Geophysics. — *On the* STONELEY-*wave equation*. I. By J. G. SCHOLTE. (Communicated by Prof. J. D. v. D. WAALS.)

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§ 1. *Introduction.*

As early as 1899 an investigation was made by KNOTT [1]) about the relations between the amplitudes of plane waves, vibrating in the plane of incidence, which are reflected and refracted at a plane surface of separation of two infinite elastic solids.

In this problem we have always to do with 5 waves, namely the incident wave, the reflected longitudinal and transversal waves and the two refracted waves. The 4 boundary conditions (continuity of the normal and tangential components of motion and those of tension) are expressed by 4 equations, which are linear with respect to the amplitudes and the coefficients of which depend on the material constants and the angle of incidence i. Therefore the 4 amplitudes of the refracted and the reflected waves (one longitudinal and one transversal) can generally be expressed in the amplitude of the incident wave.

A particular wave system is obtained if of the two reflected types of wave only one exists, the amplitude of the other one being zero. This occurs at that value of the angle of incidence i for which the determinant of the coefficients of the 4 remaining amplitudes figuring in the 4 boundary conditions is zero.

KNOTT's calculations do not hold any longer when the amplitude of the incident wave is put equal to zero. The wave system then consists of two reflected waves and two refracted waves, while the angle i is, of course, determined again by a determinant equation. It appears that this equation is equivalent to the equation of the generalised RAYLEIGH waves derived by STONELEY [2]) in 1924.

This peculiar system of waves is seismologically of importance, because the amplitudes appear to decrease in this case in both media exponential with increasing distance to the surface of separation. Strong earthquake waves which met an interface at which these STONELEY waves are possible reach the surface of the earth very much damped and are therefore registered as weak vibrations. Consequently it is of importance to investigate at what values of the material constants of the two media a STONELEY wave system can exist, *i.e.* the STONELEY equation can be solved.

In the first part of this paper the above derivation of the STONELEY equation as an extension of the theory of KNOTT will be given; in the second part an enquiry will be made into the values of the material constants for which the STONELEY equation can be solved.

§ 2. *Derivation of the* STONELEY *equation.*

We suppose the bounding surface between the two media to be the plane $z = 0$ ($z > 0$ in medium 2) and the incident wave, being longitudinal, is propagated in medium 1. Putting the angle of incidence i_1 this wave can be expressed by $A_e . F(pt - h_1 x \sin i_1 - h_1 z \cos i_1)$; the remaining 4 waves are then:

the reflected longitudinal wave: $A_r . F(pt - h_1 x \sin i_1 + h_1 z \cos i_1)$

the reflected transversal wave: $\mathfrak{A}_r . F(pt - \mathfrak{k}_1 x \sin r_1 + \mathfrak{k}_1 z \cos r_1)$

the refracted longitudinal wave: $A_d . F(pt - h_2 x \sin i_2 - h_2 z \cos i_2)$

the refracted transversal wave: $\mathfrak{A}_d . F(pt - \mathfrak{k}_2 x \sin r_2 - \mathfrak{k}_2 z \cos r_2)$

where $\dfrac{p}{2\pi}$ = the frequency,

$h = \dfrac{p}{V}$, V being the phase velocity of the longitudinal waves

$\mathfrak{k} = \dfrac{p}{\mathfrak{B}}$, $\mathfrak{B}$ being the phase velocity of the transversal waves.

The boundary conditions are that the displacement and stress are continuous at $z = 0$. We thus obtain:

$$A_e \sin i_1 + A_r \sin i_1 + \mathfrak{A}_r \cos r_1 = A_d \sin i_2 + \mathfrak{A}_d \cos i_2$$

(tangential component of the displacement)

$$A_e \cos i_1 - A_r \cos i_1 + \mathfrak{A}_r \sin r_1 = A_d \cos i_2 - \mathfrak{A}_d \sin r_2$$

(normal component of the displacement)

$$A_e \cos 2 r_1 + A_r \cos 2 r_1 - \mathfrak{A}_r \frac{\sin 2 r_1}{n_1} = \frac{\varrho_2 V_2}{\varrho_1 V_1} A_d \cos 2 r_2 - \frac{\varrho_2 \mathfrak{B}_2}{\varrho_1 V_1} \mathfrak{A}_d \sin 2 r_2$$

(normal component of the tension)

$$A_e \sin 2 i_1 - A_r \sin 2 i_1 - \mathfrak{A}_r n_1 \cos 2 r_1 = \frac{\mu_2 V_1}{\mu_1 V_2} A_d \sin 2 i_2 + \frac{\mu_2 V_1}{\mu_1 \mathfrak{B}_2} \mathfrak{A}_d \cos 2 r_2$$

(tangential component of the tension)

in which $n_1 = \dfrac{V_1}{\mathfrak{B}_1}$, ϱ = the density and μ = the rigidity.

If we suppose the incident wave to be transversal and vibrating in the plane of inci-

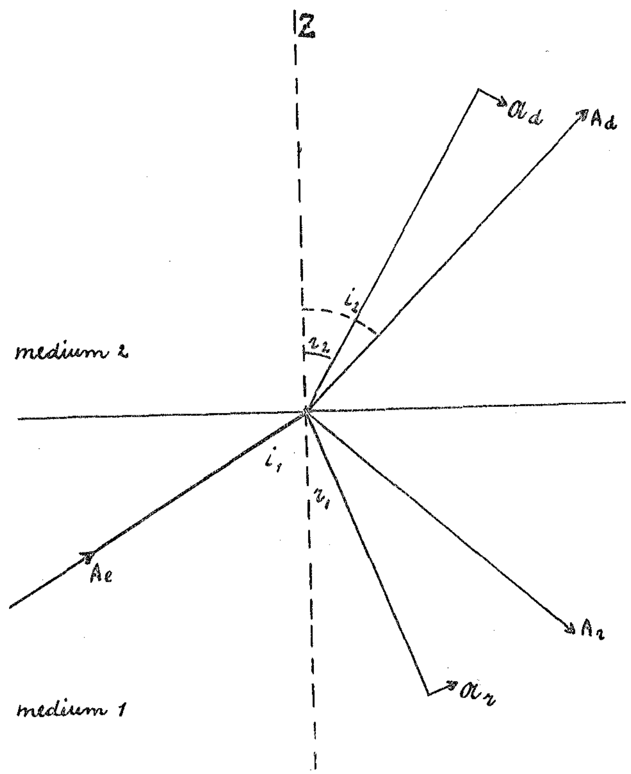

Fig. 1.

dence (figure 2), this wave can be expressed by $\mathfrak{A}_e . F\,(pt - \mathfrak{k}_1\, x \sin r_1 - \mathfrak{k}_1\, z \cos r_1)$. Writing the boundary conditions in the same order of following as above we get:

$$\mathfrak{A}_e \cos r_1 + \mathfrak{A}_r \cos r_1 + A_r \sin i_1 = \mathfrak{A}_d \cos r_2 + A_d \sin i_2$$

$$\mathfrak{A}_e \sin r_1 - \mathfrak{A}_r \sin r_1 + A_r \cos i_1 = \mathfrak{A}_d \sin r_2 - A_d \cos i_2$$

$$\mathfrak{A}_e \sin 2 r_1 + \mathfrak{A}_r \sin 2 r_1 - A_r n_1 \cos 2 r_1 = \frac{\varrho_2 \mathfrak{B}_2}{\varrho_1 \mathfrak{B}_1} \mathfrak{A}_d \sin 2 r_2 - \frac{\varrho_2 V_2}{\varrho_1 \mathfrak{B}_1} A_d \cos 2 r_2$$

$$\mathfrak{A}_e \cos 2 r_1 - \mathfrak{A}_r \cos 2 r_1 - A_r \frac{\sin 2 i_1}{n_1} = \frac{\mu_2 \mathfrak{B}_1}{\mu_1 \mathfrak{B}_2} \mathfrak{A}_d \cos 2 r_2 + \frac{\mu_2 \mathfrak{B}_1}{\mu_1 V_2} A_d \sin 2 i_2 .$$

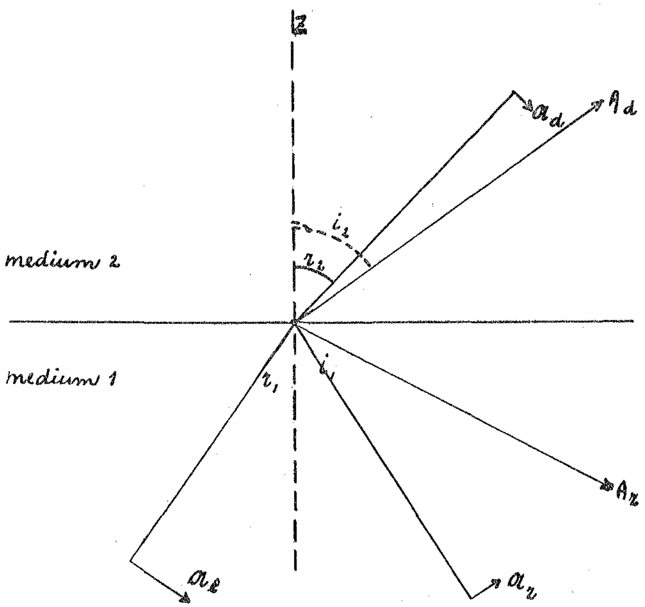

Fig. 2.

By means of such a set of equations the amplitudes of 4 waves can be expressed in the amplitude of the fifth, unless the latter is equal to zero. In that case the determinant of the coefficients of the homogeneous set of equations should be zero. If therefore we put $A_e = 0$ (fig. 1), then:

$$\begin{vmatrix} \sin i_1 & \cos r_1 & \sin i_2 & \cos r_2 \\ \cos i_1 & \sin r_1 & \cos i_2 & -\sin r_2 \\ \cos 2 r_1 & -\dfrac{\sin 2 r_1}{n_1} & \dfrac{\varrho_2 V_2}{\varrho_1 V_1} \cos 2 r_2 & -\dfrac{\varrho_2 \mathfrak{B}_2}{\varrho_1 V_1} \sin 2 r_2 \\ \sin 2 i_1 & -n_1 \cos 2 r_1 & \dfrac{\mu_2 V_1}{\mu_1 V_2} \sin 2 i_2 & \dfrac{\mu_2 V_1}{\mu_1 \mathfrak{B}_2} \cos 2 r_2 \end{vmatrix} = 0.$$

Expanding the determinant in terms of the minors of the second order formed by the first and the third row, we obtain

$$P + S + Q_1 + Q_2 + R_1 + R_2 = 0, \quad \ldots\ldots \quad (1)$$

where

$$P = n_1 n_2 \sin^2 r_2 \left(\frac{\varrho_2}{\varrho_1} \cos 2 r_2 - \cos 2 r_1 \right)^2,$$

$$S = 4 \cos i_1 \cos i_2 \cos r_1 \cos r_2 \sin^2 r_1 \left(\frac{\mu_2}{\mu_1} - 1 \right)^2,$$

$$Q_1 = n_1 \cos i_2 \cos r_2 \left\{ 2 \sin^2 r_1 \left(\frac{\mu_2}{\mu_1} - 1 \right) + 1 \right\}^2,$$

$$Q_2 = n_2 \cos i_1 \cos r_1 \left\{ 2 \sin^2 r_2 \left(\frac{\mu_1}{\mu_2} - 1 \right) + 1 \right\}^2 . \frac{\varrho_2 \mu_2}{\varrho_1 \mu_1},$$

$$R_1 = \frac{\mu_2}{\mu_1} \frac{\sin i_1 \cos i_2 \cos r_1}{\sin r_2} \quad \text{and} \quad R_2 = \frac{\varrho_2}{\varrho_1} \frac{\sin i_2 \cos i_1 \cos r_2}{\sin r_1} .$$

If we multiply these terms by μ_1/ϱ_2, it will be at once obvious that P and S are symmetrical with respect to the two suffixes 1 and 2, while Q_1 and Q_2, R_1 and R_2, if we interchange these suffixes, change into each other, so that equation (1) is symmetrical with respect to the two media.

The same equation is obtained by putting the amplitude $\mathfrak{A}_e = 0$ in the second of the two cases mentioned above (fig. 2), which is evident, because we have then the same system of waves, namely $\{A_r\ \mathfrak{A}_r\ A_d\ \mathfrak{A}_d\}$ in both cases.

Other special wave systems are possible, at wich not the incident wave, but one of the reflected waves disappears. If, for instance, in case of a longitudinal incident wave $\{A_e\}$ we put the amplitude of the longitudinal reflected wave equal to zero, we get the system $\{A_e\ \mathfrak{A}_r\ A_d\ \mathfrak{A}_d\}$; reducing the determinant equation we obtain here:

$$P - S + Q_1 - Q_2 + R_1 - R_2 = 0.$$

Further it is possible that the incident wave is propagated in the second medium (such a wave will be denoted by an accent) and that then one of the reflected waves does not exist. Therefore the following cases of special reflection occur:

$\{A_e, \mathfrak{A}_r, A_d, \mathfrak{A}_d\}$; the corresponding equation being: $P - S + Q_1 - Q_2 + R_1 - R_2 = 0$

$\{\mathfrak{A}_e, A_r, A_d, \mathfrak{A}_d\}$; the corresponding equation being: $P - S + Q_1 - Q_2 - R_1 + R_2 = 0$

$\{A'_e, \mathfrak{A}_d, A_r, \mathfrak{A}_r\}$; the corresponding equation being: $P - S - Q_1 + Q_2 - R_1 + R_2 = 0$

$\{\mathfrak{A}'_e, A_d, A_r, \mathfrak{A}_r\}$; the corresponding equation being: $P - S - Q_1 + Q_2 + R_1 - R_2 = 0$

(changing the suffixes 1 and 2 the first equation is identical with the third).

These 5 equations are all irrational, as will be obvious if we express all the circular functions in say $\sin i_1$. If we attempt to make one of the equations rational, roots are generally introduced; as all terms of these equations only differ in sign these introduced roots are roots of one of the other equations. Therefore if we rationalise equation (1) the roots introduced will be the roots of the equations for special reflection; these roots are however irrelevant. In consequence if we want to determine the value of i_1, where the wave system $\{A_r, \mathfrak{A}_r, A_d, \mathfrak{A}_d\}$ is possible, we must solve equation (1) without squaring.

Now this equation can be transformed into a more comprehensible form on account of the following remarks:

1. All terms of the left-hand side are positive if the circular functions occurring in it are real. Then equation (1) has no solution.

2. If one of the sines is imaginary, then they are all so, according to SNELLIUS' law; the terms are then partly positive partly negative, so there is a solution *à priori* possible.

3. If some, but not all, of the cosines are imaginary, the coefficients of equation (1) are partly real partly imaginary. In this case too a solution is impossible. If however all cosines are imaginary then the terms are partly negative partly positive, and the equation can, therefore, be solved. Consequently the sines of all occurring angles must be greater than 1, which is certainly true if the sinus of the smallest of these angles is greater than 1.

As $\frac{\sin i_1}{\sin r_1} = \frac{V_1}{\mathfrak{B}_1}$ and $\frac{\sin i_2}{\sin r_2} = \frac{V_2}{\mathfrak{B}_2}$ and $V_1 > \mathfrak{B}_1$, $V_2 > \mathfrak{B}_2$, r_1 and r_2 will always be smaller than i_1 and i_2.

Equation (1) being symmetrical with respect to the suffixes 1 and 2, we can henceforth assume $\mathfrak{B}_2 > \mathfrak{B}_1$ without restricting the problem; then r_1 is the smallest angle. So when all cosines are imaginary $\sin r_1 > 1$.

Summarizing these above remarks, we can assert that a solution of equation (1) is only possible if $\sin r_1$ is imaginary or is greater than 1; or putting it otherwise: $\sin^2 r_1 < 0$ or $\sin^2 r_1 > 1$.

We can condense these two inequalities into the following one: $\frac{1}{\sin^2 r_1} < 1$.

Therefore it is advisable to use $\frac{1}{\sin^2 r_1}$ as a new variable, which we shall call ζ.

If

$$\sin r_1 = \frac{1}{\sqrt{\zeta}}, \text{ then } \sin i_1 = \frac{n_1}{\sqrt{\zeta}},\ \sin i_2 = \frac{m_1}{\sqrt{\zeta}},\ \sin r_2 = \frac{m_2}{\sqrt{\zeta}}$$

where $m_1 = \frac{V_2}{\mathfrak{B}_1}$ and $m_2 = \frac{\mathfrak{B}_2}{\mathfrak{B}_1}$.

Hence

$$\cos r_1 = \frac{i\sqrt{1-\zeta}}{\sqrt{\zeta}}, \qquad \cos i_1 = \frac{i n_1 \sqrt{1-\nu_1 \zeta}}{\sqrt{\zeta}},$$

$$\cos i_2 = \frac{i m_1 \sqrt{1-\alpha\zeta}}{\sqrt{\zeta}}, \qquad \cos r_2 = \frac{i m_2 \sqrt{1-\omega\zeta}}{\sqrt{\zeta}},$$

in which

$$\nu_1 = \frac{\mathfrak{B}_1^{\,2}}{V_1^{\,2}} = \frac{\mu_1}{\lambda_1 + 2\mu_1}, \quad \omega = \frac{\mathfrak{B}_1^{\,2}}{\mathfrak{B}_2^{\,2}} = \frac{\mu_1}{\mu_2} : \frac{\varrho_1}{\varrho_2} \text{ and}$$

$$\alpha = \frac{\mathfrak{B}_1^{\,2}}{V_2^{\,2}} = \frac{\mathfrak{B}_1^{\,2}}{\mathfrak{B}_2^{\,2}} \times \frac{\mu_2}{\lambda_2 + 2\mu_2} = \omega \nu_2$$

(λ_1 and λ_2 being the constants of incompressibility)

Equation (1) then becomes

$$\left.\begin{aligned} &\left(2 - \frac{1 - \varrho_2/\varrho_1}{1 - \mu_2/\mu_1}\zeta\right)^2 + 4\sqrt{(1-\zeta)(1-\nu_1\zeta)(1-\omega\zeta)(1-\alpha\zeta)} = \\ &\sqrt{(1-\alpha\zeta)(1-\omega\zeta)} \cdot \left(2 - \frac{1}{1-\mu_2/\mu_1}\zeta\right)^2 + \sqrt{(1-\nu_1\zeta)(1-\zeta)} \cdot \left(2 + \frac{\varrho_2/\varrho_1}{1-\mu_2/\mu_1}\zeta\right)^2 + \\ &+ \frac{\varrho_2/\varrho_1}{(1-\mu_2/\mu_1)^2}\zeta^2 \cdot \left(\sqrt{(1-\zeta)(1-\alpha\zeta)} + \sqrt{(1-\omega\zeta)(1-\nu_1\zeta)}\right). \end{aligned}\right\} (2)$$

Writing the equation derived by STONELEY in this notation we get:

$$V_r^4 \{(\varrho_1-\varrho_2)^2 - (\varrho_1 x_2 + \varrho_2 x_1)(\varrho_1 y_2 + \varrho_2 y_1)\} +$$
$$+ 4 V_r^2 \{\varrho_1 x_2 y_2 - \varrho_2 x_1 y_1 - (\varrho_1 - \varrho_2)\} + 4(\mu_1-\mu_2)^2 (x_1 y_1 - 1)(x_2 y_2 - 1) = 0$$

where

$$x_1 = \sqrt{1-\nu_1 \frac{V_r^2}{\mathfrak{B}_1^2}},\ x_2 = \sqrt{1-\alpha\frac{V_r^2}{\mathfrak{B}_1^2}},\ y_1 = \sqrt{1-\frac{V_r^2}{\mathfrak{B}_1^2}},\ y_2 = \sqrt{1-\omega\frac{V_r^2}{\mathfrak{B}_1^2}},$$

V_r being the phase velocity of the STONELEY waves. Putting $\frac{V_r^2}{\mathfrak{B}_1^{\,2}} = \zeta$ this equation is identical with equation (2).

This equation reduces to a very simple form if we take

1°. $\varrho_2 = 0$: $$\left(2 - \frac{\zeta}{1-\mu_2/\mu_1}\right)^2 = 4\sqrt{(1-\nu_1\zeta)(1-\zeta)};$$

with $\mu_2 = 0$ this is the RAYLEIGH equation.

2°. $\mu_2 = 0$: $$(2-\zeta)^2 = 4\sqrt{(1-\nu_1\zeta)(1-\zeta)} - \frac{\varrho_2}{\varrho_1}\zeta\sqrt{\frac{1-\nu_1\zeta}{1-\alpha\zeta}}.$$

3°. $V_1 = V_2$ and $\mathfrak{B}_1 = \mathfrak{B}_2$ (WIECHERTS' medium):

$$\{(2-\zeta) - 2\sqrt{(1-\nu_1\zeta)(1-\zeta)}\}^2 = \left(\frac{1+\mu_2/\mu_1}{1-\mu_2/\mu_1}\zeta\right)^2 \cdot \sqrt{(1-\zeta)(1-\nu_1\zeta)}.$$