direction. When this equality of chances is not found, we say that along with the "irregular" movements there is present a certain "systematic" motion, the cause of which perhaps may be looked for in the action of forces which tend to drive the particles in a definite direction, or in the presence of a systematic flow in the medium in which the particles are dispersed.

The fact that the "irregular" movements to which the particles are subjected, notwithstanding their indifference as regards direction, can bring about a gradual transfer of matter from regions of high concentration to such of low concentration, is a statistical effect; it is a consequence of the circumstance that many more particles are set into motion from regions of high concentration than from other regions. It will be evident that in this process the intensity of the "irregular" movements likewise plays a part: when in a certain region this intensity is much higher than elsewhere, this circumstance in itself already will bring about that the diffusion from the said region will be more important than the diffusion towards it.

2. In order to translate these considerations into formulae we introduce the components u, v, w of the velocities of the particles. When the movements are exclusively of "irregular" character, the mean values of these components, taken over a suitable interval of time, should be zero:

$$\bar{u} = \bar{v} = \bar{w} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The intensity of the irregular movements can be defined by giving the mean values of the squares of these components: $\overline{u^2}$, $\overline{v^2}$, $\overline{w^2}$, or the mean value of the square of the resulting velocity: $\overline{u^2 + v^2 + w^2}$. These quantities either may be constant throughout the whole field, or they may change from point to point. — When at every point of the field

$$\overline{u^2} = \overline{v^2} = \overline{w^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the irregular motion is said to be isotropic; when these quantities differ from each other it is anisotropic.

As mentioned above the "irregular" motion may be accompanied by a systematic one, and a great number of problems refer to such cases. An important example is the combination of diffusion with a systematic downward movement due to the action of gravity on the particles; a common problem then is to find the concentration gradient which will balance the action of gravity and which thus can be stationary.

Clearly the proper distinction between "irregular" and "systematic" motions will play an important part in the analysis of such problems, for only when the "systematic" motion has been taken apart it will become possible to find the connection between diffusion, intensity of the "irregular" movements, and concentration gradient. It is the object of the present note to give some attention to this matter.

3. It will be expected that a "systematic" motion can be detected by taking the *mean values* of the velocity components u, v, w, as the "irregular" parts of these quantities then will be eliminated. In what way, however, these "mean values" should be defined?

In order to fix ideas we start from the assumption that the precise movement of every particle shall be known as a function of the time; thus *e.g.*, for the particle numbered i, the functions $u_i\,(t)$, $v_i\,(t)$, $w_i\,(t)$ will be considered as given. We then may take mean values of these quantities over a certain interval of time, say from $t = t_0$ until $t = t_0 + T$, and define the systematic part of the motion of the particle by the formula:

$$u_{is} = \frac{1}{T} \int\limits_{t_0}^{t_0+T} u_i \, dt, \quad \text{etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It will be evident that in such a mean value an amount of uncertainty, of unsharpness, must be taken for granted: the interval T must be of sufficient length compared with the "periods" of the irregular movements; on the other hand, when it is taken too long, the particle in the meantime may have wandered from one region of the field towards another, where the state of motion may be different. Some intermediate way must be found between these two extremes; where this is impossible a proper distinction between "systematic" and "irregular" motions cannot be arrived at.

It will be expected that when the same calculation is performed for various particles which at the moment t_0 were lying in the same region of the field, the result of formula (3) must be practically the same for all of them. (For simplicity it will be assumed that all particles are of the same kind.) It is possible therefore to obtain a more precise value of the "systematic" motion by taking the average of the quantity defined by (3) over a great number N of such particles:

$$u_s = \frac{1}{N} \sum_{i=1}^{i=N} u_{is} = \frac{1}{NT} \sum_{i=1}^{i=N} \int_{t_0}^{t_0+T} u_i \, dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where it is understood that the summation refers to the particles which at $t = t_0$ were lying in the same region, *i.e.* in the same element of volume ω of the field.

4. We may expect that when N is sufficiently large it will not be necessary to make the interval T very long. Hence the uncertainty present in the definition of the systematic motion when considered as a function of the time, to which reference was made above, can be reduced by considering many particles; however, in those cases where the distances between the individual particles are great, this will require the consideration of a rather extensive region and consequently may introduce an amount of unsharpness in the notion of the systematic motion as a function of the spatial coordinates.

The interval T nevertheless must not be reduced below a certain limit: it must be sufficiently ample in order that the motion of practically all particles considered will have suffered one or more "irregular" disturbances during this interval.

Indeed it will be evident that when T is diminished without limit in formula (4), we arrive at a quantity:

$$\bar{u} = \frac{1}{N} \sum_{i=1}^{i=N} u_i \, (t_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which is *not* identical with u_s. The quantity $\bar{u}$ defined by (5) is the mean value of the velocities at the instant t_0 of the particles which at that instant are lying in a given element of volume ω; in this quantity "irregular" diffusion and "systematic" flow are combined in such a way that they cannot be distinguished from each other. This can be seen for instance by imagining a case of pure diffusion, from a region A of high concentration towards a region B of low concentration, without any systematic action either of external forces or of flow of the medium in which the particles are dispersed. When at a given instant we consider the particles lying in an element of volume situated between these two regions, the chance of finding particles coming from the region A will be greater than the chance of finding particles from the region B; amongst the particles first mentioned, however, velocities in the direction from A to B will be preponderant, whereas the other type of particles will show a preponderance of velocities in the opposite direction. Hence the value of $\bar{u}$ given by (5) will not be zero, although we have supposed that there is no systematic motion.

When the mean velocity $\bar{u}$, and the other two components $\bar{v}$ and $\bar{w}$ defined by two similar equations, are multiplied by the average number of particles n per unit volume, to be found in the element of volume ω at $t = t_0$, we obtain quantities:

$$q_x = n\bar{u}, \qquad q_y = n\bar{v}, \qquad q_z = n\bar{w}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which evidently will represent the components of the "resulting mean current of particles" (measured as the number of particles which in unit time cross an element of surface of unit area). The components q_x, q_y, q_z will satisfy the equation of continuity:

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} = - \frac{\partial n}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In the particular case of stationary motion in the direction of the coordinate x only, this equation gives: $q_x = constant$. When the field is limited by impermeable boundaries, the current must be zero at these boundaries; hence we obtain: $q_x = 0$ everywhere, from which it follows that in this case: $\bar{u} = 0$. Such a case can be found when both pure diffusion and systematic motion are present, of such intensities that they balance each other.

5. Returning to the "systematic" motion as defined by eq. (4), it is possible to write down the following formula:

$$u_s = \frac{1}{N} \sum_{i=1}^{i=N} u_i(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where $t = t_0 + T$. This formula must give a good approximation to u_s, provided the interval T is not too short. It expresses the property that whereas the mean value of the velocities at the instant t_0 of the particles which at that instant are lying in a given element ω, will define the "mean resulting velocity of transportation" of the particles (related to the "mean resulting current"), on the other hand the mean value of the velocities at the instant $t_0 + T$ of the same particles will give the "systematic" velocity, provided the interval T is sufficiently long in order that practically all particles may have suffered one or more irregular disturbances during this interval.

6. The results obtained can be expressed in a different form, in which it is not necessary to know the functions $u_i(t)$, etc. for each particle explicitly, and which is more adapted to statistical considerations. Restricting for simplicity to motion in the direction of the coordinate x only, we introduce a frequency function:

$$f(\xi, l, u)\, d\xi\, dl\, du. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

which shall give the number of particles satisfying the following conditions:

(1) at the time t_0 the particles are located in the element ω, *i.e.* between ξ and $\xi + d\xi$;

(2) in the interval from t_0 until $t = t_0 + T$ the particles have moved over distances lying between l and $l + dl$;

(3) at the time $t = t_0 + T$ the velocities of the particles have values between u and $u + du$.

Theoretically, for the case of an unlimited field, we may assume that l and u may run from $-\infty$ to $+\infty$, although values exceeding certain limits practically never may occur.

Along with f we introduce a second function F defined by:

$$F(x, l, u) = f(x - l, l, u) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Then $F(x, l, u)\, dx\, dl\, du$ represents the number of particles satisfying the conditions:

(1) at the time t the particles are located in a certain element Ω extending from x towards $x + dx$;

(2) in the interval from $t_0 = t - T$ until t the particles had moved over distances lying between l and $l + dl$;

(3) at the time t the velocities of the particles have values between u and $u + du$.

Evidently we have:

$$f(x, l, u) = F(x + l, l, u) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10a)$$

With the aid of these functions the quantities considered above can be given in the form of integrals as follows (for shortness the limits $-\infty$, $+\infty$ have not been written out):

(I) The number of particles per unit of volume in the element ω at the time t_0 is given by:

$$n(\xi, t_0) = \int dl \int du\, f(\xi, l, u) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The same for the element Ω at t_0:

$$n(x, t_0) = \int dl \int du\, f(x, l, u). \quad . \quad . \quad . \quad . \quad . \quad . \quad (11a)$$

(II) The number of particles per unit of volume in the element Ω at t is given by:

$$n(x, t) = \int dl \int du\, F(x, l, u) = \int dl \int du\, f(x - l, l, u) \quad . \quad . \quad (12)$$

(III) The "mean resulting current" of particles through Ω at t is given by:

$$q(x, t) = \int dl \int du\, u\, F(x, l, u) = \int dl \int du\, u\, f(x - l, l, u) \quad . \quad (13)$$

From this quantity the "mean resulting velocity of transportation" of the particles in Ω at t is obtained by division through $n\,(x, t)$:

$$\bar{u}\,(x, t) = \frac{q\,(x, t)}{n\,(x, t)} = \frac{1}{n\,(x, t)} \int dl \int du\, u\, F\,(x, l, u) \quad . \quad . \quad . \quad (13a)$$

(IV) The "systematic velocity", defined at the time t, of the particles which at t_0 were located in the element ω, is obtained from:

$$(u_s)_\omega = \frac{1}{n\,(\xi, t_0)} \int dl \int du\, u\, f\,(\xi, l, u) \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The same for the particles which at t_0 were located in Ω:

$$(u_s)_\Omega = \frac{1}{n\,(x, t_0)} \int dl \int du\, u\, f\,(x, l, u) \quad . \quad . \quad . \quad . \quad (14a)$$

7. In order now to arrive at the distinction between "systematic motion" and "diffusion", and at the same time to obtain the connection between the latter and the intensity of the "irregular" movements, etc., we make use of a procedure which is analogous to one applied by FOKKER and by PLANCK in certain statistical considerations [1]). We introduce the hypothesis that the function $f\,(x - l, l, u)$ can be developed as follows:

$$f\,(x - l, l, u) = f\,(x, l, u) - l\,\frac{\partial f\,(x, l, u)}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Formula (13) can then be transformed into:

$$\left.\begin{aligned} q\,(x, t) &= \int dl \int du\, u\, f\,(x, l, u) - \int dl \int du\, u\, l\, \frac{\partial f\,(x, l, u)}{\partial x} = \\ &= n\,(x, t_0)\,.\,(u_s)_\Omega - \frac{\partial}{\partial x} \int dl \int du\, u\, l\, f\,(x, l, u) \end{aligned}\right\} \quad . \quad . \quad (16)$$

We shall write:

$$\overline{u\,l} = \frac{1}{n\,(x, t_0)} \int dl \int du\, u\, l\, f\,(x, l, u) \quad . \quad . \quad . \quad . \quad . \quad (17)$$

so that $\overline{u\,l}$ will be the mean value of the product $u\,l$ at the time t, calculated for the particles which at t_0 were located in Ω. With a simplification of notation (16) consequently may be written:

$$q = n_0\,.\,u_s - \frac{\partial}{\partial x}\,(n_0\,.\,\overline{u\,l}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

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1) A. D. FOKKER, Ann. d. Physik (IV) **43**, 812 (1914); and in particular: Sur les mouvements Browniens dans le champ du rayonnement noir, Archives Néerlandaises des Sciences exactes etc. (III A) **4**, 379 (1918).

M. PLANCK, Ueber einen Satz der statistischen Dynamik und seine Erweiterung in der Quantentheorie, Sitz. Ber. Berliner Akademie, 324 (1917).

In this equation a separation has been obtained between "systematic motion" and "diffusion". It will be seen that the intensity of the diffusion is dependent upon the magnitude of the quantity $\overline{ul}$, which is defined by (17). Now this is a quantity related to those which occur in the "mixture length theories" of transfer of momentum and diffusion in turbulent fluid motion. It is very gratifying that in eq. (18) it makes its appearance in a direct way, and moreover in the more exact form in which it has been defined by TAYLOR [2]).

8. It must be observed, however, that although the current q in eq. (18) refers to the instant t, the number of particles n_0 occurring in this equation refers to the instant $t_0 = t - T$. Now it has been remarked already before that the distinction between "systematic" and "irregular" motion properly can be made only when the interval T may be chosen so that during this interval the particles have not moved too far from their original positions; nor should the state of the field have changed appreciably in it. When this is the case, we may assume:

$$n(x, t) \cong n(x, t_0),$$

so that eq. (18) will become:

$$q \cong n \, . \, us - \frac{\partial}{\partial x}(n \, . \, \overline{ul}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18a)$$

In this form the equation has obtained a more convenient appearance. The degree of approximation involved in it is a necessary consequence of the unsharpness which is a priori connected with the definition of "systematic motion", as was shown before (see 3).

Nevertheless it may be useful to transform eq. (18) in a different way, which moreover makes it possible to eliminate from the second term a residue of "systematic motion", which still may be present in it.

For this purpose from eq. (11a) we deduce:

$$\begin{aligned} n(x, t_0) &= \int dl \int du \, F(x + l, l, u) = \\ &= \int dl \int du \, F(x, l, u) + \int dl \int du \, l \, \frac{\partial F(x, l, u)}{\partial x} = \\ &= n(x, t) + \frac{\partial}{dx}(n \, \overline{l}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19) \end{aligned}$$

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[2]) G. I. TAYLOR, Diffusion by continuous movements, Proc. London Math. Society (2) **20**, 196 (1922); Statistical theory of turbulence, Proc. Roy. Soc. London **A 151**, pp. 423 *seqq.* See also: S. GOLDSTEIN, Modern developments in fluid dynamics (Oxford 1938) I, pp. 205, 217.

In a similar way (17) can be transformed into

$$n\,(x,t_0)\,.\,\overline{u\,l} = \int dl \int du\; ul\,F\,(x+l,l,u) =$$

$$= \int dl \int du\; ul\,F\,(x,l,u) + \int dl \int du\; ul^2 \frac{\partial F\,(x,l,u)}{dx} =$$

$$= n\,(x,t)\,.\,\overline{ul} + \frac{\partial}{\partial x}(n\,.\,\overline{ul^2}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

In these expressions the mean values $\bar{l}$, $\overline{ul}$, $\overline{ul^2}$ refer to those particles which at the time t are situated in the element Ω (between x and $x+dx$).

By means of these results eq. (18) can be brought into the form:

$$q = n\,.\,u_s + u_s \frac{\partial}{\partial x}(n\,\bar{l}) - \frac{\partial}{\partial x}(n\,.\,\overline{ul}) - \frac{\partial^2}{\partial x^2}(n\,.\,\overline{ul^2}).$$

The last term, depending upon $\overline{ul^2}$, better can be neglected, as in making use of (15) we have already neglected quantities of the order l^2. Hence there remains:

$$q = n\,.\,u_s + u_s \frac{\partial}{\partial x}(n\,.\,\bar{l}) - \frac{\partial}{\partial x}(n\,.\,\overline{ul}). \quad . \quad . \quad . \quad . \quad (21)$$

This equation can be used to obtain an estimate of the error which may be present in eq. (18*a*). It can also be applied to eliminate the part due to the systematic motion present in $\overline{ul}$. Indeed, writing:

$$u = u_s + u' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

we have:

$$\overline{ul} = u_s\,.\,\bar{l} + \overline{u'l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and:

$$u_s \frac{\partial}{\partial x}(n\,\bar{l}) - \frac{\partial}{\partial x}(n\,.\,\overline{ul}) = -n\bar{l}\,\frac{\partial u_s}{\partial x} - \frac{\partial}{\partial x}(n\,.\,\overline{u'l}).$$

Hence (21) will become:

$$q = n\left(u_s - \bar{l}\,\frac{\partial u_s}{\partial x}\right) - \frac{\partial}{\partial x}(n\,.\,\overline{u'l}) \quad . \quad . \quad . \quad . \quad . \quad (24)$$

9. It may be useful to recapitulate the meaning of the quantities occurring in the last equation.

In the first place n is the number of particles per unit volume, to be found at the instant of time t in the element Ω (between x and $x+dx$). These particles at that instant possess velocities u; in the preceding interval from $t_0 = t - T$ until t they had moved over distances l; both u and l will vary from particle to particle.

The "systematic velocity" u_s has been defined by (14*a*); it is the mean value of the velocities u at the instant t of the particles which at t_0 were located in Ω, and thus it is not the mean value of the velocities of the particles which are to be found in Ω at t. The quantity u' has been defined as the difference $u - u_s$.

Finally $\bar{l}$ and $\overline{u'l}$ are mean values for the particles which are to be found in Ω at the instant t. The quantity $\bar{l}$ will differ from zero when a systematic motion is present. The fact that in eq. (24) there appears the combination

$$u_s - \bar{l}\,(\partial u_s/\partial x)$$

means that we should determine the "systematic velocity" rather for the element of volume from which, on the average, the particles started; it thus bears witness again to the unsharpness which is inevitable in these considerations.

10. In the preceding lines we have arrived at a direct deduction of the diffusion equation (24), which in cases where systematic motion is absent reduces to:

$$q_{\text{diffusion}} = -\frac{\partial}{\partial x}(n\,.\,\overline{u'l}) \; . \; . \; . \; . \; . \; . \; . \quad \textbf{(24a)}$$

In this deduction no assumption has been made concerning a "mean free path" of the moving particles; the only hypotheses applied are
(1) that there exists a frequency function f, as defined in eq. (9);
(2) that it is legitimate to make use of the development (15), which in case of need perhaps might be extended.

For the elaboration of a theory of the diffusion of particles in a liquid or in a gas in turbulent motion, the next step now must be to deduce expressions for u and l, and thence for the mean values considered above, starting from data concerning the state of turbulence of the moving medium, and from data concerning the action of this medium upon the particles [3]).

Appendix. — It may be that the application, first of the transformation (15) or (16), then of the one occurring in (19) and (20), which seems to be of opposite nature, is somewhat confusing at first sight, and the reader may ask why these transformations do not cancel each others effect. The following formulae perhaps can render this point more clear; at the same time they indicate how the development can be extended.

According to (13):

$$q = \int dl \int du\, u\, F(x, l, u).$$

Instead of writing this formula in the form:

$$q = n\bar{u},$$

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[3]) A few provisional considerations concerning the latter point had been brought forward by the author in a discussional remark to a paper by PRANDTL; see A. GILLES, L. HOPF, und TH. VON KÁRMÁN, Vorträge aus dem Gebiete der Aerodynamik u.s.w. (Aachen 1929), pp. 8—10. These considerations, however, are not sufficient for further work and a quantitative investigation is necessary, which it is hoped will be undertaken by Mr. TCHEN CHAN MOU in collaboration with the author.

which is of no use for further work, we write:

$$F(x, l, u) = f(x, l, u) - l\frac{\partial F}{\partial x} - \frac{1}{2}l^2\frac{\partial^2 F}{\partial x^2} - \frac{1}{6}l^3\frac{\partial^3 F}{\partial x^3}\ldots.$$

Hence the current q can be expressed in the form:

$$q = n_0\, u_s - \frac{\partial}{\partial x}(n\,\overline{ul}) - \frac{1}{2}\frac{\partial^2}{\partial x^2}(n\,\overline{ul^2}) - \frac{1}{6}\frac{\partial^3}{\partial x^3}(n\,\overline{ul^3})\ldots.$$

This formula has the advantage that the systematic velocity u_s is brought into evidence; it has the disadvantage that in the first term there appears the quantity n_0, defined by (11*a*), which refers to the instant t_0. However, as we have:

$$f(x, l, u) = F + l\,(\partial F/\partial x) + \text{ etc.},$$

it follows that:

$$n_0 = n + \frac{\partial}{\partial x}(n\bar{l}) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(n\overline{l^2}) + \frac{1}{6}\frac{\partial^3}{\partial x^3}(n\overline{l^3}) + \ldots.$$

The formula for q consequently can be brought into the form:

$$q = n\,u_s + u_s\frac{\partial}{\partial x}(n\bar{l}) - \frac{\partial}{\partial x}(n\,\overline{ul}) + \frac{1}{2}u_s\frac{\partial^2}{\partial x^2}(n\overline{l^2}) - \frac{1}{2}\frac{\partial^2}{\partial x^2}(n\,\overline{ul^2}) + \text{etc.}$$

This is the extended form of eq. (21). The separation of u_s and u', in the way as was indicated by eqs. (22) and (23), finally will lead to the extended form of eq. (24).