Mathematics. — *On the uniqueness of solutions of differential equations.* By J. G. VAN DER CORPUT.

(Communicated at the meeting of January 31, 1942.)

Theorem 1. *Consider* $n+2$ *real numbers* $\xi, \eta_1, \ldots, \eta_{n+1}$, *further a positive number* ω *and the polynomial*

$$p(x) = \sum_{\nu=0}^{n} \eta_{\nu+1} \frac{(x-\xi)^\nu}{\nu!}.$$

Let the real function $f(x, y_1, \ldots, y_n)$ *be defined in the* $(n+1)$ — *dimensional strip*

$$\xi < x < \xi + \omega, -\infty < y_1 < \infty, \ldots, -\infty < y_n < \infty$$

in such a manner that any two points $(x, y_1, \ldots, y_n)$ *and* $(x, Y_1, \ldots, Y_n)$ *of the region*

$$\xi < x < \xi + \omega; \quad \left| y_\nu - \frac{d^\nu p(x)}{dx^\nu} \right| < \omega \frac{(x-\xi)^{n-\nu}}{(n-\nu)!} \quad (\nu = 1, \ldots, n)$$

satisfy the inequality

$$|f(x, Y_1, \ldots, Y_n) - f(x, y_1, \ldots, y_n)| \leqq \max_{\nu=1,\ldots,n} \frac{(n-\nu+1)!}{(x-\xi)^{n-\nu+1}} |Y_\nu - y_\nu|. \quad (1)$$

Assertion: If the positive number ε is small enough, there exists in the interval $\xi \leqq x \leqq \xi + \varepsilon$ one function y at most, which is n times differentiable in that interval, which satisfies the differential equation

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1} y}{dx^{n-1}}\right)$$

in the interval $\xi < x \leqq \xi + \varepsilon$ *and which satisfies the initial conditions*

$$y = \eta_1, \frac{dy}{dx} = \eta_2, \ldots, \frac{d^n y}{dx^n} = \eta_{n+1} \text{ for } x = \xi.$$

The special case $n = 1$ is nearly equivalent to the theorem of Nagumo.

To prove the theorem I may assume that there exists in the interval $\xi \leqq x \leqq \xi + a$, where a is an appropiate positive number $\leqq \omega$, one function $y = \varphi(x)$ at least with the mentioned properties.

Then

$$\frac{d^{n-1} \varphi(x)}{dx^{n-1}} \text{ and } \frac{d^n \varphi(x)}{dx^n}$$

take at $x = \xi$ the values η_n and η_{n+1}, so that

$$\frac{1}{x-\xi}\left(\frac{d^{n-1}\varphi(x)}{dx^{n-1}} - \eta_n\right)$$

tends to η_{n+1}, if $x - \xi$ approaches zero. Hence we find in the interval $\xi < x \leqq \xi + \beta$, where β is a convenient positive number $\leqq a$,

$$\left| \frac{d^{n-1}\varphi(x)}{dx^{n-1}} - \eta_n - \eta_{n+1}(x-\xi) \right| < \omega (x-\xi),$$

i.e.

$$\left| \frac{d^{n-1}\varphi(x)}{dx^{n-1}} - \frac{d^{n-1} p(x)}{dx^{n-1}} \right| < \omega (x-\xi). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

I shall show that any positive $\varepsilon \leqq \beta$ possesses the required properties. Since the functions $\varphi(x)$ and $p(x)$ and their 1^{st}, 2^{nd}, …, $(n-2)^{th}$ derivatives have at $x = \xi$ the same values, it follows from (2) by repeated integration that

$$\left| \frac{d^\nu \varphi(x)}{dx^\nu} - \frac{d^\nu p(x)}{dx^\nu} \right| < \omega \frac{(x-\xi)^{n-\nu}}{(n-\nu)!} \quad (\nu = 1, 2, \ldots, n-1). \quad . \quad . \quad (3)$$

Let us now consider an arbitrary real function $\psi(x)$, which is n times differentiable in the interval $\xi \leqq x \leqq \xi + \varepsilon$, which satisfies the differential equation in the interval $\xi < x \leqq \xi + \varepsilon$ and which satisfies the initial conditions. My object in writing this article is to prove $\psi(x) = \varphi(x)$ for $\xi \leqq x \leqq \xi + \varepsilon$.

If the positive number $\gamma \leqq \varepsilon$ is small enough, the inequalities (3) hold in the interval $\xi < x \leqq \xi + \gamma$ with $\psi(x)$ for $\varphi(x)$. Let δ be the upper bound of these numbers γ.

It is sufficient to show that $\varphi(x) = \psi(x)$ in the interval $\xi \leqq x \leqq \xi + \delta$. Indeed, I assert that in that case $\delta = \varepsilon$, so that ε possesses the required properties. In fact, suppose $\delta < \varepsilon$. Since $\psi(x) = \varphi(x)$ in the interval $\xi \leqq x \leqq \xi + \delta$, it follows from considerations of continuity that (3) holds also in the vicinity of δ, contrary to the choice of δ. Hence $\delta = \varepsilon$.

The functions

$$(x-\xi)^{-n+\nu-1}\left(\frac{d^{\nu-1}\psi(x)}{dx^{\nu-1}} - \frac{d^{\nu-1}\varphi(x)}{dx^{\nu-1}}\right) \quad (\nu = 1, 2, \ldots, n)$$

tend with $x - \xi$ to zero. Hence

$$M = \max_{\substack{1 \leqq \nu \leqq n \\ \xi < x \leqq \xi + \delta}} \frac{(n-\nu+1)!}{(x-\xi)^{n-\nu+1}} \left| \frac{d^{\nu-1}\psi(x)}{dx^{\nu-1}} - \frac{d^{\nu-1}\varphi(x)}{dx^{\nu-1}} \right|$$

is a finite number. Since $\varphi(x)$ and $\psi(x)$ satisfy the differential equation, we have in the interval $\xi < x \leqq \xi + \delta$

$$\left| \frac{d^n \psi(x)}{dx^n} - \frac{d^n \varphi(x)}{dx^n} \right|$$

$$\leqq \left| f\left(x, \psi, \frac{d\psi}{dx}, \ldots, \frac{d^{n-1}\psi}{dx^{n-1}}\right) - f\left(x, \varphi, \frac{d\varphi}{dx}, \ldots, \frac{d^{n-1}\varphi}{dx^{n-1}}\right) \right|.$$

From (1) it follows that in the interval $\xi \leqq x \leqq \xi + \delta$

$$\left| \frac{d^n \psi(x)}{dx^n} - \frac{d^n \varphi(x)}{dx^n} \right| \leqq M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and therefore

$$\left| \frac{d^{n-1} \psi(x)}{dx^{n-1}} - \frac{d^{n-1} \varphi(x)}{dx^{n-1}} \right| \leqq M(x-\xi) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It is sufficient to show $M = 0$.

Let us first consider the case in which in (5) the equality holds for every $x > \xi$ in the vicinity of ξ.

The left-hand side being a continuous function of x, we have for these numbers x

$$\frac{d^{n-1} \psi(x)}{dx^{n-1}} - \frac{d^{n-1} \varphi(x)}{dx^{n-1}} = M(x-\xi) \text{ or } -M(x-\xi),$$

which implies

$$\frac{d^n \psi(x)}{dx^n} - \frac{d^n \varphi(x)}{dx^n} = M \text{ or } -M \text{ for } x = \xi;$$

hence $M = 0$, since

$$\frac{d^n \varphi(x)}{dx^n} \text{ and } \frac{d^n \psi(x)}{dx^n}$$

assume at ξ the same value η_{n+1}.

Now it is sufficient to show that the remaining case is excluded. Let x be an arbitrary number $> \xi$ and $\leqq \xi + \delta$. In the remaining case there would exist a number $\lambda > \xi$ and $< x$ satisfying the inequality

$$\left| \frac{d^{n-1} \psi(\lambda)}{d\lambda^{n-1}} - \frac{d^{n-1} \varphi(\lambda)}{d\lambda^{n-1}} \right| < M(\lambda - \xi)$$

and from (4) it would follow that

$$\left| \frac{d^{n-1} \psi(x)}{dx^{n-1}} - \frac{d^{n-1} \varphi(x)}{dx^{n-1}} - \frac{d^{n-1} \psi(\lambda)}{d\lambda^{n-1}} + \frac{d^{n-1} \varphi(\lambda)}{d\lambda^{n-1}} \right| \leqq M(x - \lambda).$$

Hence (5) would follow with $<$ for $\leqq$ and by repeated integration we should find

$$\left| \frac{d^{\nu-1} \psi(x)}{dx^{\nu-1}} - \frac{d^{\nu-1} \varphi(x)}{dx^{\nu-1}} \right| < M \frac{(x-\xi)^{n-\nu+1}}{(n-\nu+1)!} \qquad (\nu = 1, \dots, n; \xi < x \leqq \xi + \delta),$$

which is impossible, since according to the definition of M the two sides are equal one to another for conveniently chosen ν and x ($\xi < x \leqq \xi + \delta$). This establishes the theorem.

Theorem 2. Consider $n + 2$ *real numbers* $\xi, \eta_1, \dots, \eta_{n+1}$, *further a positive number* ω *and the polynomial*

$$p(x) = \sum_{\nu=0}^{n} \eta_{\nu+1} \frac{(x-\xi)^\nu}{\nu!}.$$

Let the real function $f(x, y_1, \dots, y_n)$ *be defined in the* $(n+1)$ — *dimensional strip*

$$\xi < x < \xi + \omega, \; -\infty < y_1 < \infty, \dots, -\infty < y_n < \infty.$$

Suppose that in the region

$$\xi < x < \xi + \omega, \qquad \left| y_\nu - \frac{d^\nu p(x)}{dx^\nu} \right| < \omega \frac{(x-\xi)^{n-\nu}}{(n-\nu)!} \qquad (\nu = 1, \dots, n)$$

the derivatives $f_\nu = \frac{\partial f}{\partial y_\nu}$ *of the first order of* f *exist, are continuous and satisfy the inequality*

$$\sum_{\nu=1}^{n} |f_\nu(x, y_1, \dots, y_n)| \frac{(x-\xi)^{n-\nu+1}}{(n-\nu+1)!} \leqq 1.$$

Then the assertion of theorem 1 is valid.

In fact, we have

$$|f(x, Y_1, \dots, Y_n) - f(x, y_1, \dots, y_n)| = \left| \sum_{\nu=1}^{n} \frac{\partial f(x, \lambda_1, \dots, \lambda_n)}{\partial \lambda_\nu} \cdot (Y_\nu - y_\nu) \right|.$$

where $(\lambda_1, \dots, \lambda_n)$ is a point conveniently chosen between $(y_1, \dots, y_n)$ and $(Y_1, \dots, Y_n)$ and the found expression is

$$\leqq \left\{ \max_{1 \leqq \nu \leqq n} \frac{(n-\nu+1)!}{(x-\xi)^{n-\nu+1}} |Y_\nu - y_\nu| \right\} \sum_{\nu=1}^{n} \left| \frac{\partial f(x, \lambda_1, \dots, \lambda_n)}{\partial \lambda_\nu} \right| \cdot \frac{(x-\xi)^{n-\nu+1}}{(n-\nu+1)!}$$

$$\leqq \max_{1 \leqq \nu \leqq n} \frac{(n-\nu+1)!}{(x-\xi)^{n-\nu+1}} \cdot |Y_\nu - y_\nu|,$$

so that theorem 2 is a corollary of the first proposition.

Mathematics. — *Die Kovarianten von vier Ebenen im* R_5. Von R. WEITZENBÖCK.

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Ich habe in einer früheren Arbeit [1]) die projektiven Invarianten von vier Ebenen im R_5 ermittelt; hier sollen die Kovarianten mit einer Reihe Punktkoordinaten x berechnet werden. Man findet deren acht, alle vom dritten Grade in den x, die durch drei quadratische Syzygien verknüpft sind. Zugleich ergibt sich, dass drei Ebenen des R_5 keine Kovarianten mit nur einer Reihe und daher auch keine Kontravarianten mit nur einer Reihe R_4-Koordinaten u' besitzen.

§ 1.

Wir deuten die vier Ebenen durch a, α, p, π oder 1, 2, 3, 4 an. Der Aufbau der Komitanten führt dann, genau so wie in obgenannter Arbeit, auf Klammerfaktoren der Typen

$$f_1 = (a^3 \alpha^2 p) \text{ und } f_2 = (a^3 \alpha^2 x).$$

Hier leitet f_1 zu Ketten der Gestalt

$$f_3 = (a^3 \alpha^2 p)(p^2 \pi^3 b)(b^2 \beta^3 q) \dots = 12 \cap 34 \cap 12 \cap \dots,$$

die auf Invarianten

$$A_{ik} = (i^3 k^3) \text{ und } J_{ijkl} = (i^3 j^2 k)(jk^2 l^3)$$

reduzierbar sind.

Bei f_3 setzen wir

$$P_{ik} = (i^2 k^3 x) i, \qquad (1)$$

was also neben der Reihe x die sechs Reihen

$$P_{12}, P_{13}, P_{14}, P_{34}, P_{42}, P_{23}$$

gibt. Es gilt dann

$$P_{ik} = P_{ki} - \tfrac{1}{3} A_{ik} x. \qquad (2)$$

Geometrisch ist P_{ik} der in der Ebene E_i gelegene Punkt des R_3, der x mit E_k verbindet. Es gilt also z.B.

$$(1^3 P_{1i} \xi \eta) = 0, \quad (1^3 P_{i1} \xi \eta) = -\tfrac{1}{3} A_{i1} (1^3 x \xi \eta)$$

und

$$(1^3 P_{1i} x \xi) = (1^3 P_{i1} x \xi) = 0.$$

Nimmt man in (1) statt x ein P_{rs} selbst, so ergeben sich die Reihen

$$(i^3 k^2 P_{rs}) k$$

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1) Proc. Kon. Akad. v. Wetensch., Amsterdam, **35**, 1026—1029 (1932).