Mathematics. — *Conformal Differential Geometry. Curves in conformal euclidean spaces.* By J. HAANTJES. (Communicated by Prof. W. VAN DER WOUDE.)

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Introduction.

The purpose of this paper is to develop the differential geometry of curves in a conformal euclidean space R_n of dimension $n>2$, in particular to obtain the intrinsic equations of a curve, which determine the curve up to a conformal pointtransformation in R_n, and the analogue of the Frenet-Serret formulas.

As to the method employed in this note, no polyspherical coordinates are introduced, by which the conformal transformations can be brought in a linear form. BLASCHKE [1]) and others have treated the problem for R_2 and R_3 in that way looking upon a curve as a particular system of ∞^1 spheres. Our method on the other hand is based upon the well-known fact, again proved in § 1, that the conformal properties are those properties, which are unaffected by a conformal transformation of the fundamental tensor:

$$a'_{\lambda\varkappa} = \sigma^2 a_{\lambda\varkappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

with σ satisfying the differential equation

$$\nabla_\nu s_\mu = s_\nu s_\mu - \tfrac{1}{2} a_{\mu\nu} s_\lambda s^\lambda \quad , \quad s_\mu = \partial_\mu \log \sigma . \quad . \quad . \quad . \quad . \quad (2)$$

Although in ordinary differential geometry the extension to RIEMANNian spaces raises no essential difficulty as to the geometry of curves, it is not the case in conformal differential geometry, owing to the fact that in curved spaces a conformal property is defined as a property which is unaffected by a conformal transformation of the fundamental tensor: $a'_{\lambda\varkappa} = \sigma^2 a_{\lambda\varkappa}$, σ being an *arbitrarily chosen* function of the coordinates. In curved spaces it is impossible to impose on σ the condition (2), this equation being not completely integrable. This essential difference between the conformal geometry of flat spaces and curved spaces is the reason why we restrict ourselves in this paper to flat spaces.

I hope to treat the case $n=2$ in a later communication.

§ 1. *Preliminaries.*

Let $a_{\lambda\varkappa}$ be the fundamental tensor of an n-dimensional flat space R_n, in which the coordinates are denoted by $x^\varkappa$. We may of course assume

[1]) W. BLASCHKE, Vorlesungen über Differentialgeometrie III, Springer, Berlin.

that the coordinate system is a rectangular Cartesian one, but we will not confine ourselves to these systems. The transformation

$$y^{\varkappa} = f^{\varkappa}(x^{\lambda}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

is a conformal pointtransformation if

$$a_{\lambda\varkappa}(y) \frac{\partial y^{\varkappa}}{\partial x^{\mu}} \frac{\partial y^{\lambda}}{\partial x^{\nu}} = \sigma^2 a_{\mu\nu}(x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Suppose that in the region considered the transformation (3) is reversible. So we have

$$x^{\varkappa} = F^{\varkappa}(y^{\lambda}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We now pass to another coordinate system $(\varkappa')$ by the transformation

$$x^{\varkappa'} = F^{\varkappa'}(x^{\lambda}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where the functions $F^{\varkappa'}$ are identical with $F^{\varkappa}$. Then the coordinates of the point $y^{\varkappa}$ with respect to the system $(\varkappa')$ are from (6) and (5)

$$y^{\varkappa'} = F^{\varkappa'}(y^{\lambda}) = x^{\varkappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and the components of the fundamental tensor at the point $y^{\varkappa}$ with respect to $(\varkappa')$ are by (4)

$$a_{\lambda'\varkappa'}(y) = \sigma^2 a_{\lambda\varkappa}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Therefore if the pointtransformation (3) and the coordinate transformation (6) are carried out together each point keeps its own coordinates as a result of which the equations of transformed curves and surfaces remain the same. But from (8) we see that the fundamental tensor has become a factor σ^2. Now the space is assumed to be flat. Then the curvature affinor defined by

$$K_{\nu\mu\lambda}^{\cdot\cdot\cdot\varkappa} = 2\,\partial_{[\nu}\left\{ {}^{\varkappa}_{\mu]\lambda} \right\} + 2\left\{ {}^{\varkappa}_{\varrho[\nu} \right\}\left\{ {}^{\varrho}_{\mu]\lambda} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

will vanish both for $a_{\lambda\varkappa}$ and $\sigma^2 a_{\lambda\varkappa}$. This leads for $n > 2$ to the following equation for σ [2])

$$\nabla_{\mu} s_{\lambda} = s_{\mu} s_{\lambda} - \tfrac{1}{2} a_{\mu\lambda} s_{\varrho} s^{\varrho}, \qquad (s_{\mu} = \partial_{\mu} \log \sigma), \quad . \quad . \quad . \quad (10)$$

where the covariant differentiation is taken with respect to $a_{\lambda\varkappa}$. Therefore, if the coordinates are rectangular Cartesian, ∇_{μ} is identical with ∂_{μ}.

Conversely, to every solution σ of (10) a conformal pointtransformation can be found, such that this transformation creates this special σ [3]).

We thus have the result that in order to find the conformal properties of curves, surfaces etc., one may as well determine the properties which

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[2]) Cf. SCHOUTEN–STRUIK, Einführung in die neueren Methoden der Differentialgeometrie II, (Noordhoff 1938), p. 205. In this book our factor σ^2 is denoted by σ.

[3]) SCHOUTEN–STRUIK, Einführung II, p. 205.

are invariant under a conformal transformation of the fundamental tensor $a'_{\lambda\varkappa} = \sigma^2 a_{\lambda\varkappa}$, σ *satisfying the equation* (10).

The equation (10) is completely integrable. There exists also only one solution for which s_λ and σ have given values $(s_\lambda)_0$ and σ_0 at a given point $x_0^\varkappa$.

The CHRISTOFFEL-symbols computed from the tensor $a'_{\lambda\varkappa}$ are connected with the CHRISTOFFEL-symbols belonging to $a_{\lambda\varkappa}$ by the formulas

$$\left\{ {\varkappa \atop \mu\lambda} \right\}' = \left\{ {\varkappa \atop \mu\lambda} \right\} + A_\mu^\varkappa s_\lambda + A_\lambda^\varkappa s_\mu - a_{\mu\lambda} s^\varkappa, \quad \ldots \ldots \quad (11)$$

where $A_\lambda^\varkappa$ is the unit affinor.

§ 2. *The conformal parameter and the conformal orthogonal ennuple.* Let a curve be given by the equation

$$x^\varkappa = x^\varkappa(t), \quad \ldots \ldots \ldots \ldots \quad (12)$$

t being a scalar parameter. The arc-lengths with respect to $a_{\lambda\varkappa}$ and $a'_{\lambda\varkappa}$ will be denoted by s and s', the corresponding covariant derivatives along the curve by δ/ds and δ'/ds' respectively. (These covariant derivatives are identical with the ordinary derivates with respect to euclidean coordinate systems belonging to the corresponding fundamental tensor). If $p^\varkappa$ is a contravariant vector we have from (11)

$$\frac{\delta' p^\varkappa}{ds'} = \sigma^{-1} \left\{ \frac{\delta p^\varkappa}{ds} + \left(s_\mu \frac{dx^\mu}{ds} \right) p^\varkappa + (s_\mu p^\mu) \frac{dx^\varkappa}{ds} - a_{\mu\lambda} p^\mu \frac{dx^\lambda}{ds} s^\varkappa \right\}. \quad (13)$$

We now consider the vectors

$$i^\varkappa = \frac{dx^\varkappa}{ds}, \qquad q^\varkappa = \frac{\delta i^\varkappa}{ds}, \qquad r^\varkappa = \frac{\delta^2 i^\varkappa}{ds^2}. \quad \ldots \ldots \quad (14)$$

It may be shown by direct calculation using (10) and (13) that these vectors transform under a conformal transformation (1) of the $a_{\lambda\varkappa}$ in the following way

$$\left.\begin{array}{ll} a) & i'^\varkappa = \sigma^{-1} i^\varkappa \\ b) & q'^\varkappa = \sigma^{-2} \{ q^\varkappa + (s_\mu i^\mu) i^\varkappa - s^\varkappa \} \\ c) & r'^\varkappa = \sigma^{-3} [r^\varkappa + \{ 2 s_\mu q^\mu + (s_\mu i^\mu)^2 - s_\lambda s^\lambda \} i^\varkappa]. \end{array}\right\} \quad \ldots \quad (15)$$

From this we see that the two-direction determined by the osculating plane is not conformal invariant in contrast with the two-direction (local plane) determined by $i^\varkappa$ and $r^\varkappa$ (thus for a euclidean system by $\dot{x}^\varkappa$ and $\dddot{x}^\varkappa$). This local plane, which shall be called *conformal osculating plane*, is unique except when $i^\varkappa$ and $r^\varkappa$ have the same direction. If this happens at every point of the curve, the curve is a circle. We therefore for the present exclude the circles.

The vector defined by

$$v^{\varkappa} = r^{\varkappa} + a_{\mu\nu}\, q^{\mu}\, q^{\nu}\, i^{\varkappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

lies in the conformal osculating plane and is normal to $i^{\varkappa}$ as may be shown by multiplication by $i^{\varkappa}$. The transformation of $v^{\varkappa}$ follows from (15)

$$v'^{\varkappa} = \sigma^{-3}\, v^{\varkappa}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

From $v^{\varkappa}$ can be derived a scalar, which is multiplied by a power of σ under the transformation (1), namely

$$\varrho = \sqrt{a_{\lambda\varkappa}\, v^{\lambda}\, v^{\varkappa}}\,; \quad \varrho' = \sigma^{-2}\, \varrho. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This scalar enables us to define on the curve a parameter τ invariant under conformal transformations

$$\tau = \int \sqrt{\varrho}\, ds + c \qquad \text{(c constant)}. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

This parameter is called the *conformal parameter* or the *inversion-length* [4]) of the curve. It is as follows from the definition of ϱ of the *third order*.

Let us now turn to the orthogonal ennuple. The question is to complete the directions of $i^{\varkappa}$ and $v^{\varkappa}$ to a system of n mutually orthogonal directions, which are conformal invariant.

Consider a unit vector-field $p^{\varkappa}$ along the curve normal to $i^{\varkappa}$. Then the transformation of $p^{\varkappa}$ under a conformal transformation is

$$p'^{\varkappa} = \sigma^{-1}\, p^{\varkappa}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

From this relation and (13) follows at once the transformation of the covariant derivative of $p^{\varkappa}$:

$$\frac{\delta' p'^{\varkappa}}{ds'} = \sigma^{-2} \left\{ \frac{\delta p^{\varkappa}}{ds} + (s_{\mu}\, p^{\mu})\, i^{\varkappa} \right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

So we see that the local plane determined by $\dfrac{\delta p^{\varkappa}}{ds}$ and $i^{\varkappa}$ is a conformal invariant plane and therefore the direction in this plane normal to $i^{\varkappa}$ will be a conformal invariant direction. It is easy to see that this direction is determined by the vector

$$\frac{\delta p^{\lambda}}{ds}\, (A_{\lambda}^{\varkappa} - i^{\varkappa}\, i_{\lambda}) = \frac{\delta p^{\varkappa}}{ds} - \left(i_{\lambda}\, \frac{\delta p^{\lambda}}{ds} \right) i^{\varkappa}. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Now $p^{\varkappa}$ is supposed to be a unit vector. So $\dfrac{\delta p^{\varkappa}}{ds}$ and with it the vector (22) are orthogonal to $p^{\varkappa}$.

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[4]) H. LIEBMANN, Beiträge zur Inversionsgeometrie der Kurven, Münchner Berichte (1923).

Let us apply this result to the unit vector $\underset{2}{i}^{\varkappa}$ in the direction of $v^{\varkappa}$

$$\underset{2}{i}^{\varkappa} = \varrho^{-1} v^{\varkappa}. \quad \ldots \ldots \ldots \quad (23)$$

We get

$$\frac{\delta \underset{2}{i}^{\lambda}}{ds} (A_{\lambda}^{\varkappa} - i^{\varkappa} i_{\lambda}) = \varrho_2 \underset{3}{i}^{\varkappa} \quad \ldots \ldots \ldots \quad (24)$$

where $\underset{3}{i}^{\varkappa}$ is a unit vector normal to $i^{\varkappa}$ and $\underset{2}{i}^{\varkappa}$. The algebraic sign of ϱ_2 is not determined by the latter equation, but we may choose $\underset{3}{i}^{\varkappa}$ so as to make ϱ_2 positive. If $\varrho_2 \neq 0$ the same operator can be applied to $\underset{3}{i}^{\varkappa}$. In doing so we obtain

$$\frac{\delta \underset{3}{i}^{\lambda}}{ds} (A_{\lambda}^{\varkappa} - i^{\varkappa} i_{\lambda}) = - \varrho_2 \underset{2}{i}^{\varkappa} + \varrho_3 \underset{4}{i}^{\varkappa}, \quad \ldots \ldots \quad (25)$$

where again ϱ_3 is chosen to be non-negative. This equation defines a unit vector $\underset{4}{i}^{\varkappa}$ normal to $i^{\varkappa}$, $\underset{2}{i}^{\varkappa}$ and $\underset{3}{i}^{\varkappa}$, as may be seen by multiplying (25) by $i_{\varkappa}$, $\underset{2}{i}_{\varkappa}$ and $\underset{3}{i}_{\varkappa}$ respectively. We have f. i.

$$\frac{\delta \underset{3}{i}^{\lambda}}{ds} (A_{\lambda}^{\varkappa} - i^{\varkappa} i_{\lambda}) \underset{2}{i}_{\varkappa} = - \underset{3}{i}_{\lambda} \frac{\delta \underset{2}{i}^{\lambda}}{ds} = - \varrho_2 \quad \ldots \ldots \quad (26)$$

Proceeding in this way we get (if none of the quantities $\varrho_2, \varrho_3, \ldots$ appears to be zero)

$$\left.\begin{array}{l} \dfrac{\delta \underset{4}{i}^{\lambda}}{ds} (A_{\lambda}^{\varkappa} - i^{\varkappa} i_{\lambda}) = - \varrho_3 \underset{3}{i}^{\varkappa} + \varrho_4 \underset{5}{i}^{\varkappa} \\ \cdots\cdots\cdots\cdots \\ \dfrac{\delta \underset{n}{i}^{\lambda}}{ds} (A_{\lambda}^{\varkappa} - i^{\varkappa} i_{\lambda}) = - \varrho_{n-1} \underset{n-1}{i}^{\varkappa}. \end{array}\right\} \quad \ldots \ldots \quad (27)$$

If ϱ_j vanishes the set of equations breaks off with $\dfrac{\delta \underset{j}{i}^{\lambda}}{ds}$. The directions of the vectors $i^{\varkappa}, \underset{2}{i}^{\varkappa}, \ldots, \underset{n}{i}^{\varkappa}$ are as we have seen conformal invariant. Together they form at each point of the curve what is called the *conformal orthogonal ennuple*.

It may be noted that the quantities $\varrho_2, \varrho_3, \ldots$ are not conformal invariant. In fact we have from (25), (26) and (27)

$$\varrho_i' = \sigma^{-1} \varrho_i \qquad (i = 2, 3, \ldots, n-1) \quad \ldots \ldots \quad (28)$$

But this states that the quantities

$$h_i = \varrho^{-\frac{1}{2}} \varrho_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

are $n-2$ *conformal invariants of the curve.* Since the vectors $i^\varkappa, \underset{2}{i}^\varkappa, \ldots$ are unit vectors it is clear that the vectors defined by

$$j^\varkappa = \varrho^{-\frac{1}{2}} i^\varkappa = \frac{d x^\varkappa}{d\tau}; \; \underset{i}{j}^\varkappa = \varrho^{-\frac{1}{2}} \underset{i}{i}^\varkappa \qquad (i = 2, \ldots, n) \quad . \quad . \quad (30)$$

form a system of n *mutually orthogonal conformal invariant vectors* all of the same length.

§ 3. *The conformal covariant derivative along the curve.*

We have hitherto used the covariant derivative belonging to the metric $a_{\lambda\varkappa}$. As has been pointed out in § 1 this derivative is not invariant under conformal transformations, the CHRISTOFFEL-symbols transforming in a way given by (11). If, however, we have the disposal of a covariant vector Q_μ, which transforms as follows

$$Q'_\mu = Q_\mu - s_\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

the quantities

$$\Gamma^\varkappa_{\mu\lambda} = \left\{ {\varkappa \atop \mu\,\lambda} \right\} + Q_\mu A^\varkappa_\lambda + Q_\lambda A^\varkappa_\mu - a_{\mu\lambda} Q^\varkappa \quad . \quad . \quad . \quad . \quad (32)$$

are conformal invariant as a consequence of (11) and (31). They can be used as parameters of a conformal connection.

It may be noted that the parameters $\Gamma^\varkappa_{\mu\lambda}$ only define a covariant derivative along the curve if Q_μ is a function on the curve. It is of course necessary in this case to use the invariant parameter τ in order to obtain conformal invariant derivatives.

Now it can be shown at once from (1), (15 . b) and (18) that the Q_μ defined by

$$Q_\mu = a_{\mu\lambda} \left\{ \frac{\delta i^\lambda}{d s} + \frac{1}{2} \left(\frac{d}{d s} \log \varrho \right) i^\lambda \right\} = \frac{\delta i_\mu}{d s} + \frac{1}{2} \left(\frac{d}{d s} \log \varrho \right) i_\mu \quad (33)$$

transforms under conformal transformations in the right way. The covariant derivative along the curve defined by (32) and (33) will be denoted by D_τ. It is a conformal invariant derivative. We have f.i.

$$\left. \begin{aligned} D_\tau j^\varkappa &= \varrho^{-1} \left\{ \frac{\delta i^\varkappa}{d s} - \frac{1}{2} \left(\frac{d}{d s} \log \varrho \right) i^\varkappa + 2 \left(Q_\mu i^\mu \right) i^\varkappa - Q^\varkappa \right\} \\ &= \varrho^{-1} \left\{ \frac{\delta i^\varkappa}{d s} + \frac{1}{2} \left(\frac{d}{d s} \log \varrho \right) i^\varkappa - Q^\varkappa \right\} = 0. \;^{5)} \end{aligned} \right\} \quad . \quad (34)$$

[5]) Conversely the Q_μ may be defined by the assumption $D_\tau j^\varkappa = 0$. It may be remarked that this covariant derivative is not commutative with the process of raising and lowering of indices defined by the tensor $a_{\lambda\varkappa}$ because of $D_\tau a_{\lambda\varkappa} = -2 (Q_\mu i^\mu) a_{\lambda\varkappa} \neq 0$. The processes are commutative if for the raising and lowering of indices is used the conformal invariant tensor $b_{\lambda\varkappa} = \varrho a_{\lambda\varkappa}$.

The transformation of Q_μ under conformal transformations is given by (31). Now it lies on the surface to inquire whether it is possible to find a conformal transformation such that Q'_μ becomes zero. From (10) we derive as a necessary and sufficient condition the vanishing of

$$\frac{\delta Q_\lambda}{d s} - (i^\mu Q_\mu) Q_\lambda + \tfrac{1}{2} i_\lambda Q_\mu Q^\mu. \quad . \quad . \quad . \quad . \quad . \quad (35)$$

But from the definition of Q_λ we derive by a small calculation

$$\frac{\delta Q_\lambda}{d s} - (i^\mu Q_\mu) Q_\lambda + \tfrac{1}{2} i_\lambda Q_\mu Q^\mu = \varrho\,(\underset{2}{i_\lambda} + h_1\, i_\lambda) \quad . \quad . \quad . \quad (36)$$

where

$$h_1 = \frac{1}{2\varrho}\left\{\frac{d^2}{ds^2}\log\varrho - \frac{1}{4}\frac{d}{ds}\log\varrho - \frac{\delta i_\mu}{ds}\frac{\delta i^\mu}{ds}\right\}. \quad . \quad . \quad (37)$$

So we see that the expression (35) is different from zero. It may be shown by direct calculation from (15, b) and (18) that the quantity h_1 is a *conformal invariant of the curve.*

If the covariant derivative defined by (32) is used, the equation (36) can be written as follows

$$D_\tau Q_\lambda + j^\mu Q_\mu Q_\lambda - \tfrac{1}{2} j_\lambda Q_\mu Q^\mu = \varrho\, a_{\lambda\varkappa}\,(\underset{2}{j^\varkappa} + h_1\, j^\varkappa). \quad . \quad . \quad (38)$$

§ 4. *The "Frenet-Serret" formulas.*

We return to the equations (24), (25) and (27). This set of equations may be written in a form involving the conformal covariant derivative.

Let $p^\varkappa$ be a unit vector normal to $i^\varkappa$. Then $\varrho^{-\frac{1}{2}} p^\varkappa$ is conformal invariant. From (32) and (33) we get

$$D_\tau (\varrho^{-\frac{1}{2}} p^\varkappa) = \varrho^{-1}\left\{\frac{\delta p^\varkappa}{ds} + (Q_\mu p^\mu)\, i^\varkappa\right\}. \quad . \quad . \quad . \quad . \quad (39)$$

Moreover since (comp. footnote [5]))

$$D_\tau a_{\lambda\varkappa} = -2\,(Q_\mu j^\mu)\, a_{\lambda\varkappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

we have as a consequence of (34)

$$i_\varkappa D_\tau (\varrho^{-\frac{1}{2}} p^\varkappa) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

From (39) and (41) it follows that the vector (39) apart from a factor ϱ^{-1} is equal to the component of $\dfrac{\delta p^\varkappa}{ds}$ normal to $i^\varkappa$.

Applying this result to the set of equations (24), (25) and (27) we have from (29) and (30)

$$D_\tau \underset{2}{j^\varkappa} = h_2 \underset{3}{j^\varkappa}$$

$$\left.\begin{aligned} D_\tau \underset{3}{j^\varkappa} &= -h_2 \underset{2}{j^\varkappa} + h_3 \underset{4}{j^\varkappa} \\ &\cdots\cdots\cdots \\ D_\tau \underset{n}{j^\varkappa} &= -h_{n-1} \underset{n-1}{j^\varkappa}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

These equations together with the equations (34) and (38):

$$\left.\begin{aligned} D_\tau j^\varkappa = 0 \quad ; \quad j^\varkappa = \frac{dx^\varkappa}{d\tau} \\ D_\tau Q_\lambda + (j^\mu Q_\mu) Q_\lambda - \tfrac{1}{2} j_\lambda Q_\mu Q^\mu = \frac{a_{\lambda\varkappa}(h_1 j^\varkappa + j_2^\varkappa)}{a_{\mu\nu} j^\mu j^\nu} \end{aligned}\right\} \quad . \quad . \quad (43)$$

will be called *the conformal "Frenet-Serret" formulas.*

§ 5. *The intrinsic equations of the curve.*

In this section we shall show, that the curve is determined, to within conformal representations, by the expressions $h_1, \ldots, h_{n-1}$ in terms of the conformal parameter τ.

When we substitute the expressions for $h_1, h_2, \ldots, h_{n-1}$ in the Frenet-Serret formulas, a system is obtained of $N = n(n+2)$ differential equations of the form

$$\frac{d}{d\tau} \alpha_x = F_x(\alpha_1, \ldots, \alpha_N, \tau), \qquad (x = 1, \ldots, N) \quad . \quad . \quad . \quad (44)$$

where the α_x represent the N unknown quantities

$$x^\varkappa, Q_\lambda, j^\varkappa, \underset{2}{j^\varkappa}, \ldots, \underset{n}{j^\varkappa}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

We know that a system (44) admits a unique set of solutions, whose values for $\tau = 0$ are given arbitrarily. These values are chosen so that the vectors $\underset{1}{j^\varkappa} = j^\varkappa, \underset{2}{j^\varkappa}, \ldots, \underset{n}{j^\varkappa}$ for $\tau = 0$ are mutually orthogonal and all have the same length α_0. From (42) and (43) it follows

$$\frac{d}{d\tau}(a_{\lambda\varkappa} \underset{a}{j^\lambda} \underset{b}{j^\varkappa}) = D_\tau (a_{\lambda\varkappa} \underset{a}{j^\lambda} \underset{b}{j^\varkappa}) \begin{cases} = 0 \quad (a \neq b) \quad (a, b = 1, \ldots, n) \\ = -2(Q_\mu j^\mu) a_{\lambda\varkappa} \underset{a}{j^\lambda} \underset{a}{j^\varkappa} \quad (b = a) \end{cases} \qquad (46)$$

which proves that for all values of τ the solutions $j^\varkappa$ are mutually orthogonal and have the same length α. Furthermore it has to be shown that this system of orthogonal vectors is identical with the conformal ennuple of the curve given by the solution

$$x^\varkappa = x^\varkappa(\tau). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

From the definition of conformal invariant ennuple (§ 2) and our system of differential equations (42) and (43) it is easy to see that this will be the case if the parameter τ in (47) is the conformal parameter of the curve.

In order to prove that τ is indeed the conformal parameter we introduce the arc-length s defined by the fundamental tensor $a_{\lambda\varkappa}$. Since

$$a_{\lambda\varkappa} j^\lambda j^\varkappa = a_{\lambda\varkappa} \frac{dx^\lambda}{d\tau} \frac{dx^\varkappa}{d\tau} = \alpha^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

τ and s are connected by the equation

$$d\tau = a^{-1}\, ds. \qquad (49)$$

Then equation (43) may be written as follows

$$\left.\begin{array}{ll} a) & \dfrac{\delta}{ds}\, i^{\varkappa} = Q^{\varkappa} - 2\,(Q_\mu\, i^\mu)\, i^{\varkappa} - \left(\dfrac{d}{ds}\, log\; a\right) i^{\varkappa}, \qquad \left(i^{\varkappa} = \dfrac{dx^{\varkappa}}{ds}\right) \\ b) & \dfrac{\delta}{ds}\, Q_\lambda = (Q_\mu\, i^\mu)\, Q_\lambda - \tfrac{1}{2}\, i_\lambda\, Q^{\varkappa}\, Q_{\varkappa} + a^{-2}\,(h_1\, \underset{2}{i_\lambda} + i_\lambda), \qquad (\underset{2}{i_\lambda} = a^{-1}\, \underset{2}{j_\lambda}) \end{array}\right\} \qquad (50)$$

where $\underset{2}{i^{\varkappa}}$ is a unit vector. Multiplying (50.a) by $i_{\varkappa}$ we get

$$Q_\mu\, i^\mu = -\frac{d}{ds}\, log\; a. \qquad (51)$$

Using the relations (50.b) and (51) it may be shown by direct calculation from (50.a) that

$$\frac{\delta^2}{ds^2}\, i^{\varkappa} + a_{\lambda\mu}\, \frac{\delta i^\mu}{ds}\, \frac{\delta i^\lambda}{ds}\, i^{\varkappa} = a^{-2}\, \underset{2}{i^{\varkappa}}. \qquad (52)$$

So the vector $v^{\varkappa}$ (comp. (16)) belonging to the curve (47) is equal to $a^{-2}\, \underset{2}{i^{\varkappa}}$ from which it follows by (18) that

$$\varrho = a^{-2}. \qquad (53)$$

A comparison between (19) and (49) then shows that the parameter τ is indeed the conformal invariant parameter.

The values of Q_μ and a for $\tau = 0$ may be chosen arbitrarily. The question arises what happens if we choose for these quantities other values. It is always possible to express the new values in terms of the old ones in the following form

$$(\overline{Q}_\mu)_0 = \sigma_0\,(Q_\mu)_0 + (s_\mu)_0 \quad ; \quad \overline{a}_0 = \sigma_0^{-1}\, a_0 \text{ or } (\underset{a}{\bar{j}^{\varkappa}})_0 = \sigma_0^{-1}\,(\underset{a}{j^{\varkappa}})_0. \qquad (54)$$

Generally these values will lead to another solution for $x^{\varkappa}$ and therefore to another curve. In the following it will be shown that this latter curve, obtained with the initial values (54), is a conformal representation of the curve (47).

Suppose that the coordinate system $(\varkappa)$ is euclidean with respect to the fundamental tensor $a_{\lambda\varkappa}$. The $(s_\mu)_0$ and σ_0 in (54) determine a function σ, namely the unique solution of the differential equation (10) for which s_μ and σ have the values $(s_\mu)_0$ and σ_0 at the point $x_0^{\varkappa}$ $(\tau = 0)$. We now pass to the coordinate system $(\varkappa')$ which is euclidean with respect to $\sigma^2\, a_{\lambda\varkappa}$ and whose directions of the coordinate lines at $x_0^{\varkappa}$ coincide with the directions of the coordinate lines of the system $(\varkappa)$. So we have

$$a_{\lambda'\varkappa'} = \sigma^{-2}\, \delta_{\lambda'\varkappa'} \begin{cases} = 0 \;, & \lambda' \neq \varkappa' \\ = \sigma^{-2}, & \lambda' = \varkappa' \end{cases} \qquad (55)$$

$$(A_{\varkappa}^{\varkappa'})_0 = \left(\frac{\partial x^{\varkappa'}}{\partial x^{\varkappa}}\right)_0 = \sigma_0\, \delta_{\varkappa}^{\varkappa'}. \qquad (56)$$

It may be noted that the transformation

$$x^{\varkappa'} = x^{\varkappa'}(x^{\lambda}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

is a conformal coordinate transformation.

By (55) the CHRISTOFFEL-symbols of $a_{\lambda\varkappa}$ with respect to the system $(\varkappa')$ are

$$\left\{ {\varkappa' \atop \mu' \lambda'} \right\} = - s_{\mu'} A_{\lambda'}^{\varkappa'} - s_{\lambda'} A_{\mu'}^{\varkappa'} + s^{\varkappa'} a_{\mu'\lambda'} \quad . \quad . \quad . \quad . \quad . \quad (58)$$

where $s_{\mu'}$, as a consequence of (10) satisfies the equation

$$\partial_{\nu'} s_{\mu'} + s_{\nu'} s_{\mu'} - \tfrac{1}{2} a_{\nu'\mu'} s_{\varrho'} s^{\varrho'} = 0. \quad . \quad . \quad . \quad . \quad . \quad (59)$$

The parameters of the conformal connection are therefore from (32)

$$\Gamma_{\mu'\lambda'}^{\varkappa'} = (Q_{\mu'} - s_{\mu'}) A_{\lambda'}^{\varkappa'} + (Q_{\lambda'} - s_{\lambda'}) A_{\mu'}^{\varkappa'} - (Q^{\varkappa'} - s^{\varkappa'}) a_{\mu'\lambda'} \quad . \quad . \quad (60)$$

From this we see that the equations (42) written with respect to $(\varkappa')$ can be obtained from the same equations with respect to $(\varkappa)$ by replacing Q_μ by $Q_{\mu'} - s_{\mu'}$. The same is true for the first equation (43). We proceed to show that the same can be said of the second equation (43):

$$\frac{dQ_\lambda}{d\tau} - (j^\mu Q_\mu) Q_\lambda + \tfrac{1}{2} j_\lambda Q_\mu Q^\mu = \frac{a_{\lambda\varkappa}(h_1 j^\varkappa + \overset{2}{j}{}^\varkappa)}{a_{\mu\nu} j^\mu j^\nu} \quad . \quad . \quad . \quad (61)$$

With respect to the system $(\varkappa')$ the left hand side of this equation runs as follows

$$\frac{dQ_{\lambda'}}{d\tau} - \Gamma_{\mu'\lambda'}^{\varkappa'} Q_{\varkappa'} j^{\mu'} + (j^{\mu'} Q_{\mu'}) Q_{\lambda'} - \tfrac{1}{2} j_{\lambda'} Q_{\mu'} Q^{\mu'} \quad . \quad . \quad . \quad (62)$$

As a consequence of (59) and (60) this expression may be written in the form

$$\frac{d(Q_{\lambda'} - s_{\lambda'})}{d\tau} - j^{\mu'} (Q_{\mu'} - s_{\mu'})(Q_{\lambda'} - s_{\lambda'}) + \tfrac{1}{2} j_{\lambda'} (Q_{\mu'} - s_{\mu'})(Q^{\mu'} - s^{\mu'}) \quad (63)$$

which is the same expression in $Q_{\mu'} - s_{\mu'}$ as the left hand side of (61) is in Q_μ.

The initial values of the vectors $\underset{a}{j}{}^{\lambda'}$ and $Q_{\mu'} - s_{\mu'}$, are by (54) and (56)

$$\left. \begin{array}{ll} \underset{a}{j}{}^{\varkappa'}: & \sigma_0 (\underset{a}{\bar{j}}{}^\varkappa)_0 = (\underset{a}{j}{}^\varkappa)_0 \\ Q_{\mu'} - s_{\mu'}: & \sigma_0^{-1} (\bar{Q}_\mu - s_\mu)_0 = (Q_\mu)_0. \end{array} \right\} \quad . \quad . \quad . \quad . \quad (64)$$

Therefore, the Frenet-Serret formulas (42) and (43) written with respect to the coordinate system $(\varkappa')$ can be obtained from the system with respect to $(\varkappa)$ by replacing Q_μ by $Q_{\mu'} - s_{\mu'}$. In the second case we have chosen the system $(\varkappa')$ so as to obtain the same initial values for $\underset{a}{j}^{\varkappa'}$ and $Q_{\mu'} - s_{\mu'}$ as we had in the first case for $\underset{a}{j}^{\varkappa}$ and Q_{μ}. But then the solution $x^{\varkappa'}(\tau)$ will be the same function of τ as our original solution $x^{\varkappa}(\tau)$. Since the transformation (57) is a conformal one the curve

$$x^{\varkappa'} = x^{\varkappa'}(\tau) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

is therefore a conformal representation of the curve (47).

From this it follows that a curve is determined to within conformal representations by the expressions for the conformal invariants $h_1, h_2, \ldots, h_{n-1}$ in terms of the conformal parameter. So the equations of a curve may be written in the form

$$h_1 = h_1(\tau),\ h_2 = h_2(\tau), \ldots, h_{n-1} = h_{n-1}(\tau). \quad . \quad . \quad . \quad (66)$$

They are called *its intrinsic equations.*