la densité de G est au moins égal à $1-\varepsilon$. À chacun des P une translation congruente

$$\theta_\nu = \theta'_\nu + \frac{h_\nu}{q_\nu^r} \qquad (h_\nu \text{ entier}; \quad \nu = 1, 2, \ldots, n) \;.\;\;.\;\;. \quad (18)$$

correspond, transférant les points $(\theta'_1, \ldots, \theta'_n)$ de P_0 en des points $(\theta_1, \ldots, \theta_n)$ de P. Soit $(\theta_1, \ldots, \theta_n)$ un point de P_0 appartenant à G et soit $x \geqq 1, y_1, \ldots, y_n$ une solution entière quelconque de (17). Alors pour tout point $(\theta'_1, \ldots, \theta'_n)$ avec (18) on a

$$\left.\begin{array}{r} |\theta'_\nu\, q_\nu^r\, f_\nu(x) - q_\nu^r\, y_\nu + h_\nu\, f_\nu(x)| = |\theta_\nu\, f_\nu(x) - y_\nu|\, q_\nu^r < q_\nu^r\, \Omega_\nu(x) \\ (\nu = 1, 2, \ldots, n) \end{array}\right\} . \quad (19)$$

Le système S possédant la propriété $\mathcal{M}(q_1, \ldots, q_n; C)$ nous pouvons poser

$$q_\nu^r\, f_\nu(x) = f_\nu(X), \quad \text{où} \quad X \leqq C^r x.$$

En posant en outre $Y_\nu = q_\nu^r\, y_\nu - h_\nu\, f_\nu(x)$ nous tirons de (19)

$$|\theta'_\nu\, f_\nu(X) - Y_\nu| < q_\nu^r\, \Omega_\nu\left(\frac{X}{C^r}\right)$$

et donc, si x est suffisamment grand

$$< \psi_\nu\left(\frac{X}{C^r}\,\chi\left(\frac{X}{C^r}\right)\right) < \psi_\nu(X) = \omega_\nu(X) \qquad (\nu = 1, 2, \ldots, n),$$

à cause des relations (16), $\eta(x) \to 0$, $\chi(x) \to \infty$ (quand $x \to \infty$ et (15).

Comme (17) possède une infinité de solutions entières nous concluons que le système

$$|\theta'_\nu\, f_\nu(X) - Y_\nu| < \omega_\nu(X) \qquad (\nu = 1, 2, \ldots, n)$$

admet une infinité de solutions entières de même. Ça veut dire que le point $(\theta'_1, \ldots, \theta'_n)$ appartient à G. Or, ceci entraîne que la densité de G par rapport à chacun des parallélépipèdes P est $\geqq 1-\varepsilon$. Le nombre ε étant arbitraire et G ne dépendant pas de ε, nous en concluons que la mesure mG est égale à 1. C.q.f.d.

Hydrodynamics. — *Laminar flow in radial direction along a plane surface.* By A. VAN WIJNGAARDEN. (Mededeeling N^0. 43 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hoogeschool te Delft.) (Communicated by Prof. J. M. BURGERS.)

(Communicated at the meeting of February 28, 1942.)

1. The theory of laminar boundary layer flow mostly has been used for those types of flow in which the velocity of the fluid at great distances from the wall is approximately parallel to a single given direction. A solution of this kind for the case of the flow in radial direction along an infinite plane surface has been given by HOMANN [1]). In this solution at great distances from the plane the velocity is directed towards the plane; however, at the same time it possesses a radial component which increases proportionally to the distance from the central axis. The equation of continuity then requires that the component normal to the plane shall increase indefinitely with the distance from the plane; it must be assumed, consequently, that the field is limited by another plane at some great distance. In this case the amount of fluid transported in the radial direction through the surface of a cylinder having its axis along the axis of the field increases proportionally to the square of the radius of the cylinder, in agreement with the fact that the amount of fluid supplied by the main stream likewise is proportional to the square of the radius.

In the case where a limited amount of fluid is supplied in the axis of the field, the velocity component in the radial direction must decrease inversely proportionally to the radius. We meet such kinds of flow for instance in the valves of piston pumps and in the air supported bearings of modern ultra-centrifuges. Actually the problem here is greatly complicated by the presence of a second limiting plane boundary.

The investigation of the field with decreasing radial velocity along a single plane wall will be the subject of the present paper. In a final remark a more general view on the problem has been indicated.

2. We take the plane boundary as X, Y-plane and the central axis as Z-axis. Further we denote the distance from the axis by r, and the velocities in the r- and z-directions by u and w respectively. It is supposed that the motion is independent of the time and has no tangential component. Then the equations of motion and of continuity are:

$$\left.\begin{array}{l} u\dfrac{\partial u}{\partial r} + w\dfrac{\partial u}{\partial z} = -\dfrac{\partial}{\partial r}\left(\dfrac{p}{s}\right) + \nu\left[\dfrac{\partial^2 u}{\partial z^2} + \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} - \dfrac{u}{r^2}\right] \\ u\dfrac{\partial w}{\partial r} + w\dfrac{\partial w}{\partial z} = -\dfrac{\partial}{\partial z}\left(\dfrac{p}{s}\right) + \nu\left[\dfrac{\partial^2 w}{\partial z^2} + \dfrac{\partial^2 w}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial w}{\partial r}\right] \\ u = \dfrac{1}{r}\dfrac{d\psi}{\partial z}; \qquad w = -\dfrac{1}{r}\dfrac{\partial\psi}{\partial r} \end{array}\right\} . \quad (1)$$

Here p is the pressure, s the (constant) density and ν the kinematic viscosity of the fluid. By introducing the dimensionless variables:

$$\varrho = \frac{r}{r_0};\ \zeta = \frac{z}{r_0\sqrt{\varepsilon}};\ u^* = \frac{u}{u_0};\ w^* = \frac{w}{u_0\sqrt{\varepsilon}};\ \left(\frac{p}{s}\right)^* = \left(\frac{p}{s}\right)\Big/ u_0^2;\ \psi^* = \frac{\psi}{r_0\sqrt{\varepsilon}},$$

1) F. HOMANN, Zeitschr. f. angew. Math. u. Mech. **16**, 153—164 (1936).

where r_0 and u_0 are certain convenient standard measures of length and velocity and ε is the inverse of the Reynolds number $R = u_0 r_0/\nu$, the equations (1) change into:

$$\left.\begin{aligned} & u^* \frac{\partial u^*}{\partial \varrho} + w^* \frac{\partial u^*}{\partial \zeta} = -\frac{\partial}{\partial \varrho}\left(\frac{p}{s}\right)^* + \frac{\partial^2 u^*}{\partial \zeta^2} + \varepsilon \left[\frac{\partial^2 u^*}{\partial \varrho^2} + \frac{1}{\varrho}\frac{\partial u^*}{\partial \varrho} - \frac{u^*}{\varrho^2}\right] \\ & \varepsilon\left[u^* \frac{\partial w^*}{\partial \varrho} + w^* \frac{\partial w^*}{\partial \zeta}\right] = -\frac{\partial}{\partial \zeta}\left(\frac{p}{s}\right)^* + \varepsilon \frac{\partial^2 w^*}{\partial \zeta^2} + \varepsilon^2 \left[\frac{\partial^2 w^*}{\partial \varrho^2} + \frac{1}{\varrho}\frac{\partial w^*}{\partial \varrho}\right] \\ & u^* = \frac{1}{\varrho}\frac{\partial \psi^*}{\partial \zeta};\ w^* = -\frac{1}{\varrho}\frac{\partial \psi^*}{\partial \varrho} \end{aligned}\right\} \quad (1a)$$

By crossdifferentiating these equations we eliminate the pressure; the resulting equation for ψ^* becomes:

$$\left.\begin{aligned} & \frac{\partial^4 \psi^*}{\partial \zeta^4} + \frac{1}{\varrho}\frac{\partial \psi^*}{\partial \varrho}\frac{\partial^3 \psi^*}{\partial \zeta^3} + \frac{1}{\varrho}\frac{\partial \psi^*}{\partial \zeta}\left(\frac{2}{\varrho}\frac{\partial^2 \psi^*}{\partial \zeta^2} - \frac{\partial^3 \psi^*}{\partial \varrho\, \partial \zeta^2}\right) = \\ & = \varepsilon\left[-2\frac{\partial^4 \psi^*}{\partial \varrho^2 \partial \zeta^2} + \frac{2}{\varrho}\frac{\partial^3 \psi^*}{\partial \varrho\, \partial \zeta^2} + \frac{1}{\varrho}\frac{\partial \psi^*}{\partial \zeta}\frac{\partial^3 \psi^*}{\partial \varrho^3} - \frac{1}{\varrho}\frac{\partial \psi^*}{\partial \varrho}\frac{\partial^3 \psi^*}{\partial \varrho^2 \partial \zeta} - \right. \\ & \qquad \left. - \frac{3}{\varrho^2}\frac{\partial \psi^*}{\partial \zeta}\frac{\partial^2 \psi^*}{\partial \varrho^2} + \frac{1}{\varrho^2}\frac{\partial \psi^*}{\partial \varrho}\frac{\partial^2 \psi^*}{\partial \varrho\, \partial \zeta} + \frac{3}{\varrho^3}\frac{\partial \psi^*}{\partial \varrho}\frac{\partial \psi^*}{\partial \zeta}\right] + \\ & \qquad + \varepsilon^2\left[-\frac{\partial^4 \psi^*}{\partial \varrho^4} + \frac{2}{\varrho}\frac{\partial^3 \psi^*}{\partial \varrho^3} - \frac{3}{\varrho^2}\frac{\partial^2 \psi^*}{\partial \varrho^2} + \frac{3}{\varrho^3}\frac{\partial \psi^*}{\partial \varrho}\right] \end{aligned}\right\} \quad (2)$$

Now we bound the field by a cone having its axis along the Z-axis and its top in the origin. A generating line of the cone is given by: $z = M \cdot r$. If we write $\eta = \zeta/\varrho$, this generating line is given by $\eta = M/\sqrt{\varepsilon} = N$.

The boundary conditions for $\eta = 0$ always are: $u^* = 0$ and $w^* = 0$. Along with these we give the velocity at the surface of the cone $\eta = N$ by prescribing here: $u^* = 1/\varrho$ and $w^* = C/\varrho$. In this case a solution may be found, assuming:

$$\psi^* = \varrho \cdot f(\eta). \quad (3)$$

For f we obtain the equation:

$$\left.\begin{aligned} & f^{IV} + f''' f + 3 f'' f' = \\ & = \varepsilon[-2f'' - 6\eta f''' - 2\eta^2 f^{IV} + 3ff' - 3\eta f'^2 - 3\eta^2 f' f'' - 3\eta ff'' - \eta^2 ff'''] + \\ & \qquad + \varepsilon^2 [3f - 3\eta f' - 21\eta^2 f'' - 10\eta^3 f''' - \eta^4 f^{IV}] \end{aligned}\right\} \quad (4)$$

As $u^* = f'/\varrho$ and $w^* = (\eta f' - f)/\varrho$ the boundary conditions become:

$$\text{for } \eta = 0 : f = 0,\ f' = 0$$
$$\text{for } \eta = N : f = \eta - C,\ f' = 1.$$

3. Equation (4) can be integrated once and then becomes:

$$\left.\begin{aligned} & f''' + f'^2 + ff'' + A_0 = \\ & = \varepsilon[-2\eta f'' - 2\eta^2 f''' + 2f^2 - \eta^2 f'^2 - \eta ff' - \eta^2 ff''] + \\ & \qquad + \varepsilon^2 [3\eta f - 3\eta^2 f' - 6\eta^3 f'' - \eta^4 f'''] \end{aligned}\right\} \quad (5)$$

In order to obtain an impression of the order of magnitude of the terms on the right hand

side we substitute the asymptotic expansion:

$$f = \eta - C + a_1/\eta + a_2/\eta^2 + \ldots$$

The terms between the square brackets then become, respectively:

$$-3\eta C + (2C^2 + 4a_1) + 0(1/\eta) \text{ and } -3\eta C + 0(1/\eta).$$

As the maximum value of η is given by $M/\sqrt{\varepsilon}$ and as we will consider such solutions only in which f and its derivatives remain finite for all η below this maximum value, the order of magnitude of the right hand side of equation (5) at most will be: $M\sqrt{\varepsilon}$. Consequently for every finite value of M we can make the right hand side arbitrarily small by choosing the Reynolds number R sufficiently high. For great values of R we therefore may confine ourselves to the solution of the equation:

$$f''' + f'' f + f'^2 - A_0 = 0,$$

with the boundary conditions $f = 0$, $f' = 0$ for $\eta = 0$; and $f = \eta - C$, $f' = 1$ for $\eta = \infty$. The last condition gives $A_0 = 1$. Hence:

$$f''' + f'' f + f'^2 - 1 = 0 \quad (6)$$

This is a boundary layer equation which might also have been deduced in the ordinary way by taking for the pressure the "undisturbed" pressure and dropping all terms of the order of ε. The method used above, however, has the advantage of admitting a precise check upon the possible influence of the neglected terms. The same as nearly all boundary layer equations, equation (6) is a special case of HARTREE's equation [2]):

$$f''' + f'' f - \beta(f'^2 - 1) = 0,$$

with $\beta = -1$. From HARTREE's analysis we know that, as our β has a negative value, the ordinary condition $f' = 1$ for $\eta = \infty$ is not sufficient to determine the function f, and therefore we are able to apply the more precise condition $f = \eta - C$.

In our special case the equation lends itself to a more rigorous investigation. The equation can be integrated twice and then becomes:

$$f' + \tfrac{1}{2} f^2 - \tfrac{1}{2}\eta^2 + B\eta + D = 0.$$

The boundary conditions give $D = 0$ and $B = C$; hence we obtain:

$$f' + \tfrac{1}{2} f^2 - \tfrac{1}{2}\eta^2 + C\eta = 0. \quad (7)$$

As $C = -f''(0)$ we can integrate this equation numerically, starting with a given value of C. It only has to be demonstrated that f will tend to $\eta - C$ with increasing values of η for every value of C.

If $C \neq 0$ we can make the substitutions: $\eta = Cz$, $f = CF$; the equation then is transformed into:

$$dF/dz = C^2(\tfrac{1}{2}z^2 - z - \tfrac{1}{2}F^2). \quad (8)$$

This form lends itself very well for a graphical discussion, as dF/dz has a known value on every hyperbola

$$\tfrac{1}{2}z^2 - z - \tfrac{1}{2}F^2 = A = \textit{constant}.$$

2) D. R. HARTREE, Proc. Camb. Phil. Soc. 33, 223—239 (1937).

It is easily shown (see fig. 1) that for all negative values, as well as for all positive values of C up to a certain limit, the limiting form of F indeed will be $z - 1$ as required. There is a critical value of C for which the limiting form becomes, however, $-z+1$.

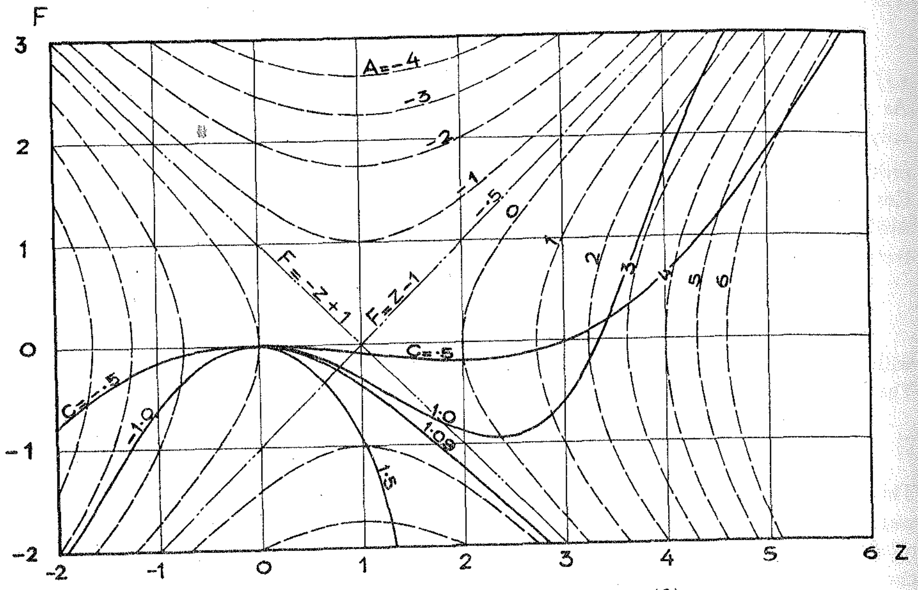

Fig. 1. Graphical solution of equation (8).

For greater values of C the solutions again will have the limiting form $z-1$, but now they have a pole for some finite value of z. Then comes a second critical value of C, for which the limiting form again is: $-z-1$; after that the solution has two poles and so on. The first critical value of C appears to be about 1,09.

4. Analytically the behaviour of f can be discussed by substituting $f = 2h'/h$, $x = \eta - C$ and $\frac{1}{4}C^2 = n + \frac{1}{2}$. Then equation (7) changes into:

$$d^2h/dx^2 + (n + \tfrac{1}{2} - \tfrac{1}{4}x^2)\,h = 0 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

This is WEBER's equation defining the parabolic cylinder functions [3]). We can choose two fundamental solutions $D_n(x)$ and $E_n(x)$ with the asymptotic expansions:

$$D_n(x) \backsim e^{-x^2/4}\left\{x^n - \frac{n(n-1)}{2}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4}x^{n-4} - \dots\right\}$$

$$E_n(x) \backsim e^{x^2/4}\left\{x^{-n-1} + \frac{(n+1)(n+2)}{2}x^{-n-3} + \right.$$

$$\left. + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4}x^{-n-5} + \dots\right\}.$$

Then:

$$f \backsim \frac{-e^{-x^2/4}.\,x^{n+1} + a\,e^{x^2/4}.\,x^{-n}}{e^{-x^2/4}.\,x^n + a\,e^{x^2/4}.\,x^{-n-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

[3]) E. T. WHITTAKER and G. N. WATSON, Modern Analysis, (Cambridge, 1920), p. 347, § 16.5.

a has to be determined in such a way that the condition $f = 0$ for $\eta = 0$ (*i.e.* for $x = -C$) will be satisfied. This requires:

$$a = -D_n'(-C)/E_n'(-C). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

Now if $a \neq 0$, we have $f \sim x$ for great values of x; however, if $a = 0$, we shall have $\sim -x$. By considering the particular cases where n is an integer it can be shown that there exists one critical value for negative n, and further one critical value between every two consecutive positive integer values of n; the number of poles of a solution is equal to the number of critical values inferior to n.

Hence we see that for every value of C up to about 1,09 we have found a solution that satisfies all boundary conditions. For higher values of C the function f and its derivatives will become infinite for one or more values of η and the neglections introduced into equation (4) no longer are valid.

The numerical computation of f may be executed best by means of a numerical integration of (7), in combination with an easily constructed series in ascending powers of η, applicable for very small η, and with an asymptotic series applicable for very large η. Both series, however, are not easily manageable.

A simple table of values of f, f' and $f - \eta f'$ has been computed for the case $C = 0$. Although for that value of C the function f will approach to its limiting form as quickly

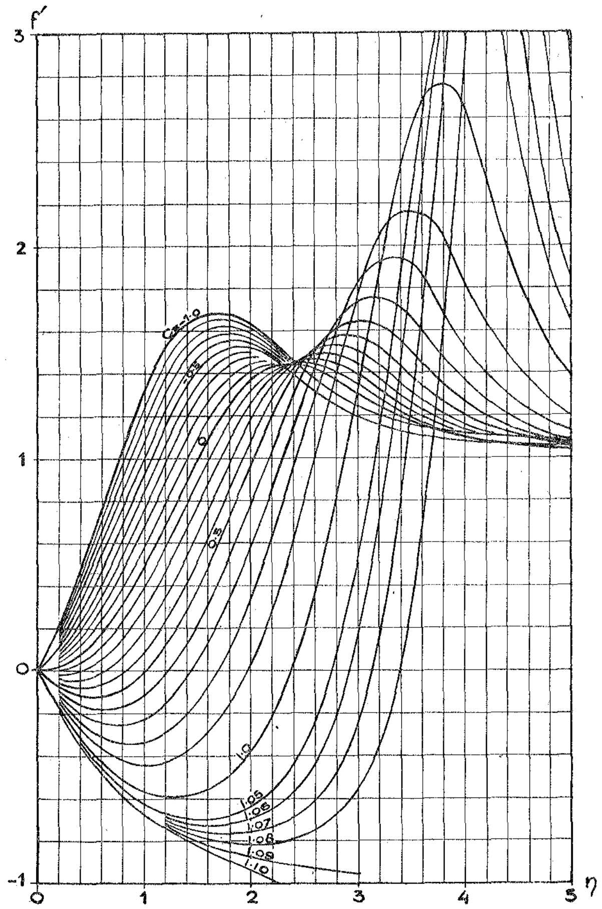

Fig. 2. Solutions of equation (7) for different values of C.

Solution of equation (7) for the case $C = 0$.

η	f	f'	$\eta f' - f$	η	f	f'	$\eta f' - f$
0	0	0	0	4.3	4.041	1.080	0.603
0.1	0.00017	0.00500	0.00043	4.4	4.149	1.074	0.577
0.2	0.00133	0.02000	0.00267	4.5	4.256	1.069	0.555
0.3	0.00450	0.04499	0.00900	4.6	4.363	1.064	0.534
0.4	0.01066	0.07994	0.02131	4.7	4.469	1.060	0.514
0.5	0.02082	0.12478	0.04157	4.8	4.575	1.057	0.498
0.6	0.03594	0.17935	0.07167	4.9	4.680	1.054	0.482
0.7	0.05700	0.24338	0.11336	5.0	4.785	1.050	0.468
0.8	0.08492	0.31639	0.16819	5.5	5.308	1.039	0.410
0.9	0.12056	0.39773	0.23740	6.0	5.826	1.032	0.367
1.0	0.16471	0.48644	0.32173	6.5	6.340	1.027	0.334
1.1	0.21805	0.58123	0.42130	7.0	6.853	1.022	0.305
1.2	0.28111	0.68049	0.53548	7.5	7.363	1.019	0.283
1.3	0.35423	0.78226	0.66271	8.0	7.872	1.017	0.263
1.4	0.43757	0.88427	0.80041	8.5	8.380	1.015	0.246
1.5	0.53101	0.98401	0.94500	9.0	8.887	1.013	0.231
1.6	0.63421	1.07889	1.09201	9.5	9.393	1.012	0.218
1.7	0.74654	1.16634	1.23624	10	9.898	1.011	0.206
1.8	0.86715	1.24403	1.37210	11	10.908	1.008	0.187
1.9	0.99495	1.31003	1.49411	12	11.916	1.007	0.170
2.0	1.12872	1.36299	1.59726	13	12.922	1.006	0.157
2.1	1.26710	1.40223	1.67758	14	13.928	1.005	0.145
2.2	1.40871	1.42777	1.73238	15	14.933	1.004	0.135
2.3	1.55222	1.44031	1.76049	16	15.937	1.004	0.126
2.4	1.69638	1.44114	1.76236	17	16.941	1.003	0.119
2.5	1.84012	1.43200	1.73988	18	17.944	1.003	0.112
2.6	1.9825	1.4148	1.6960	19	18.947	1.003	0.105
2.7	2.1229	1.3917	1.6347	20	19.950	1.002	0.101
2.8	2.2607	1.3646	1.5602	25	24.960	1.002	0.080
2.9	2.3957	1.3353	1.4767	30	29.967	1.001	0.067
3.0	2.5277	1.3053	1.3882	35	34.971	1.001	0.057
3.1	2.6568	1.2757	1.2979	40	39.975	1.001	0.050
3.2	2.7829	1.2476	1.2094	50	49.980	1.000	0.040
3.3	2.9064	1.2216	1.1249	60	59.983	1.000	0.033
3.4	3.0273	1.1978	1.0452	70	69.986	1.000	0.028
3.5	3.1460	1.1763	0.9711	80	79.988	1.000	0.025
3.6	3.2626	1.1576	0.9048	90	89.989	1.000	0.022
3.7	3.3775	1.1412	0.8449	100	99.990	1.000	0.020
3.8	3.4909	1.1268	0.7909	200	199.995	1.000	0.010
3.9	3.6029	1.1146	0.7440	300	299.997	1.000	0.007
4.0	3.7138	1.1037	0.7010	400	399.998	1.000	0.005
4.1	3.824	1.095	0.666	500	499.998	1.000	0.004
4.2	3.933	1.087	0.632	1000	999.999	1.000	0.001

as possible, the approach still is very slow. For other values of C between —1,0 and + 1,10 the equation has been solved graphically; the results are indicated in fig. 2.

5. *Final remark.* Instead of making the particular assumption that for large values of z the velocity u^* should decrease inversely proportionally to ϱ, we also may try to find a solution for the case in which u^* is proportional to an arbitrary power of ϱ, by assuming $\psi^* = \varrho^{\alpha} . f(\eta)$, where $\eta = \zeta . \varrho^{-\gamma}$. As the powers of ϱ on the left hand side of equation (2) at any rate must be the same, we obtain the condition:

$$\alpha + \gamma = 2.$$

Now the terms on the right hand side of the equation generally are not of the same order in ϱ as the terms on the left hand side. For $\alpha = 1$ (which is the case just treated), however, this condition is satisfied. For $\alpha = 2$, which is the case of HOMANN's solution, the right hand side of the equation wholly vanishes. If we bound the field again by a surface of revolution, a meridian curve of which is defined by $\eta = M/\sqrt{\varepsilon}$, then we can ask for what values of α the terms on the right hand side may be made arbitrarily small by choosing the Reynolds number high enough.

The expression between the first pair of brackets on the right hand side in (5) must not have a term with η^2 in its asymptotic expansion. We find that this is the case only for $\alpha = 1$, $\alpha = 2$ or $\alpha = -1/2$. For other values of α we may neglect the terms on the right hand side only for not too large η. The simplified equation then becomes:

$$f^{IV} + \alpha f''' f + (6 - 3\alpha) f'' f' = 0,$$

which can be integrated into:

$$f''' + \alpha f'' f + (3 - 2\alpha)(f'^2 - 1) = 0.$$

If we put,

$$\text{for positive } \alpha : F = f\sqrt{\alpha};\ y = \eta\sqrt{\alpha};$$

$$\text{for negative } \alpha : F = -f\sqrt{-\alpha};\ y = \eta\sqrt{-\alpha};$$

and in both cases: $\beta = -(3 - 2\alpha)/\alpha$, the equation becomes:

$$\frac{d^3 F}{dy^3} + F\frac{d^2 F}{dy^2} - \beta(f'^2 - 1) = 0.$$

Hence we obtain HARTREE's equation in its general form. For $\alpha = 1$, $\alpha = 2$ and $\alpha = -1/2$ the values of β are -1, $+1/2$ and $+8$ respectively. The last two values of β are positive and the solution in this case is determined by the conditions $f = 0$ and $f' = 0$ for $\eta = 0$, and $f' = 1$ for $\eta = \infty$.