$$\frac{k_3}{2} = 0{,}08475685$$

$$\frac{k_3 k_4}{4} = 0{,}00030889$$

$$1 + \frac{k_3}{2} + \frac{k_3 k_4}{4} = 1{,}08506574 \quad \rightarrow \quad log = 0{,}0354561$$

$$log\, \pi = 0{,}4971499$$

$$3\, log\, k_1 = 9{,}9800545 \quad +$$

$$0{,}5126605$$

$$log\, 16\, a_2^2\, a_4 = 0{,}4384846 \quad -$$

$$log\, k_1^3\, C_4(k_1) = \underline{0{,}0741759}$$

$$k_1^3\, C_4(k_1) = 1{,}186249$$

$$\text{Fehler} \approx k_1^3\, k_5\, C_4(k_1) \approx k_1^3\, C_4(k_1) \cdot \frac{k_4^2}{4} = 0{,}000016 \quad +$$

$$k_1^3\, C_5(k_1) = \underline{1{,}186265}$$

$$log\, k_1^3\, C_5(k_1) = \underline{0{,}0741816}$$

in genauer Uebereinstimmung mit MAXWELL (l.c. S. 319).

Mathematics. — *Conformal differential geometry.* II. *Curves in conformal two-dimensional spaces.* By J. HAANTJES. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of February 28, 1942.)

Summary.

In a former paper [1]) a method has been introduced for developing the conformal differential geometry of curves in flat spaces of dimension $n > 2$. In this note it is proved that the same theory holds also for $n = 2$ if we restrict ourselves to the conformal transformations of the Möbius group. In particular the conformal Frenet-Serret formulae, which give differential relations between the fundamental quantities of a curve, have exactly the same form. Furthermore geometrical interpretations are given of these fundamental quantities, which include among other things the conformal parameter and the inversion curvature.

The fundamental theorem.

Let $a_{\lambda\varkappa}$ be the fundamental tensor of a 2-dimensional flat space R_2, in which the coordinates are denoted by $x^\varkappa$. This coordinate system is assumed to be a rectangular cartesian one, though we need not to restrict ourselves to these systems. In C.D.G. I [1]) we have proved the following theorem:

The conformal invariant properties in a flat space are those properties, which are invariant under a conformal transformation of the fundamental tensor

$$a'_{\lambda\varkappa} = \sigma^2\, a_{\lambda\varkappa}, \qquad (1)$$

such that the space remains a flat space.

Therefore, the curvature $\varkappa'$ of the metric tensor $a'_{\lambda\varkappa}$ has to vanish. This curvature is given by the equation

$$\varkappa' = -\,\sigma^{-2}\, a^{\mu\nu}\, \partial_\mu\, \partial_\nu\, log\, \sigma, \qquad (2)$$

taken from SCHOUTEN–STRUIK [2]). Hence the function σ in (1) must satisfy the equation

$$a^{\mu\nu}\, \partial_\mu\, s_\nu = 0\,; \quad s_\nu = \partial_\nu\, log\, \sigma. \qquad (3)$$

The above theorem applies to the whole set of conformal transformations and it enables us to develop the differential geometry of this set of transformations.

In this paper however we will restrict ourselves to those conformal transformations, which transform circles into circles (the so called Möbius group). This restriction imposes an additional condition on σ, which can be deduced by requiring that a circle remains a circle, if $a'_{\lambda\varkappa}$ is taken as the fundamental tensor instead of $a_{\lambda\varkappa}$.

Let the arc-lengths of a curve C with respect to $a_{\lambda\varkappa}$ and $a'_{\lambda\varkappa}$ be denoted by s and s', the corresponding covariant derivatives along the curve by δ/ds and δ'/ds' respectively. The coordinate system being a cartesian one for the metric $a_{\lambda\varkappa}$, the covariant derivative δ/ds is identical with the ordinary derivative d/ds. If $i^\varkappa$ is the unit vector tangent to the curve, the curvature k of C may be found from the Frenet formulae

$$\frac{\delta i^\varkappa}{ds} = k\, \underset{1}{i}{}^\varkappa\,; \quad \frac{\delta \underset{1}{i}{}^\varkappa}{ds} = -\,k\, i^\varkappa, \qquad (4)$$

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[1]) Conformal differential geometry. Curves in conformal euclidean spaces, Proc. Ned. Akad. v. Wetensch., Amsterdam, **44**, 814—824 (1941), referred to as C.D.G. I.

[2]) SCHOUTEN–STRUIK, Einführung in die neueren Methoden der Differentialgeometrie. II (Noordhoff, 1938) p. 291, formel (19.1 β).

where $\underset{1}{i}^{\varkappa}$ is a unit vector normal to $i^{\varkappa}$. Since $i^{\varkappa}$ and $\underset{1}{i}^{\varkappa}$ are unit vectors, they transform under a conformal transformation (1) of the $a_{\lambda\varkappa}$ as follows

$$i'^{\varkappa} = \sigma^{-1}\, i^{\varkappa}; \quad \underset{1}{i}'^{\varkappa} = \sigma^{-1}\, \underset{1}{i}^{\varkappa}. \qquad (5)$$

These transformed vectors are related by the following differential equation

$$\frac{\delta'\, i'^{\varkappa}}{ds'} = k'\, \underset{1}{i}'^{\varkappa}, \qquad (6)$$

where k' is the curvature of C with respect to the tensor $a'_{\lambda\varkappa}$. Now if $p^{\varkappa}$ is any contravariant vector we have [3]

$$\frac{\delta'\, p^{\varkappa}}{ds'} = \sigma^{-1} \left\{ \frac{\delta\, p^{\varkappa}}{ds} + (s_{\mu}\, i^{\mu})\, p^{\varkappa} + (s_{\mu}\, p^{\mu})\, i^{\varkappa} - a_{\mu\lambda}\, p^{\mu}\, i^{\lambda}\, s^{\varkappa} \right\}. \qquad (7)$$

From the equations (4), (6) and (7) follows by a simple calculation the relation between k and k'

$$k' = \sigma^{-1}\, (k - s_{\mu}\, \underset{1}{i}^{\mu}). \qquad (8)$$

Hence we have

$$\frac{d k'}{ds'} = \sigma^{-1} \left\{ \frac{d k}{ds} - (\partial_{\nu}\, s_{\mu})\, i^{\nu}\, \underset{1}{i}^{\mu} + s_{\mu}\, s_{\nu}\, i^{\mu}\, \underset{1}{i}^{\nu} \right\}. \qquad (9)$$

Now suppose the curve C is a circle with respect to the metric $a_{\lambda\varkappa}$. Then k is constant and so will be k' if a circle will remain a circle under the conformal transformation of the fundamental tensor. This leads as is seen from (9) to the vanishing of

$$(\partial_{\nu}\, s_{\mu} - s_{\nu}\, s_{\mu})\, i^{\nu}\, \underset{1}{i}^{\mu} \qquad (10)$$

for every pair of two mutually orthogonal vectors i^{ν} and $\underset{1}{i}^{\mu}$, which condition is equivalent with the equation

$$\partial_{\nu}\, s_{\mu} - s_{\nu}\, s_{\mu} = \lambda\, a_{\mu\nu}. \qquad (11)$$

Multiplication by $a^{\mu\nu}$ gives in connection with (3)

$$\lambda = -\tfrac{1}{2}\, a^{\mu\nu}\, s_{\nu}\, s_{\mu} = -\tfrac{1}{2}\, s_{\nu}\, s^{\nu}. \qquad (12)$$

Hence σ satisfies the equation

$$\partial_{\nu}\, s_{\mu} - s_{\nu}\, s_{\mu} + \tfrac{1}{2}\, a_{\mu\nu}\, s_{\varrho}\, s^{\varrho} = 0. \qquad (13)$$

As (3) is a consequence of (13), this equation is the only condition imposed upon σ. So we have arrived at the following result.

The conformal properties, which are invariant under the transformations of the Möbius group, are those properties, which are invariant under a conformal transformation of the fundamental tensor $a'_{\lambda\varkappa} = \sigma^2\, a_{\lambda\varkappa}$, σ *satisfying the equation* (13).

By comparing this theorem with the result obtained in C.D.G. I concerning the conformal transformations in an R_n $(n > 2)$, we see that for every $n > 1$ σ satisfies the same equation. But only for $n > 2$ the equation (13) appears to be a direct consequence of the vanishing of the curvature affinor belonging to $a'_{\lambda\varkappa}$. As a result of this the conformal theory developed in C.D.G. I may be applied to the case $n = 2$. We give here a brief summary of the results specialised for $n = 2$.

[3] Compare C.D.G. I, p. 816.

Fundamental relations.

a. Let the curvatures of a curve for the metrics $a_{\lambda\varkappa}$ and $a'_{\lambda\varkappa}$ be denoted by k and k' respectively. From (9) and (13) it follows that

$$\frac{d k'}{ds'} = \sigma^{-2}\, \frac{d k}{ds}. \qquad (14)$$

We choose the direction of increasing s so that $\frac{dk}{ds}$ is positive. The relation (14) enables us to define on the curve a conformal invariant parameter τ

$$\tau = \int \sqrt{\varrho}\, ds + \text{constant}; \; \varrho = \frac{dk}{ds}. \qquad (15)$$

This parameter of the third order is called the *conformal parameter* of the curve.

b. Instead of $i^{\varkappa}$ and $\underset{1}{i}^{\varkappa}$ (comp. (4) we use the following conformal invariant vectors

$$j^{\varkappa} = \frac{d x^{\varkappa}}{d\tau} = \varrho^{-\frac{1}{2}}\, i^{\varkappa}; \; \underset{1}{j}^{\varkappa} = \varrho^{-\frac{1}{2}}\, \underset{1}{i}^{\varkappa}, \qquad (16)$$

which have the direction of the tangent and the normal respectively.

c. The covariant differentiation to s being not a conformal invariant differentiation is replaced by a *conformal covariant differentiation* to the parameter τ. This differentiation is defined by the connexion parameters

$$\Gamma^{\varkappa}_{\mu\lambda} = \left\{ {\varkappa \atop \mu\lambda} \right\} + Q_{\mu}\, A^{\varkappa}_{\lambda} + Q_{\lambda}\, A^{\varkappa}_{\mu} - a_{\mu\lambda}\, a^{\varkappa\nu}\, Q_{\nu}, \qquad (17)$$

where $A^{\varkappa}_{\lambda}$ is the unit affinor and Q_{μ} is given by the equation

$$Q_{\mu} = k\, \underset{1}{i}_{\mu} + \tfrac{1}{2} \left(\frac{d}{ds} \log \varrho \right) i_{\mu}. \qquad (18)$$

The transformation of Q_{μ} under conformal transformations is given by

$$Q'_{\mu} = Q_{\mu} - s_{\mu}, \qquad (19)$$

from which it follows at once that the parameters $\Gamma^{\varkappa}_{\mu\lambda}$ are invariant. The conformal covariant derivative to the parameter τ is denoted by the symbol D_{τ}.

d. *The conformal "Frenet-Serret" formulae* for the curve are

$$\left. \begin{aligned} & D_{\tau}\, j^{\varkappa} \equiv \frac{d j^{\varkappa}}{d\tau} + \Gamma^{\varkappa}_{\mu\lambda}\, j^{\mu} j^{\lambda} = 0 \\ & D_{\tau}\, \underset{1}{j}^{\varkappa} = 0 \\ & D_{\tau}\, Q_{\lambda} + (j^{\mu}\, Q_{\mu})\, Q_{\lambda} - \tfrac{1}{2}\, Q_{\mu}\, Q^{\mu}\, j_{\lambda} = (j^{\mu}\, j_{\mu})^{-1}\, (h\, j_{\lambda} + \underset{1}{j}_{\lambda}). \end{aligned} \right\} . \qquad (20)$$

e. The function $h(\tau)$ is a conformal invariant of the fifth order of the curve. It is called the *inversion curvature* of the curve [4]. The function $h(\tau)$ determines the curve to within conformal representations belonging to the Möbius group. So the equation of the curve may be written in the form

$$h = h(\tau). \qquad (21)$$

This equation is called its *intrinsic equation*.

[4] BLASCHKE uses the invariant $b = 2h$. (Vorlesungen über Differentialgeometrie, III).

f. An important theorem concerning touching curves is the following one. A necessary and sufficient condition in order that two curves have at a point P at least a five-point (six-point) contact is that the quantities $j^{\varkappa}$, Q_{μ} (and h) at P be the same for both curves. This theorem is an immediate consequence of the definition of the quantities involved.

The loxodromic.

An isogonal trajectory of a system of circles passing through two fixed points is called a loxodromic. In order to find the intrinsic equation of a loxodromic we choose the fundamental tensor $a_{\lambda\varkappa}$ so that the family of circles with respect to the metric $a_{\lambda\varkappa}$ is a pencil of straight lines through a fixed point P. If the constant angle under which the curve $x^{\varkappa} = x^{\varkappa}(s)$ meets the lines is denoted by α, the coordinates of P are given by

$$y^{\varkappa} = x^{\varkappa} + \lambda \cos\alpha\, i^{\varkappa} + \lambda \sin\alpha\, \underset{1}{i}^{\varkappa}. \quad \ldots\ldots \quad (22)$$

By differentiating (22) we obtain the following two equations for λ and k, the curvature of the loxodromic,

$$\frac{d\lambda}{ds} = -\cos\alpha, \qquad \lambda k = \sin\alpha \quad \ldots\ldots \quad (23)$$

from which we get

$$\lambda = -s\cos\alpha, \quad k = -\frac{tg\,\alpha}{s}, \quad \frac{dk}{ds} = \frac{tg\,\alpha}{s^2} = \varrho \text{ [5]}. \quad \ldots \quad (24)$$

Consequently

$$Q_{\mu} = -\frac{tg\,\alpha}{s}\underset{1}{i}_{\mu} - \frac{1}{s} i_{\mu}. \quad \ldots\ldots \quad (25)$$

Then the invariant h can be calculated from (20). It appears to be

$$h = cotg\, 2\alpha. \quad \ldots\ldots \quad (26)$$

Hence we have the theorem

The curves of constant inversion curvature are loxodromics.

Consider a point $x^{\varkappa}$ of the loxodromic. Besides the osculating circle at $x^{\varkappa}$ there exist several other circles connected with the curve at this point f.i. the circle through $x^{\varkappa}$ belonging to the coaxial system by which the loxodromic is defined and the circle through $x^{\varkappa}$ normal to this system.

The center of the circle through the point $x^{\varkappa}$ orthogonal to the pencil is $y^{\varkappa}$. Hence its curvature with respect to $a_{\lambda\varkappa}$ is

$$|\lambda|^{-1} = \frac{1}{s\cos\alpha} = Q_{\mu}(-\sin\alpha\, i^{\mu} - \cos\alpha\, \underset{1}{i}^{\mu}) = Q_{\mu} v^{\mu}, \quad \ldots \quad (27)$$

where v^{μ} is the unit vector pointing towards the center of the circle. This equation however is conformal invariant as may be seen from (8) and (19). So (27) gives the curvature with respect to any metric tensor obtained from $a_{\lambda\varkappa}$ by a conformal transformation.

The circle through $x^{\varkappa}$ belonging to the pencil by which the loxodromic is defined is a straight line with respect to the fundamental tensor $a_{\lambda\varkappa}$. Its direction is given by the vector

$$\sin\alpha\, \underset{1}{i}^{\varkappa} + \cos\alpha\, i^{\varkappa}. \quad \ldots\ldots \quad (28)$$

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[5]) Here $tg\,\alpha$ is supposed to be positive.

Its curvature c with respect to the metric tensor $a_{\lambda\varkappa}$ vanishes. With respect to any other metric tensor obtained from $a_{\lambda\varkappa}$ by a conformal transformation the curvature is given by the following invariant equation

$$c = Q_{\mu}(\cos\alpha\, \underset{1}{i}^{\mu} - \sin\alpha\, i^{\mu}) = Q_{\mu} w^{\mu}, \quad \ldots\ldots \quad (29)$$

where w^{μ} is the unit vector normal to the tangent of the circle. Indeed, this expression vanishes if Q_{μ} is replaced by the value (25), whereas both sides of (29) transform under conformal transformations in the same way as a consequence of (8) and (19), w^{μ} being a unit vector.

We shall use these results in the following section.

Geometrical interpretations of τ, Q_{μ} and h.

The parameter τ. If d denotes the distance between the centers of two circles C_1 and C_2 with radius r_1 and r_2, the expression

$$I = \sqrt{\frac{(r_1 - r_2)^2 - d^2}{2\, r_1 r_2}} \quad \ldots\ldots \quad (30)$$

is a conformal invariant of the curve. This invariant I is a function of the cross-ratio of the points of intersection of the two circles with an arbitrarily chosen circle orthogonal to C_1 and C_2 [6]). When the expression (30) is calculated for the osculating circles of the curves at the points s and $s + \triangle s$, we obtain

$$\triangle I = \frac{dk}{ds} \triangle s^2 + \ldots. \quad \ldots\ldots \quad (31)$$

Consequently we have from (15) to within terms of higher order

$$\triangle \tau = \sqrt{\triangle I}, \quad \ldots\ldots \quad (32)$$

which gives us at the same time a geometrical interpretation of τ.

The quantity Q_{μ}. Before we turn to the geometrical interpretation of Q_{μ}, we ask for the geometrical figure, which corresponds to a covariant vector w_{μ} at a point $P(x_0^{\varkappa})$, transforming under conformal transformations as follows

$$w'_{\mu} = w_{\mu} - s_{\mu}. \quad \ldots\ldots \quad (33)$$

Let $e^{\varkappa}$ be any contravariant unit vector at P. Then the transformation of the scalar $p = e^{\mu} w_{\mu}$ is given by

$$p' = \sigma^{-1}(p - e^{\mu} s_{\mu}). \quad \ldots\ldots \quad (34)$$

By comparing (34) with (8) we see that this transformation is identical with that of the curvature of a circle through P, whose tangent at P is orthogonal to e^{μ}. Therefore e^{μ} and w_{μ} together define a circle through P with the center

$$x^{\varkappa} = x_0^{\varkappa} + p^{-1} e^{\varkappa}. \quad \ldots\ldots \quad (35)$$

In varying the unit vector $e^{\varkappa}$ we obtain a family of ∞^1 circles all passing through the point P. It is seen from (35) that the locus of the centers of these circles is a straight line given by

$$w_{\lambda}(x^{\lambda} - x_0^{\lambda}) = 1. \quad \ldots\ldots \quad (36)$$

Hence this family of circles is a coaxial system, whose axis (36) is orthogonal to the vector $w^{\varkappa}$.

Now the transformation of Q_{μ} is identical with that of w_{μ}. Hence we have the theorem

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[6]) Comp. W. BLASCHKE, Vorlesungen über Differentialgeometrie, III, p. 41.

The covariant vector Q_μ corresponds geometrically to a coaxial system of circles.

Henceforth this system of circles will be denoted by (Q). It may be noted that the curvature of the circle of the system (Q), which is tangent to the curve at the point under consideration is given by

$$Q_\mu \underset{1}{i}^\mu = k, \quad \ldots \ldots \ldots \quad (37)$$

from which it follows that this circle is identical with the osculating circle of the curve. Hence the pencil (Q) contains the osculating circle. So its axis passes through the center of curvature and is normal to the vector $Q^\varkappa$.

In the following a geometrical property will be given of this particular system of circles, which leads at the same time to a geometrical interpretation of Q_μ.

Consider a loxodromic having at P at least a five-point contact with the given curve C. Then as we have seen the quantities $j^\varkappa$, $\underset{1}{j}^\varkappa$ and Q_μ at P are the same for both the curve and the loxodromic. If besides that the inversion curvature of both curves are equal we have at P a six-point contact. Now a curve is determined by the values of the quantities $j^\varkappa$, $\underset{1}{j}^\varkappa$ and Q_μ at one point together with the function $h(\tau)$ [7]), which is a constant for a loxodromic. Therefore, there exists only one loxodromic, which has at P a six-point contact with C and a system of ∞^1 loxodromics, which have at P at least a five-point contact with the curve C. Each of these ∞^1 loxodromics meets a coaxial system of circles under a constant angle α, which is connected by the inversion curvature of the loxodromic by formel (26). Now consider the circle through P normal to one of these coaxial systems. Its curvature is according to (27) given by

$$Q_\mu(-\sin\alpha \underset{1}{i}^\mu - \cos\alpha\, i^\mu) = Q_\mu v^\mu, \quad \ldots \ldots \quad (38)$$

its center by

$$y^\varkappa = x^\varkappa + (Q_\mu v^\mu)^{-1} v^\varkappa, \quad \ldots \ldots \ldots \quad (39)$$

from which it follows that this circle belongs to the system (Q) at P. But to every value of α corresponds according to (26) one value of h, thus one loxodromic having at P a five-point contact with the curve. Hence each circle of the system (Q) can be obtained in this way. This result enables us to state the following theorem.

There exists a family of ∞^1 loxodromics having at least a five-point contact with a given curve at a point P. Each of these loxodromics meets a pencil of circles under a constant angle. The system of circles through P each of which is normal to one of these pencils form the coaxial system (Q) at P.

Another geometrical interpretation of the pencil (Q) is obtained as follows. Consider again a loxodromic which has at P a five-point contact with the curve, together with the coaxial system of circles, which are cut by this loxodromic under a constant angle α. The center of the circle through P belonging to this system is according to (28) and (29) given by

$$x^\varkappa + (Q_\mu w^\mu)^{-1} w^\varkappa, \quad \ldots \ldots \ldots \quad (40)$$

where

$$w^\varkappa = \cos\alpha \underset{1}{i}^\varkappa - \sin\alpha\, i^\varkappa. \quad \ldots \ldots \ldots \quad (41)$$

Hence this circle too belongs to the system (Q) at P. If we had started with another loxodromic having a five-point contact with the curve, we should have obtained another circle of (Q). We may state this result thus:

Of each pencil of circles belonging to a loxodromic, which has at P at least a five-point contact with a curve, one circle passes through P. These circles through P together form the coaxial system (Q) of the curve at P.

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[7]) This theorem has been proved in C.D.G. I.

The invariant h. A geometrical interpretation of the inversion curvature h is obtained by considering the loxodromic, which has at P a six-point contact with the given curve and whose inversion curvature is therefore equal to that of the curve at P. We then arrive at the theorem.

The loxodromic which has at P a six-point contact with a given curve C meets a pencil of circles under a constant angle α. The inversion curvature of C at the point P is connected with this angle by the formel

$$h = \operatorname{cotg} 2\alpha. \quad \ldots \ldots \ldots \quad (42)$$

Another geometrical interpretation of the invariant h is obtained by considering the circle of the system (Q), which is normal to the curve at P. This circle is called the *normal circle* of the curve at P. In the following it will appear that if h is negative two consecutive normal circles have real points of intersection and therefore meet under a real angle. The center of the normal circle is given by

$$y^\varkappa = x^\varkappa + p\, i^\varkappa; \quad p = (Q_\mu i^\mu)^{-1}. \quad \ldots \ldots \quad (43)$$

Then if $\theta(\tau)$ is the angle between the normal circles of the curve at the points $P(\tau_0)$ $P'(\tau)$ we have at P

$$\left(\frac{d\theta}{d\tau}\right)^2 = \frac{1}{p^2}\left\{\Sigma\left(\frac{dy^\varkappa}{d\tau}\right)^2 - \left(\frac{dp}{d\tau}\right)^2\right\}. \quad \ldots \ldots \quad (44)$$

Since

$$\frac{dy^\varkappa}{d\tau} = i^\varkappa\left(\varrho^{-\frac{1}{2}} + \frac{dp}{d\tau}\right) + \varrho^{-\frac{1}{2}} k\, p \underset{1}{i}^\varkappa, \quad \ldots \ldots \quad (45)$$

the equation (44) reduces to

$$\left(\frac{d\theta}{d\tau}\right)^2 = \frac{\varrho^{-1}}{p^2}\left\{1 + 2\varrho^{\frac{1}{2}}\frac{dp}{d\tau} + k^2 p^2\right\}. \quad \ldots \ldots \quad (46)$$

From the fundamental formulea (20) we obtain

$$-\frac{1}{p^2}\frac{dp}{d\tau} = \varrho^{\frac{1}{2}} h + \tfrac{1}{2}\varrho^{-\frac{1}{2}}\left(k^2 + \frac{1}{p^2}\right). \quad \ldots \ldots \quad (47)$$

When this expression is substituted in (46) it is found that the inversion curvature h satisfies the equation

$$\left(\frac{d\theta}{d\tau}\right)^2 = -2h. \quad \ldots \ldots \ldots \quad (48)$$

This relation bears out the statement that θ only exists for negative h. So we have arrived at a geometrical interpretation of h, expressed by the following theorem [8])

The normal circles at the points τ and $\tau + \triangle\tau$ of a curve of negative inversion curvature meet under an angle $\triangle\theta$ for which we have to within terms of higher order

$$\triangle\theta = \sqrt{-2h}\,\triangle\tau. \quad \ldots \ldots \ldots \quad (49)$$

For curves of positive inversion curvature we obtain in much the same way

$$\left(\frac{dI}{d\tau}\right)^2 = h, \quad \ldots \ldots \ldots \quad (50)$$

where I is the conformal invariant of the two normal circles at P and P', defined by (30).

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[8]) This geometrical interpretation of h has been given by J. MAEDA, Geometrical meanings of the inversion curvature of a plane curve. Jap. J. Math. **16**, 177—232 (1940).