Mechanics. — *On the elastic behaviour of the so-called "Bourdon" pressure gauge.* II. By C. B. BIEZENO and J. J. KOCH.

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4. *The iterative method.* Provisionally we introduce in the endplanes V_1 and V_2 a system of tangential stresses σ_1, such that:

1⁰. each plane is loaded by an equilibrium system,

2⁰. the material points of V_1 and V_2 remove to planes V_1' and V_2', the relative position of which is defined by the specific quantities ε_1 and ω_1, (differing from the definitive values ε_0 and ω_0 mentioned in § 3).

3⁰. for the points of any cross-section the displacements in the direction y are given by v_1.

These stresses can easily be found by considering a fibre intersecting the meridional plane of symmetry of the ring in the point (x, y). The original length of this fibre is $(r + y)d\psi$ whereas its new length is given by:

$$(r + y + v_1)\, d\psi + \varepsilon_1\, r d\psi + (y + v_1)\, \omega_1\, d\psi.$$

The specific elongation ε therefore is:

$$\varepsilon = \frac{v_1\, d\psi + \varepsilon_1\, r d\psi + (y + v_1)\, \omega_1\, d\psi}{(r + y)\, d\psi}.$$

In the numerator we neglect the product $v_1\omega_1$ against terms of the first degree in v_1, ε_1 and ω_1; in the denominator y in all actual cases is negligible with respect to r. Hence:

$$\varepsilon = \frac{v_1}{r} + \varepsilon_1 + \frac{y}{r}\, \omega_1$$

and consequently

$$\sigma_1 = E\varepsilon = E\left[\frac{v_1}{r} + \varepsilon_1 + \frac{y}{r}\, \omega_1\right].$$

The equilibrium of the force-system produced by the stresses σ_1 requires — if dF denotes the surface element on which σ_1 acts —

$$\int_F E\left[\frac{v_1}{r} + \varepsilon_1 + \frac{y}{r}\omega_1\right] dF = 0$$

$$\int_F E\left[\frac{v_1}{r} + \varepsilon_1 + \frac{y}{r}\omega_1\right] y\, dF = 0.$$

On account of symmetry $\int_F y\, dF$ as well as $\int_F v_1\, dF$ is zero, so that the equations pass into:

$$E\,\varepsilon_1\,F = 0$$
$$\frac{E}{r}\int_F [v_1 + \omega_1\, y]\, y\, dF = 0$$

from which we derive:

$$\varepsilon_1 = 0 \qquad \omega_1 = -\frac{\int_F v_1\, y\, dF}{\int_F y^2\, dF}.$$

With these values σ_1 becomes

$$\sigma_1 = \frac{E}{r}\left[v_1 - \frac{\int_F v_1\, y\, dF}{\int_F y^2\, dF}\, y \right].$$

As a matter of course we replace the integrals $\int_F v_1\, y\, dF$ and $\int_F y^2\, dF$ by the sums $\Sigma\, v_1\, y\, \triangle F$ and $\Sigma\, y^2\, \triangle F$, where $\triangle F$ stands for the meridional surface of a segment of the length s.

The product $v_1\, y$, the sums $\frac{1}{4}\Sigma\, v_1\, y\, \triangle F$ and $\frac{1}{4}\Sigma\, y^2\, \triangle F$ and the expression

$$v_1^* = v_1 - \frac{\Sigma\, v_1\, y\, \triangle F}{\Sigma\, y^2\, \triangle F}\, y$$

can be read from the columns 20—23 of table I.

The introduction of the stress systems

$$\sigma_1 = \frac{E}{r}\, v_1^*$$

in the endplanes V_1 and V_2 unfortunately does not bring our problem to an end. For the forces $\sigma_1\, \triangle F$ in two corresponding elements of the planes V_1 and V_2 include an angle $d\psi$, and therefore produce a resultant force

$$\sigma_1\, \triangle F\, .\, d\psi$$

in the direction of the negative y-axis.

Consequently the ring is submitted to a new flexure due to forces P_{2y}

TABLE II. 2nd iteration.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
joint	x	y	$\triangle x$	$\triangle y$	$P_{2y} = -\frac{E}{r^2} \triangle F v_1^*$	K_{2y}	$K_{2y} \triangle x$	M_2^*	M_2^{**}	M_2	$\triangle \varphi_2$	φ_2	$\triangle v_2 = -\varphi_2 \triangle x$	v_2	$v_2 y \triangle F$	$\frac{\Sigma v_2 y \triangle F}{\Sigma y^2 \triangle F} y$	$v_2^* = v_2 - \frac{\Sigma v_2 y \triangle F}{\Sigma y^2 \triangle F} y$
0	0	262			-1703			0	-1972	-1972	- 986			-2331	-305	-1490	-841
			75	- 1		1703	128					- 986	74				
1	75	261			-3164			128	-1973	-1845	-1845			-2257	-589	-1485	-772
			75	- 3		4867	365					-2831	212				
2	150	258			-2457			493	-1973	-1480	-1480			-2045	-528	-1468	-577
			75	- 6		7324	549					-4311	323				
3	225	252			-1364			1042	-1973	- 931	- 931			-1722	-434	-1433	-289
			75	- 9		8688	652					-5242	393				
4	300	243			- 94			1694	-1973	- 279	- 279			-1329	-323	-1382	53
			74	-11		8782	650					-5521	409				
5	374	232			1434			2344	-1972	372	372			- 920	-213	-1320	400
			74	-17		7348	544					-5149	381				
6	448	215			2858			2888	-1973	915	915			- 539	-116	-1223	684
			70	-28		4490	314					-4234	296				
7	518	187			3741			3202	-1973	1229	1229			- 243	- 45	-1064	821
			58	-49		749	43					-3005	174				
8	576	138			3467			3245	-1973	1272	1272			- 69	- 10	- 785	716
			36	-65		-2718	-98					-1733	62				
9	612	73			2055			3147	-1973	1174	1174			- 7	- 1	- 415	408
			12	-73		-4773	-57					- 559	7				
10	624	0			0			3090	-1972	1118	559			0	0	0	0
								$\Sigma = 19728$							$\Sigma = -2564$		
	$10^{-3}\alpha$	$10^{-3}\alpha$	$10^{-3}\alpha$	$10^{-3}\alpha$	$10^{-5}\frac{E}{E^*}\frac{s^2h}{Ir^2}\frac{\alpha^5}{\beta^2\gamma^2 P}$	$10^{-5}\frac{E}{E^*}\frac{s^2h}{Ir^2}\frac{\alpha^5}{\beta^2\gamma^2 P}$	$10^{-5}\frac{E}{E^*}\frac{s^2h}{Ir^2}\frac{\alpha^6}{\beta^2\gamma^2 P}$	$10^{-5}\frac{E}{E^*}\frac{s^2h}{Ir^2}\frac{\alpha^6}{\beta^2\gamma^2 P}$	$10^{-5}\frac{E}{E^*}\frac{s^2h}{Ir^2}\frac{\alpha^6}{\beta^2\gamma^2 P}$	$10^{-5}\frac{E}{E^*}\frac{s^2h}{Ir^2}\frac{\alpha^6}{\beta^2\gamma^2 P}$	$10^{-5}\frac{E}{E^{*2}}\frac{s^3h}{I^2r^2}\frac{\alpha^7}{\beta^5\gamma^2 P}$	$10^{-5}\frac{E}{E^{*2}}\frac{s^3h}{I^2r^2}\frac{\alpha^7}{\beta^5\gamma^2 P}$	$10^{-5}\frac{E}{E^{*2}}\frac{s^3h}{I^2r^2}\frac{\alpha^8}{\beta^5\gamma^2 P}$	$10^{-5}\frac{E}{E^{*2}}\frac{s^3h}{I^2r^2}\frac{\alpha^8}{\beta^5\gamma^2 P}$	$10^{-5}\frac{E}{E^{*2}}\frac{s^4h^2}{I^2r^2}\frac{\alpha^{10}}{\beta^4\gamma^2 P}$	$10^{-5}\frac{E}{E^{*2}}\frac{s^3h}{I^2r^2}\frac{\alpha^8}{\beta^5\gamma^2 P}$	$10^{-5}\frac{E}{E^{*2}}\frac{s^3h}{I^2r^2}\frac{\alpha^8}{\beta^5\gamma^2 P}$

TABLE III. 3nd iteration.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
joint	x	y	$\triangle x$	$\triangle y$	$P_{3y} = -\frac{E}{r^2}\triangle F v_2^*$	K_{3y}	$K_{3y}\triangle x$	M_3^*	M_3^{**}	M_3	$\triangle\varphi_3$	φ_3	$\triangle v_3 = -\varphi_2\triangle x$	v_3	$v_3 y\triangle F$	$\frac{\Sigma v_3 y\triangle F}{\Sigma y^2\triangle F}y$	$v_3^* = v_3 - \frac{\Sigma v_3 y\triangle F}{\Sigma y^2\triangle F}y$
0	0	262			420			0	4610	4610	2305			5329	698	3388	1941
			75	− 1		− 420	− 315					2305	−173				
1	75	261			772			− 315	4610	4295	4295			5156	1346	3375	1781
			75	− 3		−1192	− 894					6600	−495				
2	150	258			577			−1209	4609	3400	3400			4661	1203	3336	1325
			75	− 6		−1769	−1327					10000	−750				
3	225	252			289			−2536	4610	2074	2074			3911	986	3258	653
			75	− 9		−2058	−1544					12074	−906				
4	300	243			− 53			−4080	4610	530	540			3005	730	3142	− 137
			74	−11		−2005	−1484					12604	−933				
5	374	232			−400			−5564	4610	− 954	− 954			2072	481	3000	− 928
			74	−17		−1605	−1188					11650	−862				
6	448	215			− 684			−6752	4609	−2143	−2143			1210	260	2780	−1570
			70	−28		− 921	− 645					9507	−665				
7	518	187			−821			−7397	4610	−2787	−2787			545	102	2418	−1873
			58	−40		− 100	− 58					6720	−390				
8	576	138			−716			−7455	4610	−2845	−2845			155	21	1784	−1629
			36	−65		616	221					3875	−140				
9	612	73			−408			−7234	4610	−2624	−2624			15	1	944	− 929
			12	−73		1024	123					1251	− 15				
10	624	0			0			−7111	4609	−2502	−1251			0	0	0	0
								$\Sigma = -45097.5$							$\Sigma = 5828$		
	$10^{-3}\alpha$	$10^{-3}\alpha$	$10^{-3}\alpha$	$10^{-3}\alpha$	$10^{-5}\frac{E^2}{E^{*2}}\frac{s^4h^2}{I^2r^4}\frac{\alpha^9}{\beta^4\gamma^4}P$	$10^{-5}\frac{E^2}{E^{*2}}\frac{s^4h^2}{I^2r^4}\frac{\alpha^9}{\beta^4\gamma^4}P$	$10^{-6}\frac{E^2}{E^{*2}}\frac{s^4h^2}{I^2r^4}\frac{\alpha^{10}}{\beta^4\gamma^4}P$	$10^{-6}\frac{E^2}{E^{*2}}\frac{s^4h^2}{I^2r^4}\frac{\alpha^{10}}{\beta^4\gamma^4}P$	$10^{-6}\frac{E^2}{E^{*2}}\frac{s^4h^2}{I^2r^4}\frac{\alpha^{10}}{\beta^4\gamma^4}P$	$10^{-6}\frac{E^2}{E^{*3}}\frac{s^4h^2}{I^2r^4}\frac{\alpha^{10}}{\beta^4\gamma^4}P$	$10^{-6}\frac{E^2}{E^{*3}}\frac{s^5h^2}{I^3r^4}\frac{\alpha^{11}}{\beta^7\gamma^4}P$	$10^{-6}\frac{E^2}{E^{*3}}\frac{s^5h^2}{I^3r^4}\frac{\alpha^{11}}{\beta^7\gamma^4}P$	$10^{-6}\frac{E^2}{E^{*3}}\frac{s^5h^2}{I^3r^4}\frac{\alpha^{12}}{\beta^7\gamma^4}P$	$10^{-6}\frac{E^2}{E^{*3}}\frac{s^5h^2}{I^3r^4}\frac{\alpha^{12}}{\beta^7\gamma^4}P$	$10^{-6}\frac{E^2}{E^{*3}}\frac{s^6h^3}{I^3r^4}\frac{\alpha^{10}}{\beta^6\gamma^4}P$	$10^{-6}\frac{E^2}{E^{*3}}\frac{s^5h^2}{I^3r^4}\frac{\alpha^{12}}{\beta^7\gamma^4}P$	$10^{-6}\frac{E^2}{E^{*3}}\frac{s^5h^2}{I^3r^4}\frac{\alpha^{12}}{\beta^7\gamma^4}P$

parallel to the y-axis of the magnitude (pro unit of surface):

$$P_{2y} = -\frac{\sigma_1 \triangle F \, d\psi}{r \, d\psi} = -\frac{E \triangle F}{r^2} v_1^* .$$

The displacements v_2 caused by this load can be calculated after the scheme of table I (see table II, column 1—15). They renew the problem in as far as they are responsible for a new distortion of the endplanes V_1 and V_2. Therefore a new set of normal stresses has to be introduced in these planes, which on their part give rise to other displacements v_3 a.s.o.

If the process of iteration herewith described, converges, we get three series of quantities M_i, ω_i, σ_i, which summed up, give rise to the ultimate bending moment M, the ultimate ω_0, and the ultimate tangential stress σ.

As will be seen from the two tables II, III, the set of values M_3 is nearly proportional to the set of values M_2.

Therefore the iteration needs not to be continued any longer; for if we put $M_3 : M_2 = \lambda$, obviously $M_4 : M_3$ will be λ too, a.s.o. so that

$$M = M_1 + M_2 + M_3 + \ldots = M_1 + M_2 + \frac{M_3}{1-\lambda}.$$

Analogous remarks hold for ω and σ.

5. *Numerical results.* It follows from table I, II and III that

$$\omega_1 = -\frac{\Sigma v_1 \, y \triangle F}{\Sigma y^2 \triangle F} = -\frac{13014}{4508} \cdot \frac{10^{-1} . s \, a^3}{E^* I \beta^3} p = -0{,}2887 \frac{s \, a^3}{E^* I \beta^3} p$$

$$\omega_2 = -\frac{\Sigma v_2 y \triangle F}{\Sigma y^2 \triangle F} = +\frac{2564}{4508} \cdot \frac{10^{-1} E s^3 h a^7}{E^* I^2 r^2 \beta^5 \gamma^2} p = 0{,}05688 \frac{s a^3}{E^* I \beta^3} \cdot \frac{E s^2 h a^4}{E^* I r^2 \beta^2 \gamma^2} p$$

$$\omega_3 = -\frac{\Sigma v_3 y \triangle F}{\Sigma y^2 \triangle F} = -\frac{5828}{4508} \cdot \frac{10^{-2} E^2 s^5 h^2 a^{11}}{E^{*2} I^3 r^4 \beta^7 \gamma^4} p = -0{,}01293 \frac{s a^3}{E^* I \beta^3} \cdot \left[\frac{E s^2 h a^4}{E I r^2 \beta^2 \gamma^2}\right]^2 p$$

$$\lambda = \frac{\omega_3}{\omega_2} = -\frac{0{,}01293}{0{,}05688} \cdot \frac{E s^2 h a^4}{E^* I r^2 \beta^2 \gamma^2} .$$

With $r = 3{,}15$ cm, $h = 0{,}06$ cm, $s = 0{,}075$ cm, $E = 10^6$ kg/cm² (the data of the pressure gauge under consideration), we find:

$$\omega = \omega_1 + \omega_2 + \frac{\omega_3}{1-\lambda} = \left[-1{,}127 + 0{,}3933 \frac{a^4}{\beta^2 \gamma^2} - \frac{0{,}1584 \left(\frac{a^4}{\beta^2 \gamma^2}\right)^2}{1 + 0{,}403 \frac{a^4}{\beta^2 \gamma^2}} \right] . \, 10^{-3} \frac{a^3}{\beta^3} p.$$

For the standard pressure gauge itself, we get — putting $a = \beta = \gamma = 1$ —

$$\omega = 0{,}8465 \, . \, 10^{-3} \, p.$$

Experiments made on this gauge gave

$$\omega = 0{,}7260 \, . \, 10^{-3}\, p.$$

The agreement of both results is quite satisfactory, and probably would have still been better if we had used the real modulus of elasticity instead of the standard value 10^6 kg/cm^2.

For the moment M we get in a similar manner:

$$M = \left[M_1 + 0{,}1772 \frac{\alpha^4}{\beta^2 \gamma^2} M_2 + \frac{0{,}0314 \left(\frac{\alpha^4}{\beta^2 \gamma^2} \right)^2}{1 + 0{,}403 \frac{\alpha^4}{\beta^2 \gamma^2}} M_3 \right] 10^{-4} \alpha^2 p.$$

In this formula the values of M_1, M_2, M_3, following from table I, II, III, for the point (i) have to be substituted if the total bending moment at this point (i) is required. The greatest bending stress σ_b is given by

$$\sigma_b = \frac{M}{\frac{1}{6} h^2 \beta^2} = \frac{10^4 M}{6 \beta^2}.$$

Finally the total tangential normal stress is given by

$$\sigma_t = \frac{E}{\gamma r} \left[v_1^* + 1{,}772 \frac{\alpha^4}{\beta^2 \gamma^2} v_2^* + \frac{0{,}314 \left(\frac{\alpha^4}{\beta^2 \gamma^2} \right)^2}{1 + 0{,}403 \frac{\alpha^4}{\beta^2 \gamma^2}} v_3^* \right] \frac{10^{-5} s \alpha^4}{E^* I \beta^3} p.$$

Again the values v_1^*, v_2^*, v_3^* following from table I, II, III for the point (i) have to be substituted in this formula if the total tangential stress at this point (i) is required.

To conclude we reproduce in table IV the stress distribution of the standard gauge, characterized by the parameters $\alpha = \beta = \gamma = 1$.

The columns 1, 2, 3 contain the values M_1, M_2 and M_3: $(1 - \lambda)$; the value of λ used in the third column agrees with the value of λ, used with the quantity ω $(\lambda = -\frac{0{,}01293}{0{,}05688} = -0{,}403)$. The total moment M is inserted in column 5. The common multiplying factor of all columns 2—5 is $10^{-4}\, p$. The quantities occurring in the following columns are expressed in kg/cm^2, and relate to the pressure $p = 60$ kg/cm^2, for which the gauge was said to be designed. Column 6 gives the greatest bending stress, column 10 the total tangential stress as a sum of the stresses σ_1, σ_2 and $\sigma_3 : (1 - \lambda) = \sigma_3 : 1{,}403$, and column 11 the so-called "ideal" stress, which is equal to the absolute value of the difference of the extreme principal stresses. In calculating this stress, it must be observed, that σ_b (column 6) bas been calculated for the outer surface of the gauge, and that an equal bending stress, but of opposite sign, occurs at the inner surface. From two corresponding points of the outer and inner surface, *that* point is to be considered as the most dangerous one, for which the difference of the two

TABLE IV.

1	2	3	4	5	6	7	8	9	10	11
	M_1	M_2	$\frac{M_3}{1{,}403}$	M	σ_b	σ_1	σ_2	$\frac{\sigma_3}{1{,}403}$	σ_t	$\sigma_{\text{id.}}$
0	722	−349	103	476	4760	2534	−1109	323	1748	6508
1	697	−327	96	466	4660	2354	−1018	286	1622	6282
2	621	−262	76	435	4350	1828	− 761	221	1288	5638
3	494	−165	46	375	3750	1015	− 381	109	743	4493
4	319	− 49	12	282	2820	70	70	− 23	117	2937
5	97	66	−21	142	1420	−1067	527	−154	− 694	2114
6	−170	162	−48	− 56	− 560	−2126	902	−261	−1485	2045
7	−452	218	−62	−296	−2960	−2783	1082	−312	−2013	4973
8	−689	225	−64	−528	−5280	−2580	944	−271	−1907	7187
9	−836	208	−59	−687	−6870	−1529	538	−155	−1146	8016
10	−884	198	−56	−742	−7420	0	0	0	0	7420
	$10^{-4}p$	$10^{-4}p$	$10^{-4}p$	$10^{-4}p$						

extreme principal stresses is the greater; and it is seen at once that this is the point for which σ_b and σ_t have opposite signs. With this remark in mind the absolute value of σ_{id} is obtained by adding the corresponding *absolute* values of σ_b and σ_t.

A glance at column 11 learns that very high stresses indeed occur, and that the manifold ruptures of pressure-gauges which are registered in practice, are in full accordance with what has to be expected. It is worth while to state, that the greatest ideal stress occurs in the neighbourhood of point 10, and not in the point 10 itself, which is in full agreement with the experiment.

Finally it must be remarked that our calculations only hold for gauges, the cross section of which is similar to that of our standard gauge. To get reliable results for other cross sections, the calculations given here are to be repeated.