in a following communication in connection with results obtained about the transport of other substances.

The data obtained on the strength of the transport enable us to calculate the strength of the transport in the pedicels. When 6 leaves wih 1200 tentacles take up 300 γ nitrogen in 24 hours, $\frac{1}{4}$ γ N. is transported per tentacle, i.e. $\frac{33}{28}$ γ asparagine. The diameter of a tentacle just below the gland amounts to about 0.04 mm. So $\frac{33}{28}\gamma$ asparagine is transported through a surface of 0.00126 mm^2 in 24 hours, i.e. 0.039 mg. asparagine per mm^2 per hour. If the transport takes place through the protoplasm, this figure rises considerably. Reliable data on the transport in parenchyma cells in root, stalk or leaf are not known to me. So we come to comparing the transport in the tentacles with that in the sieve tubes. For transport in the stalk (MÜNCH) 10.7—63.3 mg. per mm^2 per hour was found, for transport of assimilates from a beanleaf (BIRCH–HIRSCHFELD) 5 mg. per mm^2 per hour, for supply of assimilates to fruit (MÜNCH) 4.7—6 mg. per mm^2 per hour, for the rate of transport of sugars in the stalk in cotton (MASON and MASKELL) 2.3 mg per mm^2 per hour. From this it appears that the rate of transport in the sieve-tubes is more than 100 times faster than the transport in the parenchyma cells of the tentacles. Therefore it appears on comparison with the transport in the sieve-tubes that the rate of the latter is much greater. For the present there is no reason to assume that the transport in the sieve-tube would be a process that is analogous with the active transport in the Drosera tentacles.

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Hydrodynamics. — *On the influence of the concentration of a suspension upon the sedimentation velocity* (in particular for a suspension of spherical particles) *).

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15. With the aid of the results obtained in sections **11.**—**14.** we will now attempt to calculate the influence which a given particle experiences from all the surrounding particles, in a field extending indefinitely in all directions and everywhere possessing the same average number of particles per unit volume. It will be seen in **17.** that a difficulty still remains in the problem, in so far as there occurs an integral, the value of which depends upon the way the integration is carried out. By prescribing a certain definite way a particular value is obtained, which to the author would appear the one best adapted for the present purpose, but the problem cannot yet be considered as being wholly settled.

We begin with the summation of the velocities induced in a particle A in consequence of the presence of the other particles ("particles B"). These particles can be taken together in groups, each group being situated at some definite distance r_i from A; the number of particles per group being n_i. The contribution by each group will be calculated upon the assumption that we may use the mean value of (54) over a surface $r = constant$. Restoring the factor $F/8\pi\eta$ the total amount becomes:

$$\delta u_{\mathrm{I}} = \Sigma u_i = -\frac{5Fa^3}{24\pi\eta}\sum\frac{n_i}{r_i^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

In working out the sum it is not necessary to proceed far: from a certain distance r_m onward it is sufficiently accurate to make use of the integral:

$$n\int_{r_m}^{\infty} 4\pi r^2\,dr\,(1/r^4) = 4\pi n/r_m \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

where n is the average number of particles per unit of volume. The distance r_m is defined by:

$$n\,.\,(4\pi r_m^3/3) = 1 + \Sigma n_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

the summation extending just as far as we take separate terms in (55).

For purposes of comparison we write:

$$\delta u_{\mathrm{I}} = -\lambda_{\mathrm{I}}\, n\, s\, u_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58a)$$

where $s = 4\pi a^3/3$, $u_0 = F/6\pi\eta a$ (compare 30c), and:

$$\lambda_{\mathrm{I}} = \frac{15a}{4r_m} + \frac{15a}{16\pi n}\sum\frac{n_i}{r_i^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (58b)$$

16. The evaluation of (58b) is possible only when we possess a statistical theory of the distribution of the particles in the neighbourhood of a given one. As it is not a part of our task to develop such a theory here, we shall restrict to the consideration of a few typical cases.

We might assume in the first place that the surrounding particles may take all positions

*) Continued from these Proceedings **44**, 1941, p. 1184.

relatively to A with equal probability, provided they do not penetrate into A. The minimum distance of the centre of a particle B from the centre of A then will be: $2a$; hence we apply eq. (58*b*) with $r_m = 2a$, leaving out the sum. This gives:

$$\lambda_I = 15/8 \quad \ldots \ldots \ldots \ldots \quad (59a)$$

A second assumption is that owing to the action of repulsive forces there may be a minimum value βa for r_m, exceeding $2a$, while otherwise there shall be no restriction, nor any preference for the possible positions of B. In that case we find:

$$\lambda_I = 15/(4\beta). \quad \ldots \ldots \ldots \ldots \quad (59b)$$

When the number of particles per unit volume becomes large, the repulsive forces between them may enforce a type of distribution in which the average distances between neighbouring particles become approximately equal. For purposes of calculation the arrangement may be compared with certain types of regular arrangements. We may assume, *e.g.*, that the average values of r_i and n_i for the first few groups of particles surrounding A approximately are the same as those which are found in a simple cubical lattice with spacing l. In that case we have: $n l^3 = 1$, while the first few groups are determined by:

$$r_1 = l; \qquad r_2 = l\sqrt{2}; \qquad r_3 = l\sqrt{3}; \qquad r_4 = 2l$$
$$n_1 = 6; \qquad n_2 = 12; \qquad n_3 = 8; \qquad n_4 = 6.$$

Equation (57) then gives: $r_m = 1{,}990\ l$, and from (58*b*) we obtain: $\lambda_I = 4{,}96\ a/l = 4{,}96\ a\,n^{1/3}$. Instead of the simple cubical arrangement we also might consider the face-centred lattice as a possible picture for the average arrangement of the particles. If the spacing has the value l, we have: $n l^3 = 4$; and the first few groups are determined by:

$$r_1 = \tfrac{1}{2} l\sqrt{2}; \qquad r_2 = l; \qquad r_3 = l\sqrt{3/2}; \qquad r_4 = l\sqrt{2}$$
$$n_1 = 12; \qquad n_2 = 6; \qquad n_3 = 24; \qquad n_4 = 12.$$

In this case eq. (57) gives: $r_m = 1{,}486\ l$; from (58*b*) we obtain: $\lambda_I = 7{,}57\ a/l = 4{,}77\ a\,n^{1/3}$. As the latter value does not differ greatly from the one found with the simple cubical arrangement, we can write:

$$\lambda_I \simeq 4{,}9\ a\,n^{1/3} \quad \ldots \ldots \ldots \ldots \quad (59c)$$

as an approximate expression to be applied in the case of a more or less regular average spatial distribution of the particles.

As will be evident from sections **2.**, **11.** and **12.**, and from footnote 11), the resultant effect of the "induced velocities" corresponds to what usually is described as the result of an apparent increase of the viscosity. When the retardation δu_I had been calculated by means of EINSTEIN's formula for the specific increase of the viscosity, the result would have been:

$$\delta u_I = \frac{F}{6\pi\eta a}\left(\frac{1}{1 + 2{,}5\,ns} - 1\right) \simeq -\,2{,}5\ ns\,u_0,$$

so that in that case we should find: $\lambda_I = 2{,}5$. It will be seen that the values of λ_I given by eqs. (59*a*)—(59*c*) all are smaller. This is due to the circumstance that EINSTEIN's formula refers to the motion in bulk of the suspension, and applies to those cases where the field of flow does not change greatly over distances of the order of the mean distance between two particles. In the case treated here it is otherwise: the velocity gradients have their greatest values immediately around a given particle, *i.e.* in a part of space where a second particle cannot penetrate as easily as elsewhere. Hence in the immediate neighbourhood of a given particle the "effective viscosity" remains below the

EINSTEIN value, and the difference becomes more marked when repulsive forces lead to an increase of the extent of the "empty region".

17. We now come to the summation of the velocities described by (37*a*). For this purpose we consider a definite particle B, and take in view the velocities communicated to it by all other particles, each of which in turn must be considered as a particle A. In a similar way as was done in **16.** the particles A surrounding a given particle B can be taken together in groups, each group lying at a definite distance r_i from B. The contributions derived from all particles can be united into the expression:

$$\delta u_{II} = \Sigma\, n_i\,(\bar{u})_i + n\int_{r_m}^{\infty} 4\pi r^2\,dr\,\bar{u}\,. \quad \ldots \ldots \quad (60)$$

where $\bar{u}$ is the mean value of u_m over a surface r = *constant*; $(\bar{u})_i$ in particular being the mean value for the surface $r = r_i$; while r_m is defined by (57).

The expression (60) can be written:

$$\delta u_{II} = \{\Sigma\, n_i\,(\bar{u})_i - n\int_a^{r_m} 4\pi r^2\,dr\,\bar{u}\} + n\int_a^{\infty} 4\pi r^2\,dr\,\bar{u} \quad \ldots \quad (60a)$$

The last integral on the right hand side will be replaced by: $n\iiint dx\,dy\,dz\,u_m$, the integration being extended over the whole space outside of the spherical surface $r = a$. Making use of (37*a*) this integral can be decomposed into:

$$n\iiint dx\,dy\,dz\,u + \tfrac{1}{6} n a^2 \iiint dx\,dy\,dz\,\triangle u.$$

The first integral of this sum is not a convergent expression: its value depends upon the way in which the integration is carried out. This apparently means that the problem of the unlimited field is not determinate in itself, and a particular condition must be supplied to specify the value of this integral. We adopt the value given in eq. (34), which was obtained by performing the integration with respect to dy and dz first, the integration with respect to dx coming afterwards, as the value given in (34) is the same as the one which is obtained for the case of a field enclosed by impenetrable walls (compare equation (19), section **6.**). In this way the first term of the sum becomes: $-\,n\,s\,u_0$ [14]. — The second integral of the sum can be reduced to a surface integral, extended over the spherical surface $r = a$, the integrand being: $-\,\partial u/\partial r$, the value of which can be derived from (30*b*). This second integral reduces to: $+\,\frac{1}{2}\,n\,s\,u_0$. Combining the results we obtain:

$$n\int_a^{\infty} 4\pi r^2\,dr\,\bar{u} = -\,\tfrac{1}{2}\,n\,s\,u_0 \quad \ldots \ldots \quad (60b)$$

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14) When the mean value u over a surface r = *constant* is calculated immediately from (29) one obtains: $(F\varkappa/12\pi\eta)\,e^{-\sigma}\,(1 + 2/\sigma)$. Substitution of this expression into the integral $n\int_a^{\infty} 4\pi r^2\,dr\,u$ leads to the result: $4nF/3\varkappa^2\eta$.

When in calculating $n\iiint dx\,dy\,dz\,u$ the integration first is performed over a surface y = *constant*, the integration with respect to dy coming last, the result becomes: $2nF/\varkappa^2\eta$.

These results cannot have any sense, as both of them depend upon the parameter $\varkappa$, which to a certain extent is arbitrary.

In this way we have got around the difficulties encountered in section **7.** It is possible, however, that the result is not final, and the circumstance that in any actual case the field is bounded by the walls of the vessel containing the suspension as yet may play a deciding part.

The value of $\bar{u}$ to be used in the expression between the { } in (60*a*) can be derived from eq. (37*b*) of section **14.** After restoration of the factor $F/8\,\pi\,\eta$ it becomes: $\bar{u} = u_0\, a/r$. Making use of this result and of (60*b*) it is found that (60*a*) can be written:

$$\delta u_{II} = -\lambda_{II}\, n\, s\, u_0 \quad \ldots \ldots \ldots \quad (61a)$$

where:

$$\lambda_{II} = \frac{3}{2}\frac{r_m^2}{a^2} - 1 - \frac{3}{4\pi a^2 n}\sum \frac{n_i}{r_i} \quad \ldots \ldots \quad (61b)$$

This expression will be worked out for the same cases as have been considered in connection with the calculation of λ_I. When the surrounding particles may take all positions relatively to B with equal probability, from a distance $r_m = 2a$ onward, we can discard the sum and find: [15])

$$\lambda_{II} = 5. \quad \ldots \ldots \ldots \ldots \quad (62a)$$

In the case where the minimum distance is $\beta\, a$ (with $\beta > 2$):

$$\lambda_{II} = \tfrac{3}{2}\,\beta^2 - 1 \quad \ldots \ldots \ldots \ldots \quad (62b)$$

For a distribution of the distances of the nearest neighbours corresponding to that found in a simple cubical lattice: $\lambda_{II} = 0{,}663\; l^2/a^2 - 1 = 0{,}66\; a^{-2}\, n^{-2/3} - 1$; and for a distribution corresponding to that found in a face-centred cubical lattice: $\lambda_{II} = 0{,}267\; l^2/a^2 - 1 = 0{,}67\; a^{-2}\, n^{-2/3} - 1$; giving as an average value:

$$\lambda_{II} \simeq 0{,}67\; a^{-2}\, n^{-2/3} - 1 \quad \ldots \ldots \quad (62c)$$

18. The quantity designated by $\delta\, u_{II}$ represents the velocity which would be acquired by a particle B, of density equal to that of the liquid, in consequence of the fields of flow which are produced by the sedimenting particles A surrounding it. When the original density is restored to B, this particle moreover will acquire the velocity $u_0 = F/6\,\pi\,\eta\, a$ under the action of its own weight (compare eq. 30c); at the same time it will also experience the resultant effect of the "induced velocities", indicated by $\delta\, u_I$. Hence the resulting velocity of the particle will become:

$$u_{res} = F/6\,\pi\,\eta\, a + \delta u_I + \delta u_{II} \quad \ldots \ldots \quad (63)$$

Although terms of the second order, such as may be called forth by the combination of the effects considered, have not been taken into account, it is probable that the accuracy of the result expressed by (63) will be increased, when in the expressions (58*a*) and (61*a*), for $\delta\, u_I$ and $\delta\, u_{II}$ respectively, we replace u_0 by the resulting velocity u_{res}. Indeed, the effects denoted by $\delta\, u_I$ and $\delta\, u_{II}$ refer to fields of flow set up by sedimenting

[15]) The same value is obtained from eq. (24) of the first part of this paper (these Proceedings **44**, 1941, p. 1051), when k_2 and k_3 are replaced by zero. With the values given in **7.** we then have:

$$(8\pi\eta/F)\;\delta u_{II} = (N/\Omega)\iiint_G dx\, dy\, dz\; \Phi_m = -\,n\,.\,80\,\pi\, a^2/9,$$

where N is the total number of particles contained in the vessel, so that $n = N/\Omega$. Introducing s and u_0 we obtain: $\delta\, u_{II} = -5\, n\, s\, u_0$.

particles, and thus are proportional to the actual velocity acquired by a particle. Making this substitution, we obtain:

$$u_{res} = F/6\,\pi\,\eta\, a - (\lambda_I + \lambda_{II})\, n s\, u_{res} \quad \ldots \ldots \quad (63a)$$

from which:

$$u_{res} = \frac{F}{6\pi\eta a}\,\frac{1}{1 + (\lambda_I + \lambda_{II})\, n s} \quad \ldots \ldots \quad (64)$$

where, according to (35): $F = s(\varrho_p - \varrho)\, g$.

19. Now that we have obtained a provisional expression for the value of the sedimentation velocity in an infinitely extending field, it would be necessary to return to the case of a suspension enclosed in a vessel. However, we will first give attention to the motion of a *cloud of particles of finite extent*, in a field which itself is unlimited. The influence exerted by the particles upon each other's motion in this case will increase the velocity of fall, which may acquire values greatly exceeding the sedimentation velocity of a single particle. It is possible — and it actually occurs in many cases — that the velocity acquired by the whole mass becomes of such magnitude, that it is no more allowed to leave out the inertia terms from the equations of motion. Nevertheless we shall provisionally assume that the linear equations, in which the inertia terms have been neglected, can be applied (cases can be constructed in which no serious error is to be expected); afterwards some attention will be given to the possibilities for a more general treatment.

When we keep to the linear equations of motion, the resulting velocity of any particle in principle can be found by adding together (*a*) the velocity it derives from the force acting upon the particle itself; (*b*) the velocities induced in consequence of the presence of the surrounding particles; and (*c*) the velocities which it will derive from the fields of flow set up by the surrounding particles. This third contribution is given by:

$$\delta u_{II} = \Sigma\, u_m \quad \ldots \ldots \ldots \ldots \quad (65)$$

where the sum extends over all particles of the cloud, with the exception of the particle B for which the velocity must be found. The value of this sum depends upon the dimensions and the form of the cloud; upon the distribution of the particles through the cloud; and upon the position of the particle B within the cloud. We assume that the number of particles per unit volume (n) has the same value everywhere in the cloud, and that the form of the cloud is spherical, with radius R_0. When the particle B is situated not too near to the surface of the cloud (in the following lines we will limit ourselves to the consideration of such particles), the expression (65) can be written:

$$\delta u_{II} = \Sigma n_i\,.\,(\bar{u})_i + n\iiint_{r > r_m} dx\, dy\, dz\; u_m \quad \ldots \ldots \quad (66)$$

where the integral extends over the space outside of a spherical surface with radius r_m, again defined by (57). On account of (37*a*) we have:

$$\iiint_{r > r_m} dx\, dy\, dz\; u_m = \iiint_{r > r_m} dx\, dy\, dz\; u + \tfrac{1}{6}\, a^2 \iiint_{r > r_m} dx\, dy\, dz\; \triangle\, u.$$

By means of a direct calculation it is found that the second integral on the right hand side has the value zero in the case considered. Consequently it is possible to transform (66) into:

$$\delta u_{II} = \{\Sigma n_i\,.\,(\bar{u})_i - n\int_0^{r_m} 4\pi r^2\, dr\, \bar{u}\} + n\iiint dx\, dy\, dz\; u \quad . \quad (67)$$

The triple integral here is extended over the whole cloud; in the integral occurring between the { } it is necessary therefore to take $r = 0$ as the lower limit (instead of $r = a$, as was done in (60a) above). As $\bar{u} = u_0 a/r$, this latter integral remains convergent for $r = 0$, and has the value: $2\pi a r_m^2 u_0$; hence the quantity between the { } in (67) can be written:

$$-\lambda^* n s u_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68a)$$

where:

$$\lambda^* = \lambda_{II} + 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68b)$$

20. In calculating the value of:

$$U = n \iiint dx\, dy\, dz\, u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

it is to be observed that in the present case, where the number of particles is finite, difficulties concerning the convergence will not occur. Hence it is not necessary to make use of the solution applied in the case of an infinitely extending assemblage of particles, which was given in **10**., and we can base our calculations immediately upon the formulae developed by STOKES.

The most convenient way is to make use of the expression for u, given in **9**., *viz.*: $u = u_I + u_{II} = \triangle\Psi - \partial^2\Psi/\partial x^2 + \partial^2\varphi/\partial x^2$. We first calculate the integrals of the functions Ψ and φ; the velocity U afterwards can be derived by means of differentiations. Instead of working with the function Ψ given in (26) we now can use the much simpler expression: [16])

$$\Psi_{\text{Stokes}} = F r / 8 \pi \eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

It must be observed that a construction of the type as was proposed in **10**. can be applied also to the present case; we come back to this point in **22**.

An elementary calculation gives:

$$\iiint dx\, dy\, dz\, \frac{Fr}{8\pi\eta} = \frac{F}{8\eta}\left(R_0^4 + \tfrac{2}{3} R^2 R_0^2 - \tfrac{1}{15} R^4\right) \quad . \quad . \quad (71a)$$

$$-\iiint dx\, dy\, dz\, \frac{Fa^2}{24\pi\eta r} = \frac{-Fa^2}{12\eta}\left(R_0^2 - \tfrac{1}{3} R^2\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (71b)$$

where R is the distance of the particle B from the centre of the spherical cloud (R_0 being the radius of the cloud itself). The necessary differentiations can be performed when we write: $R^2 = x^2 + y^2 + z^2$, the origin of the system of coordinates being taken at the centre of the cloud. We then find:

$$U = \frac{nF}{15\eta}\left(5R_0^2 - x^2 - 2y^2 - 2z^2\right) + \frac{nFa^2}{18\eta} \quad . \quad . \quad . \quad . \quad (72a)$$

and in a similar way for the components in the directions of the other axes:

$$V = \frac{nF}{15\eta} xy; \qquad W = \frac{nF}{15\eta} xz \quad . \quad . \quad . \quad . \quad . \quad (72b, c)$$

21. The quantities U, V, W are of an order of magnitude quite different from that of the quantities which thus far have played a part in our calculations. Discarding all terms of less importance we can say that the motion of the particles of the cloud to a first approximation is described by the equations (72a)—(72c), in which, moreover, the last term of (72a) safely can be neglected. This motion can be decomposed into a *general motion of the whole cloud* with the constant velocity:

$$u_{\text{cloud}} = \frac{4nF}{15\eta} R_0^2 = \frac{4}{3}\pi R_0^3 \,.\, nF \,.\, \frac{1}{5\pi\eta R_0} \quad . \quad . \quad . \quad . \quad (73)$$

and an *interior motion* with the components:

$$\left.\begin{aligned} u_{\text{interior}} &= \frac{nF}{15\eta}\left(R_0^2 - R^2 - y^2 - z^2\right) \\ v_{\text{interior}} &= \frac{nF}{15\eta} xy; \quad w_{\text{interior}} = \frac{nF}{15\eta} xz \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (74)$$

These latter quantities satisfy the equation of continuity. At the surface of the cloud:

$$x\, u_{\text{interior}} + y\, v_{\text{interior}} + z\, w_{\text{interior}} = 0 \quad . \quad . \quad . \quad . \quad . \quad (75)$$

from which it appears that the interior motion is tangential to this surface. Hence the spherical form of the cloud and the constant value of the number of particles per unit volume are retained during the motion.

It will be evident that the quantities given by (72a)—(72c) do not only represent the velocities of the particles in the cloud, but also *that of the liquid itself*. The liquid in the interior of the cloud thus is carried along by the particles it contains.

The motion described by eqs. (72a)—(72c) is the same as which is found for a *liquid sphere* of radius R_0, acted upon by a continuously distributed force of effective magnitude nF per unit volume, and falling in another liquid, provided both liquids possess the same viscosity [17]). Actually we must expect that owing to the presence of the particles in the sphere, the latter will possess an effective viscosity greater than that of the surrounding liquid. That this is not apparent from the equations developed must be ascribed to the circumstance that in calculating δu_{II} by means of (65) we simply have summed the amounts u_m, without considering the influence of all the other particles upon each term of this sum. Now that the sum has assumed a magnitude much larger than all other velocities, this influence certainly can no longer be neglected.

22. The results arrived at make it appear more promising to start from a different point of view, related to that of section **10**. The system of forces acting upon the liquid and the cloud of particles can be analysed into the following components:

a) a force ϱg per unit volume, acting throughout the whole field, and balanced by a pressure gradient $\partial p/\partial x = \varrho g$ (assumed to be present also in the particles);

b) a continuous force of magnitude nF per unit volume, assumed to act throughout the volume of the cloud of particles;

c) a set of "equilibrium systems" of the type considered in **9**., each system having its centre at the centre of a particle.

In order to reduce as far as possible the difficulties which may arise at the boundaries of the cloud, it is necessary to choose the parameter $\varkappa$, which occurs in the formulae describing the equilibrium systems, in such a way that $1/\varkappa$, while still being large in comparison with the average distance between neighbouring particles, at the same time is small compared with the dimensions of the cloud.

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16) Compare eq. (30a).

17) Compare H. LAMB, Hydrodynamics (6th Ed., Cambridge 1932), Art. **337**, 2⁰ (p. 600). The resistance experienced by a liquid sphere, moving with the velocity U_s in another liquid, is given by: $6\pi\eta a U_s (2\eta + 3\eta')/(3\eta + 3\eta')$, η' being the viscosity of the liquid of the sphere. When $\eta' = \eta$, this formula reduces to: $5\pi\eta a U_s$.

The "effective force" nF mentioned in the text is the total force acting per unit volume of the cloud, diminished by ϱg, as follows from: $nF = ng(\varrho_P - \varrho)\, s$.

We will not work out the calculation of the field of motion according to the scheme indicated, and restrict to the following observations:

The field of force considered under *b*) will produce a motion of the cloud as a whole, which motion will be the same as that of a mass of liquid with density $\varrho + nF/g = \varrho + ns(\varrho_P - \varrho)$, moving amidst a liquid of density ϱ. It is reasonable to assume that the liquid represented by the cloud will possess the effective viscosity $\eta' = \eta\,(1 + 2{,}5\,n\,s)$. In many cases which are encountered in actual circumstances, the motion of this mass of liquid will be such that inertia effects, both in its interior and in the surrounding liquid, cannot be neglected. A theoretical calculation then may become impossible, and experimental investigation often must be called to assistance.

Superposed upon the motion of the cloud as a whole, there will be the motion of the particles relatively to the liquid under the action of the force systems, mentioned under *c*). When the particles are sufficiently small, the sedimentation velocity usually will be extremely small in comparison with that of the cloud as a whole. The relative motion then can be calculated upon lines, similar to those followed in **15.**—**18.** There may be found some difference in the value of λ_{II}, connected with the fact that the cloud is of finite extent; also the corrections for particles near to the boundary of the cloud will be different.

Examples of the motion of such clouds of particles, carrying along with themselves the liquid contained in the cloud, are often found in nature. We mention the motion of the fog; that of clouds heavily loaded with dust particles (beautiful demonstration experiments can be made with cold smoke); the phenomena presented by certain clouds which sometimes emerge from volcanic lavas and are loaded so heavily with ashes or scoriae, that they flow down the slopes of the mountain with very great velocities [18]; water currents loaded with silt such as have been considered in DALY's theory of the formation of submarine canyons and are illustrated by beautiful experiments made by KUENEN [19]. Attention also should be called to the phenomenon known as eviction [20].

In many of these cases the particles will be so heavy that STOKES' law of resistance no longer can be applied to them, and a different law (ultimately a quadratic law) of resistance should be used. Moreover, in the motion of such clouds and currents turbulence usually plays a large part; apart form the influence it has upon the motion of the mass as a whole, it is of importance as it brings about an intense mixing and diffusion, which counteracts the sedimentation of the particles and thus keeps them much longer suspended. In all these cases a decomposition of the system of forces into three parts in the way as indicated above, and the consideration of the general motion of the suspension as that of a liquid of increased density and viscosity, will afford a valuable help in analysing the phenomena presented.

It must be remarked that when it is necessary to consider the frictional forces due to the turbulent motion, attention should be given also to the influence of the suspended particles upon the magnitude of these forces.

In the last part of this paper we hope to come back to the problem of the sedimentation in a suspension enclosed in a vessel.

(To be continued.)

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[18]) The explanation of the "nuées ardentes" as the flow of turbulent clouds of ashes down the slopes of the mountain in consequence of the force of gravity has been given by G. L. L. KEMMERLING; compare *e.g.* his paper: "De controverse uitgeschoten gloedwolken (nuées ardentes d'explosion dirigées) of lawinen gloedwolken (nuées ardentes d'avalanche)", De Ingenieur **47**, 1932, p. A 129.

[19]) Compare: PH. H. KUENEN, Experiments in connection with DALY's hypothesis on the formation of submarine canyons, Leidsche Geologische Mededeelingen **8**, 1937, p. 327; Density currents in connection with the problem of submarine canyons, Geological Magazine **75**, 1938, p. 241.

[20]) Compare: N. SHAW, The air and its ways (Cambridge 1923), p. 103.

Mathematics. — *Zur projektiven Differentialgeometrie der Regelflächen im R_4.* (Achte Mitteilung). Von R. WEITZENBÖCK und W. J. BOS.

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Wir behandeln in dieser Mitteilung einige Sätze über die Flächen F_2^3 des R_4, die durch drei gegebene Geraden allgemeiner Lage gehen.

§ 24.

Es seien a^2, α^2 und p^2 drei Geraden allgemeiner Lage. Ihre Transversale L schneidet sie in den drei Punkten

$$\left.\begin{array}{l} P_1 = (\alpha^2 p^2 a)(au') = 0, \quad P_2 = (p^2 a^2 \alpha)(\alpha u') = 0 \\ \text{und} \\ (a^2 \alpha^2 p)(pu') = -P_1 - P_2 = 0 \end{array}\right\} \quad . \quad . \quad (221)$$

Es seien P_3, P_4 und P_5 drei weitere Punkte mit $(P_1P_2P_3P_4P_5) \neq 0$, P_3 auf a^2, P_4 auf α^2 und P_5 auf der dritten Geraden p^2 gelegen. Auf jeder Regelfläche F_2^3, von der a^2, α^2 und p^2 Erzeugende sind, liegt ein durch P_3 und P_4 gehender Kegelschnitt K, der p^2 in einem Punkte $P_1 + P_2 + \alpha P_5$ trifft. Die Punkte von K sind dann durch die drei Erzeugenden a^2, α^2 und p^2 projektiv auf die der Leitlinie $P_1 P_2 = L$ bezogen.

Als Parameterdarstellung für K erhalten wir, wenn $t = 0$ dem Punkte P_3, $t = \infty$ dem Punkte P_4 und $t = 1$ dem Punkte $P_1 + P_2 + \alpha P_5$ entspricht:

$$\varrho\, P_K = \beta \,.\, P_3 + t\,[-\beta \,.\, P_3 - \gamma \,.\, P_4 + P_1 + P_2 + \alpha \,.\, P_5] + t^2 \gamma \,.\, P_4 \quad (222)$$

Für die Punkte von L setzen wir

$$\varrho\, P_L = P_1 + t \,.\, P_2 \,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (223)$$

sodass der allgemeine Flächenpunkt x auf F_2^3 gegeben ist durch

$$x = P_L + \lambda \,.\, P_K;$$

d.h. wir haben

$$\left.\begin{array}{l} \varrho\, x = P_1 \,.\, (1 + \lambda t) + P_2 \,.\, (t + \lambda t) + P_3 \,.\, (\lambda \beta - \lambda t \beta) + \\ \qquad\qquad P_4 \,.\, (-\lambda t \gamma + \lambda t^2 \gamma) + P_5 \,.\, \lambda t \alpha \end{array}\right\} (224)$$

Nehmen wir also das Simplex der fünf Punkte P_i als Koordinatensimplex, so sind die Punkte $X(t, \lambda)$ der allgemeinsten Fläche F_2^3 durch die drei Geraden a^2, α^2 und p^2 dargestellt durch

$$\left.\begin{array}{ll} \sigma X_1 = 1 + \lambda t & \\ \sigma X_2 = t + \lambda t & \text{Die drei Erzeugenden } a^2, \alpha^2, p^2 \text{ gehören} \\ \sigma X_3 = \lambda \beta (1 - t) & \text{zu den Werten } t = 0, \infty, 1;\ \gamma = 0 \text{ gibt} \\ \sigma X_4 = \lambda t \gamma (t - 1) & \text{die Leitlinie } L. \\ \sigma X_5 = \lambda t \alpha & \end{array}\right\} \quad . \quad (225)$$

Es gibt also ∞^3 Flächen F_2^3 durch die drei gegebenen Geraden, entsprechend den drei Parametern α, β, γ.