the order a^2; in that case the order of magnitude of the correction to be applied to the sedimentation velocity remains the same as that given in equation (64), but the value of the coefficient λ_{II} is not known.

It is probable that the value of the integral $\iint dS_e\, \partial\Phi/\partial n$, occurring in (22), is connected with the resultant of the frictional forces acting upon the walls of the vessel. In general the weight of the suspension will be carried partly by pressures, partly by frictional forces acting upon the walls. When the whole weight is carried by the resultant of the pressures (this is the case for a suspension enclosed between two parallel plane walls, perpendicular to the x-axis, as in the case considered in section **23.**), it is possible that formula (64) will apply with the value of λ_{II} as given in **17.** When the weight is partly or wholly carried by the frictional forces, the result perhaps may be different, and in this way an influence of the shape of the vessel could be experienced.

The application of a point of view, related to that of section **10.**, does not appear to be more promising. We might decompose the system of forces acting upon the liquid and the particles into the following components:

a) a continuous field of force having the intensity $\varrho g + nF$ per unit volume, acting through the whole space, and balanced by a pressure gradient of magnitude $\partial p/\partial x = \varrho g + nF$;

b) a set of "equilibrium systems" of the type considered in **9.**, each system having its centre at the centre of a particle;

c) a continuous field of force acting in a thin layer along the walls, making up for the "diffuse fields" of those "equilibrium systems" which would influence the field inside the vessel, if the suspension was imagined to extend also through and beyond the walls. It should be assumed again that the parameter $\varkappa$ is chosen in such a way, that $1/\varkappa$, while being large in comparison with the average distance between neighbouring particles, at the same time will be small in comparison with the dimensions of the vessel and with the radius of curvature of the walls.

In attempting to work out the equations for the motion of the liquid upon this basis, there again occur difficulties with integrals of the type $\iint dS_e\, u$ (dS_e being an element of the wall).

The difficulties probably will increase, when the number of particles per unit volume in the immediate neighbourhood of the wall should be different from the number in the more interior part of the vessel.

Provisionally the problem must be left here, in the hope that a more efficient method may be found at some later time.

Mathematics. — *A remarkable family*. By J. G. VAN DER CORPUT.

(Communicated at the meeting of January 31, 1942.)

CHAPTER III.

On analytical solutions of functional systems [1]).

Let us consider a functional system of the form

$$g_\varrho\{x, f_\nu(l_\varkappa(x))\} = 0.$$

In this chapter $\varkappa$ runs through the values $1, 2, \ldots, k$, where k denotes an integer $\geqq 2$, while ν, ϱ and σ run through the values $1, 2, \ldots, n$, where n is a positive integer. The functional system involves k given functions $l_\varkappa(x)$ of a variable x, in addition n given functions $g_\varrho(x, y_{\varkappa\nu})$ of the $1 + kn$ variables x, $y_{\varkappa\nu}$ and finally n unknown functions $f_\nu(x)$ of x. I say that the functional system possesses a solution $(f_\nu(x))$, analytical and vanishing at the origin $x = 0$, if the n funtions $f_\nu(x)$ are analytical at the origin, take at that point the value zero and satisfy the considered functional system in the vicinity of the origin.

The following examples show that several different cases are possible.

A. Dealing with the functional equation

$$f_1(x) = f_1\left(\frac{x}{2}\right) + \log\frac{1-x}{1-\frac{x}{2}},$$

we have

$$n = 1,\ k = 2,\ l_1(x) = \frac{x}{2},\ l_2(x) = x$$

and

$$g_1(x, y_{11}, y_{21}) = y_{21} - y_{11} - \log\frac{1-x}{1-\frac{x}{2}}.$$

Trying

$$f_1(x) = \sum_{\alpha=1}^{\infty} F(\alpha)\, x^\alpha. \quad \ldots \ldots \ldots \quad (1)$$

we obtain

$$F(\alpha)\left(1 - \frac{1}{2^\alpha}\right) = \frac{-1}{\alpha}\left(1 - \frac{1}{2^\alpha}\right) \quad (\alpha \geqq 1),$$

hence

$$F(\alpha) = \frac{-1}{\alpha} \quad \text{and} \quad f_1(x) = \log(1-x).$$

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[1]) Chapter I and the first part of chapter II have been published in Euclides **18** (1941—42), p. 50—78; the rest of chapter II is about to appear in the same periodical. For the well understanding of this paper it is not necessary that the reader is acquainted with the chapters I and II. The remarkable family consists of the functions characterised by functional equations.

The functional equation possesses one and only one solution analytical and vanishing at the origin.

B. Considering the functional equation

$$f_1(x+x^2)=f_1(x)+f_1\left(\frac{x}{2}\right)-\frac{x}{2},$$

we obtain

$$n=1,\ k=3,\ l_1(x)=x+x^2,\ l_2(x)=x,\ l_3(x)=\frac{x}{2}$$

and

$$g_1(x, y_{11}, y_{21}, y_{31})=y_{11}-y_{21}-y_{31}+\frac{x}{2}.$$

Trying again (1) we find for $\alpha \geqq 2$

$$F(\alpha)=2^\alpha \sum_{\frac{\alpha}{2}\leqq\beta<\alpha}\binom{\beta}{2\beta-\alpha}F(\beta),$$

hence

$$F(\alpha)\geqq 2^\alpha(\alpha-1)F(\alpha-1).$$

From $F(1)=1$ it follows that

$$F(\alpha)\geqq 2^{\frac{1}{2}(\alpha-1)(\alpha+2)}(\alpha-1)!\quad(\alpha\geqq 1).$$

The coefficients $F(\alpha)$ are defined unambiguously, but the radius of convergence of the found power series equals zero. The functional equation does not possess any solution analytical and vanishing at the origin.

C. Dealing with $f_1(x)-f_1(-x)=x^2$ and trying (1), we find relations involving the coefficients $F(\alpha)$, but these relations are contradictory. Indeed, consideration of the coefficient of x^2 gives $0=1$. It is even clear that the functional equation does not possess any solution in the vicinity of the origin, the left-hand side being an odd, the right-hand side an even function of x.

D. The functional equation $f(x)=-f(-x)$ possesses an infinity of solutions, analytical and vanishing at the origin, for any odd function is a solution.

Let us now return to the original functional system. By $\left(\frac{\delta g_e}{\delta y_{\varkappa\nu}}\right)_0$ I denote the value of that partial derivative at the origin $x=y_{\varkappa\nu}=0$ of the $(1+kn)$— dimensional espace; $\triangle$ denotes the determinant of n rows and columns, in which the constituent in the ϱ^{th} row and ν^{th} column has the value $\left(\frac{\delta g_e}{\delta y_{k\nu}}\right)_0$; finally D_α denotes the determinant of n rows and columns, in which the constituent in the ϱ^{th} row and ν^{th} column is

$$\sum_\varkappa\left(\frac{\partial g_\varrho}{\partial y_{\varkappa\nu}}\right)_0(l_\varkappa'(0))^\alpha.$$

Theorem I. *Conditions.* (1) *Suppose that the k given functions $l_\varkappa(x)$ are analytical at the origin $x=0$ and assume at that point the value zero, that $g_\varrho(x, y_{\varkappa\nu})$ is analytical and vanishes at the origin $x=y_{\varkappa\nu}=0$,* that $\triangle\neq 0$ and

$$|l_\mu'(0)|<|l_k'(0)|\quad(\mu=1,2,\dots,k-1).$$

(2) *Let us suppose in addition*

$$D_\alpha\neq 0\quad(\alpha=1,2,\dots).$$

Then the functional system possesses one and only one solution analytical and vanishing at the origin.

The proof runs as follows. The notation

$$p(x) << q(x)$$

signifies that

$$p(x)=P(0)+P(1)x+P(2)x^2+\dots$$

and

$$q(x)=Q(0)+Q(1)x+Q(2)x^2+\dots$$

are functions of x analytical at the origin with the property

$$|P(\alpha)|\leqq Q(\alpha)\quad(\alpha=0,1,\dots).$$

The notation

$$p(x, y_{\varkappa\nu}) << q(x, y_{\varkappa\nu})$$

signifies that

$$p(x, y_{\varkappa\nu})=\sum_{\beta,\gamma_{\varkappa\nu}}P(\beta,\gamma_{\varkappa\nu})x^\beta\prod_{\varkappa,\nu}y_{\varkappa\nu}^{\gamma_{\varkappa\nu}}$$

and

$$q(x, y_{\varkappa\nu})=\sum_{\beta,\gamma_{\varkappa\nu}}Q(\beta,\gamma_{\varkappa\nu})x^\beta\prod_{\varkappa,\nu}y_{\varkappa\nu}^{\gamma_{\varkappa\nu}}$$

are functions of the $1+kn$ variables x, $y_{\varkappa\nu}$ analytical at the origin $x=y_{\varkappa\nu}=0$ with the property

$$|P(\beta,\gamma_{\varkappa\nu})|\leqq Q(\beta,\gamma_{\varkappa\nu});$$

β and $\gamma_{\varkappa\nu}$ $(\varkappa=1,\dots,k;\ \nu=1,\dots,n)$ run through the sequence of the integers $\geqq 0$.

By hypothesis there exists a $(1+kn)$ — dimensional vicinity of the origin $x=y_{\varkappa\nu}=0$ with the property that the n functions $g_\varrho(x, y_{\varkappa\nu})$ are defined and analytical at every point $(x, y_{\varkappa\nu})$ of V. The determinant $\triangle$ being $\neq 0$, there exists a $(1+(k-1)n)$-dimensional vicinity V_1 of the point $x=y_{\mu\nu}=0$ (in this chapter μ runs always through the values $1, 2, \dots, k-1$) such that n analytical functions $h_\varrho(x, y_{\mu\nu})$ may be found in V_1 with the following properties; (1) if $(x, y_{\mu\nu})$ is an arbitrary point in V_1 and we put

$$y_{k\varrho}=h_\varrho(x, y_{\mu\nu}),$$

then the point $(x, y_{\varkappa\nu})$ lies in V and satisfies the n relations

$$g_\varrho(x, y_{\varkappa\nu})=0.$$

(2) The n functions $h_\varrho(x, y_{\mu\nu})$ assume at $x=y_{\mu\nu}=0$ the value zero.

(3) The n analytical functions $h_\varrho(x, y_{\mu\nu})$ are defined unambiguously in V_1 by the properties (1) and (2).

The given functional system is in the vicinity of the origin $x=0$ equivalent to

$$f_\varrho(l_k(x))=h_\varrho\{x, f_\nu(l_\mu(x))\},$$

in other words: any system of n analytical functions $f_\nu(x)$ vanishing at the origin and satisfying one of both functional systems, satisfies also the other.

From condition (1) it follows that $l'_k(0) \neq 0$, so that the substitution $l_k(x) = t$ gives in the vicinity of the origin an analytical (1, 1) transformation. Hence $x = q(t)$ and $l_\mu(x) = w_\mu(t)$ are analytical functions of t at $t = 0$ and the functional system reduces to

$$f_\varrho(t) = h_\varrho\{q(t), f_\nu(w_\mu(t))\}.$$

We can write

$$w_\mu(t) = \sum_\beta W_\mu(\beta)\, t^\beta \quad \text{and} \quad q(t) = \sum_\gamma Q(\gamma)\, t^\gamma,$$

where β and γ run through the sequence of the positive integers, and

$$h_\varrho(x, y_{\mu\nu}) = \sum_{\delta, \zeta_{\mu\nu}} H_\varrho(\delta, \zeta_{\mu\nu})\, x^\delta \prod_{\mu,\nu} y_{\mu\nu}^{\zeta_{\mu\nu}},$$

where δ and $\zeta_{\mu\nu}$ run through the sequence of the integers $\geqq 0$.

First, assume that the functional system possesses a solution

$$f_\nu(t) = \sum_\alpha F_\nu(\alpha)\, t^\alpha,$$

analytical and vanishing at the origin; α runs through the sequence of positive integers. Then we have in the vicinity of the origin

$$\sum_\eta F_\varrho(\eta)\, t^\eta = \sum_{\delta, \zeta_{\mu\nu}} H_\varrho(\delta, \zeta_{\mu\nu}) \left(\sum_\gamma Q(\gamma)\, t^\gamma\right)^\delta X(\zeta_{\mu\nu}),$$

where

$$X(\zeta_{\mu\nu}) = \prod_{\mu,\nu} \left\{\sum_\alpha F_\nu(\alpha) \left(\sum_\beta W_\mu(\beta)\, t^\beta\right)^\alpha\right\}^{\zeta_{\mu\nu}};$$

η runs through the sequence of the positive integers. The expansion of the right-hand side in powers of t produces the coefficient $F_\varrho(\eta)$ of t^η written as a sums of terms. To find one of these terms I consider a certain μ $(1 \leqq \mu \leqq k-1)$ and a certain v $(1 \leqq v \leqq n)$ and I take $\delta = 0$, $\beta = 1$, $\alpha = \eta$, $\zeta_{\mu\nu} = 1$, the other exponents $\zeta = 0$. In the term found in this manner we have

$$W_\mu(\beta) = w'_\mu(0) \quad \text{and} \quad H_\varrho(\delta, \zeta_{\mu\nu}) = \left(\frac{\partial h_\varrho}{\partial y_{\mu\nu}}\right)_0$$

and the term in question is therefore

$$\left(\frac{\partial h_\varrho}{\partial y_{\mu\nu}}\right)_0 F_\nu(\eta)\, (w'_\mu(0))^\eta.$$

In this manner we find that $F_\varrho(\eta)$ is equal to

$$\sum_{\mu,\nu} \left(\frac{\partial h_\varrho}{\partial y_{\mu\nu}}\right)_0 F_\nu(\eta)\, (w'_\mu(0))^\eta$$

augmented by the sum of the other terms; this sum is a polynomial $u_\varrho(F_\sigma(\alpha))$ in the numbers $F_\sigma(\alpha)$, where σ runs through $1, 2, \ldots, n$ and α runs through $1, 2, \ldots, \eta - 1$. Thus we obtain

$$F_\varrho(\eta) = \sum_\nu F_\nu(\eta) \sum_\mu \left(\frac{\partial h_\varrho}{\partial y_{\mu\nu}}\right)_0 (w'_\mu(0))^\eta + u_\varrho(F_\sigma(\alpha)). \quad . \quad . \quad . \quad 2)$$

If the coefficients $F_\sigma(\alpha)$ $(\alpha < \eta)$ are already known, we find n linear equations with n

unknown coefficients $F_\nu(\eta)$. The determinant $E(\eta)$ of this system of equations possesses n rows and columns; the constituent in the ϱ^{th} row and ν^{th} column is

$$\varepsilon_{\varrho\nu} - \sum_\mu \left(\frac{\partial h_\varrho}{\partial y_{\mu\nu}}\right)_0 (w'_\mu(0))^\eta,$$

where

$$\begin{aligned} \varepsilon_{\varrho\nu} &= 1 \quad \text{for} \quad \varrho = \nu \\ &= 0 \quad \text{for} \quad \varrho \neq \nu. \end{aligned}$$

Using

$$w'_\mu(0) = \frac{l'_\mu(0)}{l'_k(0)},$$

we observe that the constituent in the ϱ^{th} row and ν^{th} column of the determinant $(l'_k(0))^{n\eta} E(\eta)$ has the value

$$(l'_k(0))^\eta\, \varepsilon_{\varrho\nu} - \sum_\mu \left(\frac{\partial h_\varrho}{\partial y_{\mu\nu}}\right)_0 (l'_\mu(0))^\eta.$$

Since $y_{k\varrho} = h_\varrho(x, y_{\mu\nu})$ satisfies the system $g_\sigma(x, y_{\varkappa\nu}) = 0$, we have

$$\left(\frac{\partial g_\sigma}{\partial y_{\mu\nu}}\right)_0 + \sum_\varrho \left(\frac{\partial g_\sigma}{\partial y_{k\varrho}}\right)_0 \left(\frac{\partial h_\varrho}{\partial y_{\mu\nu}}\right)_0 = 0.$$

$\triangle$ being a determinant of n rows and columns, in which the constituent in the σ^{th} row and ϱ^{th} column equals $\left(\frac{\partial g_\sigma}{\partial y_{k\varrho}}\right)_0$, the product of $(l'_k(0))^{n\eta} E(\eta)$ and $\triangle$ is the determinant, whose constituent in the σ^{th} row and ν^{th} column has the value

$$\begin{aligned} (l'_k(0))^\eta \sum_\varrho \left(\frac{\partial g_\sigma}{\partial y_{k\varrho}}\right)_0 \varepsilon_{\varrho\nu} - \sum_\mu (l'_\mu(0))^\eta \sum_\varrho \left(\frac{\partial g_\sigma}{\partial y_{k\varrho}}\right)_0 \left(\frac{\partial h}{\partial y_{\mu\nu}}\right)_0 \\ = \left(\frac{\partial g_\sigma}{\partial y_{k\nu}}\right)_0 (l'_k(0))^\eta + \sum_\mu \left(\frac{\partial g_\sigma}{\partial y_{\mu\nu}}\right)_0 (l'_\mu(0))^\eta = \sum_\varkappa \left(\frac{\partial g_\sigma}{\partial y_{\varkappa\nu}}\right)_0 (l'_\varkappa(0))^\eta, \end{aligned}$$

hence

$$D(\eta) = E(\eta)\, (l'_k(0))^{n\eta}\, \triangle \qquad (\eta = 1, 2, \ldots). \quad . \quad . \quad . \quad . \quad (3)$$

$D(\eta)$ being $\neq 0$, we find $E(\eta) \neq 0$. Hence: if the coefficients $F_\sigma(\alpha)$ $(\alpha < \eta)$ are already known, the n coefficients $F_\nu(\eta)$ are defined unambiguously by the n relations (2). In this manner we have proved:

The functional system possesses at most one solution analytical and vanishing at the origin.

By means of the recurrent relations (2) we can determine the coefficients $F_\nu(\eta)$. Thus we obtain n formal power series $\sum_\alpha F_\nu(\alpha)\, t^\alpha$. To prove the theorem it is sufficient to show that each of these power series possesses a positive radius of convergence. In fact, these power series give then a solution, analytical and vanishing at the origin.

As we have shown, the determinants $E(\eta)$ $(\eta = 1, 2, \ldots)$ differ from zero. It follows from $|w'_\mu(0)| < 1$, that for $\eta \to \infty$ each constituent in the principal diagonal of $E(\eta)$ tends to 1, each other constituent to zero. Therefore $E(\eta)$ tends to 1 and $|E(\eta)|$

possesses a positive lower bound independent of η. Each constituent of $E(\eta)$ being bounded, we deduce from (2)

$$|F_\nu(\eta)| \leqq \frac{A}{n} \sum_{\varrho} |u_\varrho(F_\sigma(\alpha))|, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where A denotes an appropiate number independent of η.
Since $|w'_\mu(0)| < 1$, we have

$$|w'_\mu(0)| \leqq \Theta < 1,$$

where Θ is a conveniently chosen positive number < 1. Since $w_\mu(x)$ and $q(x)$ are analytical at $x = 0$, and $h_\varrho(x, y_{\mu\nu})$ at $x = y_{\mu\nu} = 0$, we have for sufficiently large K and B

$$w_\mu(x) << \frac{\Theta x}{1-Kx}, \qquad q(x) << \frac{Bx}{1-Kx}$$

and

$$h_\varrho(x, y_{\mu\nu}) << \frac{B}{(1-Kx) \prod_{\mu,\nu} (1-K y_{\mu\nu})}.$$

I choose the positive number Γ so small and then the positive number Λ so large that

$$(1+2\Gamma)\left(\Theta + \frac{K}{\Lambda}\right) < 1 \quad \text{and} \quad \Gamma(K + \Lambda\Theta) \geqq KB \quad . \quad . \quad . \quad (5)$$

For each z with absolute value $< \frac{1}{1+\Gamma}$ the function

$$(1-z)^{1+(k-1)n} \{1-(1+\Gamma)z\}^{-1-(k-1)n}$$

possesses an expansion in powers of z. This expansion being valid for $z = \frac{1}{1+2\Gamma}$, we obtain for sufficiently large M

$$(1-z)^{1+(k-1)n} \{1-(1+\Gamma)z\}^{-1-(k-1)n} << \frac{M}{1-(1+2\Gamma)z}. \quad . \quad . \quad (6)$$

From (5) it appears that

$$\Lambda B M K (1+2\Gamma)^{N+1} \left(\Theta + \frac{K}{\Lambda}\right)^{N+1} < \Gamma \quad . \quad . \quad . \quad . \quad (7)$$

for sufficiently large positive integer N. Finally I choose $H \geqq \Lambda$ so large that the inequality

$$|F_\nu(\eta)| \leqq \frac{\Gamma}{K} H^\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

holds for $\nu = 1, 2, \ldots, n$ and $\eta = 1, 2, \ldots, N$. It is sufficient to show that this inequality holds for $\nu = 1, 2, \ldots, n$ and every positive integer η. In fact, then the n power series $\sum_\alpha F_\nu(\alpha) t^\alpha$ possess a radius of convergence $\geqq \frac{1}{H}$. I may assume $\eta \geqq N+1$ and suppose that the inequalities

$$|F_\nu(\alpha)| \leqq \frac{\Gamma}{K} H^\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

are proved already for any $\alpha < \eta$.

The following argument is based on inequality (4); $u_\varrho(F_\sigma(\alpha))$ is the coefficient of t^η in the expansion of $h_\varrho\{q(t), j_\nu(w_\mu(t))\}$ where

$$j_\nu(x) = \sum_{\alpha<\eta} F_\nu(\alpha) x^\alpha.$$

From (9) it follows that

$$j_\nu(x) << \frac{\Gamma H x}{K(1-Hx)},$$

hence

$$K j_\nu(w_\mu(t)) << \frac{\Gamma H \dfrac{\Theta t}{1-Kt}}{1 - H\dfrac{\Theta t}{1-Kt}} = \frac{\Gamma H \Theta t}{1-(K+H\Theta)t} << \frac{\Gamma z}{1-z},$$

where $z = (K + H\Theta)t$. Also from (5)

$$\Gamma(K + H\Theta) \geqq \Gamma(K + \Lambda\Theta) \geqq KB,$$

hence

$$K q(t) << \frac{KBt}{1-Kt} << \frac{\Gamma z}{1-z}.$$

In this manner we obtain

$$h_\varrho\{q(t), j_\nu(w_\mu(t))\} << \frac{B}{\left(1 - \dfrac{\Gamma z}{1-z}\right)^{1+(k-1)n}}$$

$$= B(1-z)^{1+(k-1)n}\{1-(1+\Gamma)z\}^{-1-(k-1)n} << \frac{BM}{1-(1+2\Gamma)z}$$

by (6)

$$= \frac{BM}{1-(1+2\Gamma)(K+H\Theta)t} << \frac{BM}{1-(1+2\Gamma)H\left(\Theta + \dfrac{K}{\Lambda}\right)t}$$

$$<< BM \sum_{\alpha=0}^{N} H^\alpha t^\alpha + \frac{\Gamma}{\Lambda K} \sum_{\alpha=N+1}^{\infty} H^\alpha t^\alpha$$

according to (5) and (7). The absolute value of the coefficient $u_\varrho(F_\sigma(\alpha))$ of t^η in this expansion is therefore $\leqq \frac{\Gamma}{\Lambda K} H^\eta$ for every $\eta > N$, so that it follows from (4), that (8) holds for every $\eta > N$. This establishes the theorem.

To be continued.