Wir sind jetzt imstande den Beweis des folgenden Satzes zu geben:

27. *Die Multiplikation ist distributiv inbezug auf die Addition:*

$$\alpha(\beta+\gamma)=\alpha\beta+\alpha\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1,23)$$

Dem Beweis dieser Behauptung schicken wir folgende Bemerkung voraus. Es sei $\mathfrak{B}$ irgend eine Bewegung und $\mathfrak{S}$ eine Spiegelung inbezug auf die Gerade g. Dann ist $\mathfrak{B}\mathfrak{S}\mathfrak{B}^{-1}$ die Spiegelung inbezug auf die Gerade, welche durch die Bewegung $\mathfrak{B}$ aus der Geraden g hervorgeht. Dabei ist $\mathfrak{B}^{-1}$ wie üblich die inverse Bewegung von $\mathfrak{B}$.

Es sei nun α ein positives Element aus $\mathfrak{h}$, und β und γ irgend zwei Elemente aus $\mathfrak{h}$. Da die Bewegungen $\mathfrak{P}_{\sqrt{\alpha}}\mathfrak{P}_1$ und $\mathfrak{P}_1\mathfrak{P}_{\sqrt{\alpha}}$ einander invers sind, gilt offenbar auf Grund der soeben gemachten Bemerkung:

$$\mathfrak{P}_{\sqrt{\alpha}}\,\mathfrak{P}_1\,\mathfrak{S}_\xi\,\mathfrak{P}_1\,\mathfrak{P}_{\sqrt{\alpha}}=\mathfrak{S}_{\alpha\xi}. \quad . \quad . \quad . \quad . \quad . \quad (1,24)$$

Wenden wir nun die Formel (1,24) an auf die Beziehung:

$$\mathfrak{S}_{\beta+\gamma}=\mathfrak{S}_\gamma\,\mathfrak{S}_0\,\mathfrak{S}_\beta$$

dann finden wir:

$$\mathfrak{S}_{\alpha(\beta+\gamma)}=\mathfrak{P}_{\sqrt{\alpha}}\,\mathfrak{P}_1\,\mathfrak{S}_{\beta+\gamma}\,\mathfrak{P}_1\,\mathfrak{P}_{\sqrt{\alpha}}$$
$$=\mathfrak{P}_{\sqrt{\alpha}}\,\mathfrak{P}_1\,\mathfrak{S}_\gamma\,\mathfrak{P}_1\,\mathfrak{P}_{\sqrt{\alpha}}\cdot\mathfrak{P}_{\sqrt{\alpha}}\,\mathfrak{P}_1\,\mathfrak{S}_0\,\mathfrak{P}_1\,\mathfrak{P}_{\sqrt{\alpha}}\cdot\mathfrak{P}_{\sqrt{\alpha}}\,\mathfrak{P}_1\,\mathfrak{S}_\beta\,\mathfrak{P}_1\,\mathfrak{P}_{\sqrt{a}}$$
$$=\mathfrak{S}_{\alpha\gamma}\,\mathfrak{S}_0\,\mathfrak{S}_{\alpha\beta}=\mathfrak{S}_{\alpha\beta+\alpha\gamma}.$$

Damit ist die Richtigkeit des Satzes für ein positives Element α erwiesen. Für $\alpha=0$ ist die Behauptung trivial und für ein negatives Element α sieht man die Richtigkeit des Satzes ohne Mühe durch formelles Rechnen ein.

Wir können folgendermaszen zusammenfassen:

28. *Die Elemente von $\mathfrak{h}$ bilden gegenüber den oben erklärten Verknüpfungen einen Körper.*

Denn für $\mathfrak{h}$ sind ja die Körperpostulate erfüllt.

Wegen des Satzes 26 gilt:

29. *Im Körper $\mathfrak{h}$ ist jede quadratische Gleichung mit nicht-negativer Diskriminante lösbar.*

Weiter haben wir:

30. *Der Körper $\mathfrak{h}$ ist angeordnet.*

Für die Elemente von $\mathfrak{h}$ ist nämlich die Eigenschaft, positiv zu sein, definiert und für jedes Element α von $\mathfrak{h}$ gilt genau eine der Aussagen: α ist positiv, α ist Null, $-\alpha$ ist positiv. Dabei kommt es auf dasselbe hinaus, wenn man sagt: α ist positiv, oder $-\alpha$ ist negativ. Weiter sind mit α und β auch $\alpha+\beta$ und $\alpha\beta$ positiv. Für das Produkt leuchtet dies sofort ein auf Grund der Definition der Multiplikation. Für die Summe kann man die Richtigkeit der Behauptung folgendermaszen einsehen. Es sei P irgend ein Punkt auf der Geraden $(0,\infty)$ und A bezw. B das Bild dieses Punktes bei der Bewegung $\mathfrak{S}_\alpha$ bezw $\mathfrak{S}_\beta$. Die Bewegung $\mathfrak{S}_\beta\mathfrak{S}_0\mathfrak{S}_\alpha$ führt offenbar A in B über und folglich liegen A und B symmetrisch inbezug auf die Gerade $(\alpha+\beta,\infty)$. Das bedeutet aber, dasz diese Gerade ebenso wie die Punkte A und B auf derselben Seite der Geraden $(0,\infty)$ liegen wie das Ende 1, und das wollten wir zeigen. Damit ist aber auch der Beweis des Satzes schon erbracht.

Wir können nun in bekannter Weise die Beziehungen „gröszer als" und „kleiner als" einführen und dafür die üblichen Eigenschaften herleiten.

Zum Schlusz bemerken wir noch, dasz auch das Ende ∞ in die Rechnungen hineingezogen werden kann, wenn man die üblichen Verabredungen trifft.

(To be continued.)

Mathematics. — *On the affirmative content of* PEANO's *theorem on differential equations.* By D. VAN DANTZIG. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of March 28, 1942.)

1. **An example.** — The right member of the differential equation

$$\frac{dx}{dt}=3\,\mu(x)^{\frac{2}{3}}, \qquad \mu(x)=Max\,(x,0),$$

is defined[1]) and continuous for every real x. The solution passing through $t=0$, $x=a$, a being a given real number, is

$$\begin{aligned} x&=Max\,((t+a^{\frac{1}{3}})^3,\ 0) && \text{if } a>0,\\ x&=Max\,((t-b)^3,0) && \text{if } a=0 \quad (b\geqq 0 \text{ arbitrary}),\\ x&=a && \text{if } a<0.\end{aligned}$$

If, however, neither $a>0$, nor $a=0$. nor $a<0$ can be ascertained[2]), and if t is any positive number, then the value of x can not be localised closelier than within an interval containing $\langle 0,t\rangle$. In particular an *arbitrary* close approximation of x is not possible for any $t>0$.

PEANO's famous theorem[3]) states the existence of at least one solution, passing through a given point $(t_0, x_{0\mu})$, of the differential equations $dx_\lambda/dt=f_\lambda(t,x_\mu)$, provided $f_\lambda(t,x_\mu)$ is continuous in a neighbourhood of $(t_0,x_{0\mu})$. Its demonstration[3a]) uses a repeated application of BOLZANO-WEIERSTRASS' theorem, which is known[4]) not to correspond with a generally explicitly achievable construction, and therefore is not recognised by the intuitionists. The example shows that the same holds true, not only for the proof of PEANO's theorem, but also for the theorem itself. If more stringent conditions (e.g. of LIPSCHITZ' type, or even somewhat weaker ones) are imposed on $f_\lambda(t,x_\mu)$, it can be shown that the ordinary methods of CAUCHY-LIPSCHITZ or of CAUCHY-PICARD can be brought into a constructive form. If $f_\lambda(t,x_\mu)$ is only continuous, our example shows this to be impossible. In that case we can only try to construct the whole set of solutions, passing through a given point, at once.

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1) The maximum m of two real numbers x and y is a well-defined real number, even though it may not be possible to prove either $m=x$ or $m=y$. Cf [2]).

2) The existence of such numbers, which was first proved by L. E. J. BROUWER (Cf. e g. Begründung der Mengenlehre I, Verh. Koninkl. Akad. v. Wet., Amsterdam, XII; Wis- en Natuurk. Tijdschr. 2, 1923; Monatsh. f. Math. en Physik 36, 1929) is well known to-day.

3) G. PEANO, Démonstration de l'intégrabilité des équations différentielles ordinaires. Math. Ann. 37, 182—228. 1890. Cf. also G. MIE, id. 43, 553—568, 1893. The proof is reproduced e.g. by C. CARATHEODORY, Variationsrechnung, and E. KAMKE, Differentialgleichungen reeller Funktionen. Our present demonstration is much more like PEANO's original one, which uses CANTOR's instead of BOLZANO–WEIERSTRASS' theorem. Cf. D. VAN DANTZIG, A remark and a problem concerning the intuitionistic form of CANTOR's intersection theorem, these Proceedings, 45, 374—375, 1942 [10]).

3a) As it is simplified by C. ARZELÁ, Bologna Mem. (5) 5, 257—270, 1895; (5) 6, 131—140, 1896, P. MONTEL, Ann. Ec. Norm. 24, 264—283, 1907.

4) Cf. e.g. L. E. J. BROUWER, l.c. [2]).

2. **Difference inequalities.** — We consider the system of r differential equations

$$\frac{dx_\lambda}{dt} = f_\lambda (t, x_\mu) \qquad (\lambda, \mu = 1, \ldots, r), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $|t - t_0| \leq a$, $|x_\mu - x_{0\mu}| \leq b_\mu$, whereas for $(t_1, x_{1\mu})$, $(t_2, x_{2\mu})$ lying in this range, $|f_\lambda(t_1, x_{1\mu}) - f_\lambda(t_2, x_{2\mu})| \leq \varepsilon_\lambda$ if $|t_1 - t_2| \leq \delta(\varepsilon_\lambda)$, $|x_{1\mu} - x_{2\mu}| \leq \delta_\mu(\varepsilon_\lambda)$. Then the functions f_λ are bounded: $|f_\lambda(t, x_\mu)| \leq N_\lambda$. Putting $a' = \mathrm{Min}\,(a, N' b_1/N_1, \ldots, N' b_r/N_r)$, $t' = (t - t_0)/a'$, $x'_\mu = N'(x_\mu - x_{0\mu})/a' N_\mu$, $f'_\lambda(t', x'_\mu) = N' f_\lambda(t, x_\mu)/N_\lambda$, $\delta'(\varepsilon) = \mathrm{Min}\,(\delta(\varepsilon N_\lambda/N')/a'),\ N' \delta_\mu(\varepsilon N_\lambda/N')/a' N_\mu)$, where $0 < N' < 1$, e.g. $N' = \frac{3}{4}$, we find that, dropping the accents again, the equations (1) are invariant. The range becomes $|t| \leq 1$, $|x_\mu| \leq 1$. In this range $|f_\lambda(t_1, x_{1\mu}) - f_\lambda(t_2, x_{2\mu})| \leq \varepsilon$ if $|t_1 - t_2| \leq \delta(\varepsilon)$, $|x_{1\mu} - x_{2\mu}| \leq \delta(\varepsilon)$, and $|f_\lambda(t, x_\mu)| \leq N = \frac{3}{4} < 1$.

Let further $l(k)$ be natural numbers with $l = l(k) \geq \mathrm{Max}\,(-{}^2log\,\delta(2^{-k}\beta), l(k-1))$, where $0 < \beta < \mathrm{Min}\,(\frac{1}{6}, 1 - N)$, e.g. $\beta = \frac{1}{16}$, so that $\beta + N = \frac{13}{16} < 1$.

Then

$$\left.\begin{array}{l} |t_1 - t_2| \leq 2^{-l} \\ |x_{1\mu} - x_{2\mu}| \leq 2^{-l} \end{array}\right\} \rightarrow |f_\lambda(t_1, x_{1\mu}) - f_\lambda(t_2, x_{2\mu})| \leq 2^{-k}\beta. \quad . \quad . \quad (2)$$

If T, X_μ are integers with $|T| \leq 2^l$, $|X_\mu| \leq 2^{k+l}$ $(l = l(k))$, then integers $F_{k\lambda}(T, X_\mu)$ exist, such that

$$|F_{k\lambda}(T, X_\mu) - 2^k f_\lambda(2^{-l}T, 2^{-k-l}X_\mu)| \leq \alpha, \quad . \quad . \quad . \quad . \quad (3)$$

where $\frac{1}{2} < \alpha < \frac{2}{3} - \beta$, e.g. $\alpha = \frac{9}{16}$, so that $3(\alpha + \beta) = \frac{15}{8} < 2$. From the inequalities (3) follows $|F_{k\lambda}(T, X_\mu)| \leq 2^k$, the left member being an integer $\leq \alpha + 2^k N < 1 + 2^k$. Moreover

$$|F_{k+1,\lambda}(2^{l(k+1)-l(k)}T, 2^{l(k+1)-l(k)+1}X_\mu) - 2F_{k\lambda}(T, X_\mu)| \leq 1, \quad . \quad (4)$$

$$\left.\begin{array}{l} |T - T'| \leq 2 \\ |X_\mu - X'_\mu| \leq 2^{k+1} \end{array}\right\} \rightarrow |F_{k\lambda}(T, X_\mu) - F_{k\lambda}(T', X'_\mu)| \leq 1, \quad . \quad (5)$$

the left members being integers $\leq 2(\alpha + \beta) < 2$. More generally we have with $l' = l(k')$:

$$\left.\begin{array}{l} \left.\begin{array}{l} |2^{-l}T - 2^{-l'}T'| \leq 2^{-l} + 2^{-l'} \\ |2^{-k-l}X_\mu - 2^{-k'-l'}X'_\mu| \leq 2^{-l} + 2^{-l'} \end{array}\right\} \leftarrow \\ \rightarrow |2^{-k}F_{k\lambda}(T, X_\mu) - 2^{-k'}F_{k\lambda}(T', X'_\mu)| \leq (\alpha + \beta)(2^{-k} + 2^{-k'}), \end{array}\right\} \quad (6)$$

as is seen by twice applying (2) and (3) with $t_1 = 2^{-l}T$, $x_{1\mu} = 2^{-k-l}X_\mu$ and $t_1 = 2^{-l'}T'$, $x_{1\mu} = 2^{-k'-l'}X'_\mu$ respectively, and $t_2 = (T + T')/(2^l + 2^{l'})$, $x_{2\mu} = (2^{-k}X_\mu + 2^{-k'}X'_\mu)/(2^l + 2^{l'})$.

The system of differential equations (1) is now replaced by the sequence of systems of inequalities in finite differences

$$\boxed{|\Delta X_{k\lambda}(T) - F_{k\lambda}(T, X_{k\mu}(T))| \leq 1,} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where $0 \leq T \leq 2^l - 1$, $l = l(k)$, $\Delta X_{k\lambda}(T) = X_{k\lambda}(T+1) - X_{k\lambda}(T)$. For each fixed value of the integer $k \geq 0$ they have a finite number of solutions in integers $X_{k\lambda}(T)$, among which at least one satisfies the initial condition $X_{k\lambda}(0) = 0$; it is determined by $X_{k\lambda}(T+1) = X_{k\lambda}(T) + F_{k\lambda}(T, X_{k\lambda}(T))$. The solutions correspond with CAUCHY-polygons, having their edges at $t = 2^{-l}T$, $x_\lambda = 2^{-k-l}X_{k\lambda}(T)$, $0 \leq T \leq 2^l$.

3. **Top-functions.** We define:

$$l_0 = 0, \ \lambda_0 = 1, \ l_n = [{}^2 log\, n]\,^{5)}, \ \lambda_n = 2^{-l_n - 1} \qquad (n \geq 1), \quad . \quad . \quad (8)$$

$$r_{-1} = 0, \ r_0 = 1, \ r_n = (2n + 1)\lambda_n - 1. \ (n \geq 1), \quad . \quad . \quad (9)$$

Further, I_n for $n \geq 1$ is the closed interval $I_n = \langle r_n - \lambda_n, r_n + \lambda_n \rangle$. In particular $I_1 = \langle 0, 1 \rangle$. An interval I_n is contained in I_m if and only if integers $p \geq 0$ and q exist, $0 \leq q \leq 2^p - 1$, such that

$$n = 2^p m + q, \quad \text{i.e.} \quad m = [2^{-p} n]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

In this case $l_n = l_m + p$, $\lambda_n = 2^{-p}\lambda_m$,

$$r_n - r_m = \{2^{-p}(2q + 1) - 1\}\lambda_m = (2q + 1)\lambda_n - \lambda_m \quad . \quad . \quad (11)$$

The well-known "top-functions" $u_n(t)$ $(0 \leq t \leq 1,\ n \geq -1)$ are defined by

$$u_{-1}(t) = 1, \ u_n(t) = Max\,(0, 1 - |r_n - t|\,\lambda_n^{-1}) \qquad (n \geq 0). \quad . \quad (12)$$

In particular $u_m(r_n) = 0$, unless n has the form (10), e.g. if $m > n$. If

$$\varphi(t) = \sum_{-1}^{\infty} c_n u_n(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

is uniformly convergent for $0 \leq t \leq 1$, then $c_{-1} = \varphi(0)$, $c_0 = \varphi(1) - \varphi(0)$,

$$c_n = \varphi(r_n) - \tfrac{1}{2}\{\varphi(r_n + \lambda_n) - \varphi(r_n - \lambda_n)\} \qquad (n \geq 1). \quad . \quad . \quad (14)$$

Hence the c_n are uniquely determined by $\varphi(t)$. If necessary we write $c_n[\varphi]$ instead of c_n. If and only if $c_n = 0$ for $n \geq 2^l$, the graph of $\varphi(t)$ is a polygon with its edges on $t = 2^{-l}j$, $0 \leq j \leq 2^l$.

If $|\varphi(t)| \leq M$ for every $t \in I_1$, then by (14) $|c_n[\varphi]| \leq 2M$ for every $n \geq -1$. If therefore $|\varphi(t) - \psi(t)| \leq \varepsilon$ $(0 \leq t \leq 1)$, then $|c_n[\varphi] - c_n[\psi]| = |c_n[\varphi - \psi]| \leq 2\varepsilon$ $(n \geq -1)$. Hence, if $\varphi(t) = lim\,\varphi_\nu(t)$ uniformly in t, then $c_n[\varphi] = lim\,c_n[\varphi_\nu]$ uniformly in n. If the variation of φ is bounded on I_1:

$$\left|\frac{\Delta\varphi}{\Delta t}\right| = \left|\frac{\varphi(t_2) - \varphi(t_1)}{t_2 - t_1}\right| \leq N \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$(t_1 \in I_1,\ t_2 \in I_1,\ t_1 \neq t_2)$, then

$$|c_n[\varphi]| \leq N\lambda_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

for every $n \geq 0$. In fact, for $n = 0$ (16) is trivial, and for $n \geq 1$ we have

$$|c_n| \leq \tfrac{1}{2}|\varphi(r_n) - \varphi(r_n - \lambda_n)| + \tfrac{1}{2}|\varphi(r_n) - \varphi(r_n + \lambda_n)| \leq N\lambda_n.$$

Hence in this case $\sum_{-1}^{\infty}|c_n|^2$ is uniformly convergent, as for $k \geq 1$

$$\sum_{2^k}^{\infty}{}_n\,|c_n|^2 = \sum_{k}^{\infty}{}_l \sum_{0}^{2^l - 1}{}_j\,|c_{2^l + j}|^2 \leq \sum_{k}^{\infty}{}_l\, 2^l(2^{-l-1}N)^2 = 2^{-k-1}N^2.$$

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[5]) Though the entier-function $[x]$ can not be determined for *every* real x, it can, e.g. if x is a rational number or its logarithm, upon which it will only be applied here.

Hence the coefficients of the development of a function of bounded variation are the coordinates of a point of HILBERT space, belonging to the so-called "compact quadra" $\sum_0^\infty |c_n|^2 \leqq \frac{3}{2} N^2$.

If, at the other hand, (16) holds, then for $0 \leqq j \leqq 3 \,.\, 2^p - 1$

$$\left| \sum_{2^p+j}^{2^{p+2}-1} {}_n \, c_n [\varphi] \, u_n (t) \right| \leqq 2^{-p-1} N \sum_{2^p}^{2^{p+1}-1} {}_n \{ u_n (t) + \tfrac{1}{2} u_{2n} (t) + \tfrac{1}{2} u_{2n+1} (t) \} \leqq 2^{-p-1} N,$$

as for every t at most one of the terms between the curved brackets can be > 0 and then remains $\leqq 1$. Hence

$$\left| \sum_k^\infty c_n [\varphi] \, u_n (t) \right| \leqq \sum_0^\infty {}_q \, 2^{-l_k - 2q - 1} = \tfrac{4}{3} N \lambda_k,$$

so that (13) converges uniformly and then represents a uniformly continuous function $\varphi (t)$ with coefficients (14). In particular, in this case

$$| \varphi (t) | \leqq | c_{-1} [\varphi] | + \tfrac{4}{3} N. \quad \ldots \ldots \quad (17)$$

The variation of φ, however, need not be bounded, as is seen from the example $c_n = \lambda_n$ $(n \geqq 0)$, where $\{ \varphi (2^{-l}) - \varphi (0) \} \, 2^l = l + 1$.

The direction coefficients

$$m_n = m_n [\varphi] = (\Delta \varphi / \Delta t)_{I_n} = \{ \varphi (r_n + \lambda_n) - \varphi (r_n - \lambda_n) \} / 2 \lambda_n \quad . \quad (18)$$

can be expressed by the coefficients c_n:

$$m_n = \sum_0^{l_n} {}_j \, (-1)^{[2^{-j} j_n]} \, c_{[2^{-j-1} n]} \, \lambda^{-1}_{[2^{-j-1} n]} \qquad (n \geqq 1) \quad . \; . \; . \quad (19)$$

where $j_n = n - 2^{l_n}$; $\lambda^{-1}_{[2^{-j-1} n]} = 2^{l_n - j}$. In fact, for $n = 1$ (19) states that $m_1 = c_0$, which is trivial. If (19) holds for a certain value of n, then $r_{2n+q} = r_n + (q - \frac{1}{2}) \lambda_n$, $0 \leqq q \leqq 1$ (Cf. (11) with $p = 1$), and $m_{2n+q} = m_n + (-1)^q c_n \lambda_n^{-1}$. As $j_{2n+q} = 2 j_n + q$ and $[2^{-i} q] = 0$ for $i \geqq 1$, (19) is found to hold for $2n + q$ instead of n. Hence it holds for every n.

At the other hand, $c_n = \frac{1}{2} \lambda_n (m_{2n} - m_{2n+1})$, $m_n = \frac{1}{2} (m_{2n} + m_{2n+1})$. Hence, generally for every $g \geqq 1$, $n \geqq 1$:

$$m_n = 2^{-g} \sum_0^{2^g - 1} {}_j \, m_{2^g n + j}, \quad \ldots \ldots \quad (20)$$

$$c_n = 2^{-g} \lambda_n \sum_0^{2^g - 1} {}_j \, (-1)^{[2^{-g+1} j]} \, m_{2^g n + j} \,. \quad \ldots \quad (21)$$

By (21) we can determine the c_n for $n < 2^l$ if the m_n are known for $2^l \leqq n < 2^{l+1}$. For, taking $n = 2^{l-g} + h$, $1 \leqq g \leqq l$, $0 \leqq h \leqq 2^{l-g} - 1$, (21) becomes

$$c_{2^{l-g}+h} = 2^{-l-1} \sum_0^{2^g - 1} {}_j \, (-1)^{[2^{-g+1} j]} \, m_{2^l + 2^g h + j}, \quad c_0 = 2^{-l} \sum_0^{2^l - 1} {}_j \, m_{2^l + j}. \quad (22)$$

In particular (22) determines *all* coefficients of a polygonal line, having its edges on $t = 2^{-l} j$ $(0 \leqq j \leqq 2^l)$, since the coefficients c_n with $n \geqq 2^l$ then vanish.

Finally we conclude from (21) that a variation $\leqq \varepsilon$ of the m_n $(n \geqq 1)$ leads for each $k \geqq 0$ to a variation $\leqq \varepsilon \lambda_k$ of c_k. At the other hand, we can by (19) only conclude from a variation $\leqq \varepsilon \lambda_k$ of the c_k $(k \geqq 0)$ to a variation $\leqq (l_n + 1) \varepsilon$ of m_n.

4. **The theorem.** [6]) — Let S'_k be the set of all systems of real numbers $\gamma_{n\lambda}$, such that $\gamma_{-1,\lambda} = 0$ and that for each $n \geqq 1$ and for at least one $\varphi_\lambda (t)$ corresponding with a solution $X_{k\lambda} (T)$ with $X_{k\lambda} (0) = 0$ of (7)

$$\left| \sum_0^{l_n} {}_j \, (-1)^{[2^{-j} j_n]} \, 2^{-j + l_n} \, \gamma_{[2^{-j-1} n], \lambda} - m_n [\varphi_\lambda] \right| \leqq 2^{-k+1} \quad . \quad . \quad (23)$$

S'_k contains at least one element, viz $\gamma_{n\lambda} = c_n [\varphi_\lambda]$, where φ_λ corresponds with a solution of (7) which certainly exists, as was said before. Further let k_h be natural numbers with $k_{h+1} \geqq Max (k_h + f, \; l (k_h + 1) + 2)$, where $f \geqq 3 - {}^2 log \, \beta$, e.g. $f = 7$, and $S_h = S'_{k_h}$. Then S'_k and S_h each consist of a finite number of „compact" parallelotopes in HILBERT space. We prove now:

A. $S_{h+1} \subset S_h$.

B. *If* a solution $x_\lambda = \psi_\lambda (t)$ of (1) with $\psi_\lambda (0) = 0$ exists, its sequence of coefficients $c_n [\psi_\lambda]$ belongs to every S_h.

C. *If* $\gamma_{n\lambda}$ belongs to every S_h, they are the coefficients of a solution of the differential equations (1), satisfying $\psi_\lambda (0) = 0$.

This will be proved by means of two lemma's.

Lemma 1. If the functions $\psi_\lambda (t)$ are continuous for $0 \leqq t \leqq 1$, $|\psi_\lambda (t)| \leqq 1$, $\psi_\lambda (0) = 0$, and

$$\left| [\psi_\lambda (t)]_{t_1}^{t_2} - \int_{t_1}^{t_2} f_\lambda (t, \psi_\mu (t)) \, dt \right| \leqq 2^{-k} \beta \, | t_2 - t_1 | \quad . \; . \; . \quad (24)$$

for arbitrary t_1, t_2 in I_1, and $k = k_h$, then the coefficients $c_n [\psi_\lambda]$ belong to S_h.

Lemma 2. If the real numbers $\gamma_{n\lambda}$ belong to S_{h+1}, they are the coefficients $c_n [\psi_\lambda]$ of continuous functions $\psi_\lambda (t)$ with $|\psi_\lambda (t)| \leqq 1$, $\psi_\lambda (0) = 0$, and satisfying the inequalities (24).

It is trivial that the two lemma's imply statement *A*. Further, $\psi_\lambda (t)$ satisfying (1) is equivalent with the identity $[\psi_\lambda (t)]_{t_1}^{t_2} = \int_{t_1}^{t_2} f_\lambda (t, \psi_\mu (t)) \, dt$, for all t_1, t_2 in I_1 hence with validity of the integral unequalities (24) for every k. This leads immediately to statements *B* and *C*.

5. **Proof of Lemma 1.** — Let $\psi_\lambda (t)$ be continuous with $\psi_\lambda (0) = 0$, $|\psi_\lambda (t)| \leqq 1$ and satisfy (24) for $k = k_h$. Then integers $X_{k\lambda} (T)$ $(0 \leqq T \leqq 2^l - 1)$ exist, satisfying $X_{k\lambda} (0) = 0$ and

$$| X_{k\lambda} (T) - 2^{k+l} \psi_\lambda (2^{-l} T) | \leqq \alpha . \quad \ldots \ldots \quad (25)$$

[6]) The present result was found in 1939; a few simplifications date from 1940. I express my gratitude to my then assistants Mr. J. C. BOLAND and Mr. J. DE IONGH, who have checked some of the calculations and improved the demonstration by useful remarks. Mr. DE IONGH moreover contributed the formulae (19), (21) as a substitute for two somewhat lengthier formulae of the same type.

Then $|X_{k\lambda}(T)| \leqq 2^{k+l}$, and we obtain successively for $2^{-l}T \leqq t_1 \leqq t \leqq t_2 \leqq 2^{-l}(T+1)$:

$$1^\circ.\quad \left|\int_{t_1}^{t_2} f_\lambda(t, \psi_\mu(t))\, dt\right| \leqq N\,|t_2 - t_1|.$$

$$2^\circ.\quad |\psi_\lambda(t) - \psi_\lambda(2^{-l}T)| \leqq 2^{-k}\beta(t - 2^{-l}T) +$$

$$+ \left|\int_{2^{-l}T}^{t} f_\lambda(t, \psi_\mu(t))\, dt\right| \leqq (2^{-k}\beta + N)(t - 2^{-l}T) < 2^{-l}.$$

$$3^\circ.\quad |2^k f_\lambda(t, \psi_\mu(t) - F_{k\lambda}(T, X_{k\mu}(T))| \leqq \alpha + \beta \quad \text{by} \quad (2), (3).$$

$$4^\circ.\quad |[\psi_\lambda(t)]_{t_1}^{t_2} - 2^{-k} F_{k\lambda}(T, X_{k\mu}(T))(t_2 - t_1)| \leqq 2^{-k}\beta\,|t_2 - t_1| +$$

$$+ \int_{t_1}^{t_2} |f_\lambda(t, \psi_\mu(t)) - 2^{-k} F_{k\lambda}(T, X_{k\mu}(T))|\, dt \leqq 2^{-k}(\alpha + 2\beta)\,|t_2 - t_1|.$$

in particular $|2^{k+l}\{\psi_\lambda(2^{-l}(T+1)) - \psi_\lambda(2^{-l}T)\} - F_{k\lambda}(T, X_{k\mu}(T))| \leqq \alpha + 2\beta$.

$$5^\circ.\quad |\triangle X_{k\lambda}(T) - F_{k\lambda}(T, X_{k\mu}(T))| \leqq 1$$

the left members being integers $\leqq 3\alpha + 2\beta < 2$. Hence the integers $X_{k\lambda}(T)$ satisfy the difference inequalities (7). If $\varphi_\lambda(t)$ are the corresponding CAUCHY polygons, $m_{2^l+T}[\varphi_\lambda] = 2^{-k}\Delta X_{k\lambda}(T)$ being their direction coefficients, we obtain:

$$6^\circ.\quad |2^k f_\lambda(t, \psi_\mu(t)) - \triangle X_{k\lambda}(T)| \leqq 1 + \alpha + \beta \quad \text{by} \quad 3^\circ, 5^\circ.$$

Hence for $t_1 \neq t_2$:

$$7^\circ.\quad \left|\frac{\triangle\psi_\lambda}{\triangle t} - \frac{\triangle\varphi_\lambda}{\triangle t}\right| = \left|\frac{[\psi_\lambda(t)]_{t_1}^{t_2}}{t_2 - t_1} - 2^{-k}\triangle X_{k\lambda}(T)\right| \leqq$$

$$\leqq \frac{1}{|t_2 - t_1|}\int_{t_1}^{t_2} |f_\lambda(t, \psi_\mu(t)) - 2^{-k}\triangle X_{k\lambda}(T)|\, dt + 2^{-k}\beta \leqq$$

$$\leqq 2^{-k}(1 + \alpha + 2\beta) < 2^{-k+1}.$$

Hence $|m_n[\psi_\lambda] - m_n[\varphi_\lambda]| \leqq 2^{-k+1}$ for each $n \geqq 2^l$ and then by (20) for $n \geqq 1$. It follows then from (19) that $\gamma_{n\lambda} = c_n[\psi_\lambda]$ satisfies (23), hence belongs to S_h, which proves the lemma.

6. **Proof of Lemma 2.** — Let $\gamma_{n\lambda}$ belong to S_{h+1}. Then CAUCHY polygons $\varphi_\lambda(t)$ with edges $(2^{-l'}T, X_{k'\lambda}(t))$, $0 \leqq T \leqq 2^{l'}$, $k' = k_{h+1}$, $l' = l(k')$ exist, satisfying (23) with k' instead of k, for $n \geqq 2^{l'}$ and then by (20) also for $n \geqq 1$. Hence by (21): $|\gamma_{n\lambda} - c_n[\varphi_\lambda]| \leqq 2^{-k'+1}\lambda_n$. Hence by an argument like before, $\psi_\lambda(t) = \sum_0^\infty \gamma_{n\lambda} u_n(t)$ converges uniformly and by (17)

$$|\psi_\lambda(t) - \varphi_\lambda(t)| = \left|\sum_0^\infty (\gamma_{n\lambda} - c_n[\varphi_\lambda])\, u_n(t)\right| \leqq \tfrac{1}{3} \cdot 2^{-k'+3} < 2^{-l(k+1)}, \qquad (26)$$

as $k' \geqq l(k+1) + 2$.

Hence, with $2^{-l'}T \leqq t_1 \leqq t \leqq t_2 \leqq 2^{-l'}(T+1)$,

$$X_{k'\lambda}(T) = 2^{k'+l'}\varphi_\lambda(2^{-l'}T),\ |\triangle X_{k'\lambda}(T) - F_{k'\lambda}(T, X_{k'\mu}(T))| \leqq 1,$$

we have successively

$$1^\circ.\quad |\triangle X_{k'\lambda}(T)| \leqq 1 + \alpha + 2^{k'}N < (\beta + N)\,2^{k'} < 2^{k'},$$

as $1 + \alpha < \frac{5}{3} < 2^{f-2}\beta < 2^{k'}\beta$. Hence $|X_{k'\lambda}(T)| \leqq (1 + \alpha + 2^{k'}N)\ 2^{l'}$, $|\varphi_\lambda(t)| \leqq N + 2^{-k'}(1 + \alpha)$ and $|\psi_\lambda(t)| \leqq N + 2^{-k'}(\frac{11}{3} + \alpha) < N + 2^{-f+3} \leqq N + \beta < 1$.

$$2^\circ.\quad |\varphi_\lambda(t_2) - \varphi_\lambda(t_1)| = 2^{-k'}|\triangle X_{k'\lambda}(T)(t_2 - t_1)| < (\beta + N)|t_2 - t_1| \leqq 2^{-l'}.$$

$$3^\circ.\quad |\triangle X_{k'\lambda}(T) - 2^{k'} f_\lambda(t, \varphi_\mu(t))| \leqq |\triangle X_{k'\lambda}(T) - F_{k'\lambda}(T, X_{k'\mu}(T))| +$$

$$+ |F_{k'\lambda}(T, X_{k'\mu}(T)) - 2^{k'} f_\lambda(2^{-l'}T, 2^{-k'-l'}X_{k'\mu}(T))| +$$

$$+ 2^{k'}|f_\lambda(2^{-l'}T, \varphi_\mu(2^{-l'}T)) - f_\lambda(t, \varphi_\mu(t))| \leqq 1 + \alpha + \beta < 2.$$

$$4^\circ.\quad \left|[\varphi_\lambda(t)]_{t_1}^{t_2} - \int_{t_1}^{t_2} f_\lambda(t, \varphi_\mu(t))\, dt\right| \leqq$$

$$\leqq \int_{t_1}^{t_2} |2^{-k'}\triangle X_{k'\lambda}(T) - f_\lambda(t, \varphi_\mu(t))|\, dt \leqq 2^{-k'+1}|t_2 - t_1|.$$

$$5^\circ.\quad |f_\lambda(t, \varphi_\mu(t)) - f_\lambda(t, \psi_\mu(t))| \leqq 2^{-k-1}\beta \quad \text{by} \quad (26), (2).$$

If $t_1 = r_n - \lambda_n$, $t_2 = r_n + \lambda_n$ with $n \geqq 2^{l'+1}$ we have moreover by (23):

$$6^\circ.\quad |[\psi_\lambda(t) - \varphi_\lambda(t)]_{t_1}^{t_2}| = |t_2 - t_1|\,|m_n[\psi_\lambda - \varphi_\lambda]| \leqq 2^{-k'+1}|t_2 - t_1|.$$

$$7^\circ.\quad \left|[\psi_\lambda(t)]_{t_1}^{t_2} - \int_{t_1}^{t_2} f_\lambda(t, \psi_\mu(t))\, dt\right| \leqq |[\varphi_\lambda(t)]_{t_1}^{t_2} -$$

$$- \int_{t_1}^{t_2} f_\lambda(t, \varphi_\mu(t))\, dt| + |[\psi_\lambda(t) - \varphi_\lambda(t)]_{t_1}^{t_2}| + \int_{t_1}^{t_2} |f_\lambda(t, \varphi_\mu(t)) - f_\lambda(t, \psi_\mu(t))|\, dt \leqq$$

$$\leqq (2^{-k'+1} + 2^{-k'+1} + 2^{-k-1}\beta)|t_2 - t_1| \leqq 2^{-k}(2^{-f+2} + \tfrac{1}{2}\beta)|t_2 - t_1| \leqq 2^{-k}\beta|t_2 - t_1|.$$

Hence (24) has been proved for every sufficiently small interval with dyadic rational endpoints. But both sides of (24) are the absolute values of additive interval-functions, hence (24) then holds for *every* interval $\subset I_1$, which proves the lemma, and therewith the theorem.
