Applied Mechanics. — *On a special case of bending.* By C. B. BIEZENO.

(Communicated at the meeting of April 25, 1942.)

1. *Introduction.* In a recent paper R. SONNTAG [1]) draws attention to a special case of bending, which occurs if a highly elastic beam, freely supported in two points of prescribed distance l is subjected to a transverse load P, acting in the middle of the span. The beam is supposed to slide freely — and without friction — over its supports, so that great deflections are to be expected even under relatively small loads. In an ingenious, but rather artificial way the author succeeds in deducing the relations between the load P, the corresponding deflection f, and the length of the deflected beam as far as it lies between its two supports. The artificiality of the method lies in the linearisation of the problem, which — at the expense of a rather laborious control — is proved to hold within well-mentioned limits.

In the following remarks it is shown that the problem admits a quite natural solution, which for all possible beams requires the construction of one single graph from which SONNTAG's conclusions — inclusive the interesting remark about the stability of the beam — can all be derived.

2. *The graphical method.* In this section the central line of the girder under consideration will be constructed by means of a method which with a slight modification is identical with the so-called method of elastic joints [2]). As to the notation it may be stated, that:

M	stands for the bending moment in any section of the girder
EI	for its flexural rigidity
$1/\varrho$	for the curvature of the central line
l_e	for its effective length (the length between the points of support A and B)
l	for the length of the span AB
f	for the maximum deflection of the central line
P	for the central load
N	for the reactions of the supports
β	for the slope of the central line at A and B
"a"	for the length of an "element" of the girder
EI/a	for the elastic stiffness of such an element.

Fig 1 represents the bent girder under the action of the load P, giving rise to the normal reactions $N = P/2\cos\beta$ of the supports. If P is prescribed, it is required to determine N, β, l_e and f in terms of P and it is easily seen which way has to be followed if we endeavour a direct solution of this problem. However elliptic integrals will be involved in the calculations, which on account of the undeterminate length l_e will become rather troublesome. It therefore recommends itself to have rescue to an indirect method of attack by making the problem dependent on the normal reaction N. If an arbitrary value of N, say $N = 100$ kg be assumed, the central line of the girder can be constructed step by step, as illustrated in fig. 2.

Let the construction of the central line have been performed up to the endpoint T_{23} of the "element", $T_{12}T_{23}$ of length a, let S_2S_3 represent the tangent to the central line

1) Comp. R. SONNTAG, Der beiderseitig gestützte, symmetrisch belastete gerade Stab mit endlicher Durchbiegung und seine Stabilität. Ingenieur-Archiv, Bnd. XII, S. 283—307.

2) Comp. f.i. C. B. BIEZENO und R. GRAMMEL, Technische Dynamik, III, 4, 19, p. 172.

in this endpoint and let S_3 be acquired by making $T_{23}S_3$ equal to $a/2$. Then with high approximation the mean value of the bending moment M_3 occurring in the next element $T_{23}T_{34}$ of the girder will be given by the value $M_3 = Nb_3$, and consequently the radius of curvature ϱ_3 of this element is represented by $EI : M_3 = EI : Nb_3$. The centre of curvature N_3 of this element therefore can be constructed. A circle with this point as centre and with radius ϱ_3 can be drawn, and the next point T_{34} of the central line,

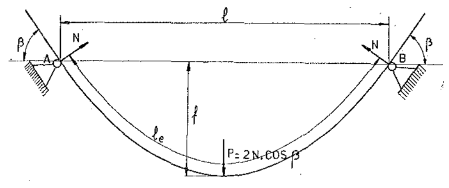

Fig. 1.

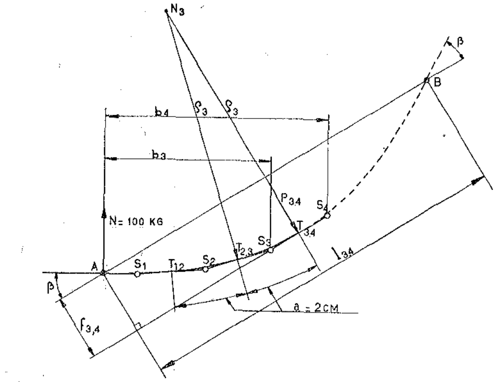

Fig. 2.

together with the corresponding tangent in this point is obtained by pacing the length $T_{23}T_{34}$ = "a" on the circle or by rotating the radius N_3T_{23} through the angle $\triangle\beta = Nba/EI$. By making $T_{34}S_4$ equal to $a/2$, and by reading off the distance b_4 the mean bending moment of the next element of the girder is found, a.s.o.

If for a moment the construction be stopped at T_{34} and if N_3T_{34} be made a line of symmetry for a curve one half of which is represented by $AT_{12}T_{23}T_{34}$, a central line is obtained, the span and deflection of which are represented by l_{34} and f_{34}. The load P_{34} necessary to maintain the prescribed deflection is fixed by the equation of equilibrium $P_{34} = 2N\cos\beta$. The construction hitherto described has been carried out in fig. 3 up to the point 15 for a girder, whose flexural rigidity amounts to 25000 kg cm². The elementary length "a" has been chosen equal to 2 cm; the normal reaction N in A has again been chosen 100 kg. We read from this construction that a deflection $f_{7,8}$ occurs if the span AB (fig. 1) is equal to $l_{7,8}$ and if the girder is centrally loaded by $P_{7,8}$ (see the upper left corner of fig. 3, where a half-circle has been drawn on a vertical diameter representing 200 kg, and where $P_{7,8}$ is parallel to the normal, passing through the points N_7 and N_8 of the main figure).

The accuracy of the construction can be checked by calculating the deflection f corresponding to that point of the central line for which the tangent is parallel to the normal n_1 of the point A. If a rectangular system of coordinates be assumed, the y-axis of which coincides with the normal n_1, the x-axis with the line $A\,1$, then the differential epuation of the central line is given by:

$$\frac{EI\,y''}{(1+y'^2)^{3/2}} = Nx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The first integral of this equation is found in an elementary way by putting $y' = tg\,t$. Taking into account that y' must be zero for $x = 0$, we find

$$y' = \frac{\alpha x^2}{\sqrt{1-\alpha^2 x^4}} \quad \text{with} \quad \alpha = \frac{N}{EI}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The derivative y' becomes infinite if $x = \alpha^{-1/2}$, which in our case ($EI = 25000$ kg cm², $N = 100$ kg) leads to $x = 22{,}37$ cm. In figure 3 the point for which y' is infinite can accurately be constructed. Its distance to the y-axis is (in the fullscale drawing) undistinguishable from the calculated value.

If we should have started our construction with another normal reaction say N^*, and if the flexural rigidity of the girder should have been EI^* instead of EI, another

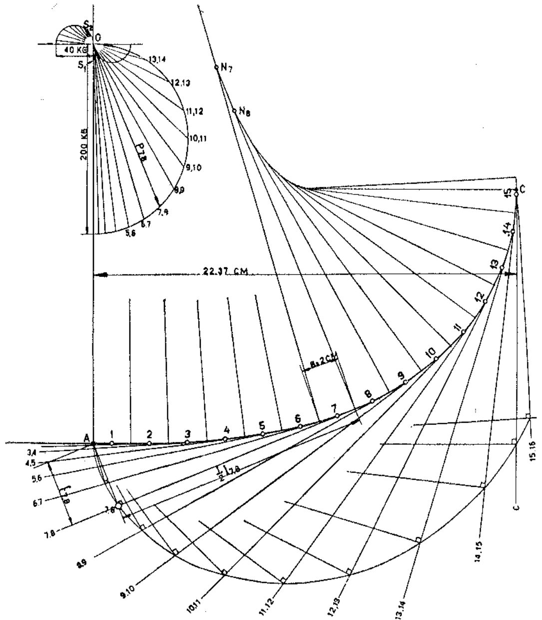

Fig. 3.

central line would have been obtained, provided the same length-scale should have been used. But it is easily understood that the central line of fig. 3 would serve our aim as well if only the scale to wich the figure has been designed, is suitably changed. Assuming that 1 cm of the figure represents p cm in reality, the quantity $\triangle \beta^* = N^*.pb.pa./EI^*$,

TABLE 1.

Girder-element	β	P	l_e	l	f	$\lambda = Pl^2/8\,EI$	f/l	l_e/l
1— 2	0.008	200.0	4	4	0.00	0.016	0	1
2— 3	0.032	200.0	8	8	0.10	0.064	0.0125	1
3— 4	0.0718	199.8	12	12	0.25	0.144	0.0208	1
4— 5	0.1276	198.7	16	16	0.47	0.254	0.0294	1
5— 6	0.1993	196.4	20	19.80	1.21	0.385	0.0612	1.01
6— 7	0.2866	192.1	24	23.54	2.20	0.533	0.0934	1.02
7— 8	0.3897	185.1	28	27	3.50	0.674	0.1296	1.037
8— 9	0.5079	174.8	32	30	5.20	0.786	0.1733	1.067
9—10	0.6404	160.1	36	32.40	7.30	0.840	0.2253	1.111
10—11	0.7855	141.3	40	33.86	9.80	0.809	0.2894	1.182
11—12	0.9416	117.9	44	34.40	12.45	0.697	0.3620	1.280
12—13	1.1068	90.0	48	33.94	15.22	0.518	0.4485	1.414
13—14	1.2792	57.6	52	32.14	18.09	0.298	0.5630	1.618
14—15	1.4567	23.0	56	29.04	20.87	0.097	0.7180	1.929
15—16	1.6359	—12.0	60	25.54	23.17	—0.39	0.9070	2.350

will only be identical with $\triangle \beta = Nb.a/EI$ if

$$p = \sqrt{\frac{N}{N^*}\,\frac{EI^*}{EI}}\left(= \sqrt{\frac{100\,.\,EI^*}{N^*\,.\,25000}} = \sqrt{\frac{EI^*}{250\,N^*}}\right). \quad . \quad . \quad . \quad (3)$$

Consequently fig. 3 holds for every girder and every load P. The results which under the assumption $N = 100$ kg, $EI = 25000$ kg cm², can be read from this figure are assembled in the first five columns of table 1. The last three columns contain those dimensionless quantities which are essential for the problem. The quantity λ has been chosen to enable a comparison with the results of SONNTAG. As can be seen from fig. 4, in which λ, f/l and l_e/l are plotted against β, and in which the results obtained by SONNTAG have been marked by points, the agreement is a remarkably good one. It may be remarked, that β has been chosen as the independent variant only to simplify the connection with SONNTAG's work. A more satisfying representation, as far as λ is regarded, is given in fig. 5, where λ is represented as function of the deflection-ratio f/l. (In both figures 4 and 5 the middle curve λ is in discussion here; in the last one SONNTAG's results are represented by the dotted line). It follows from this figure that only to a certain amount of the load P, the girder is in stable equilibrium. The load for which instability occurs can easily be read from the graph.

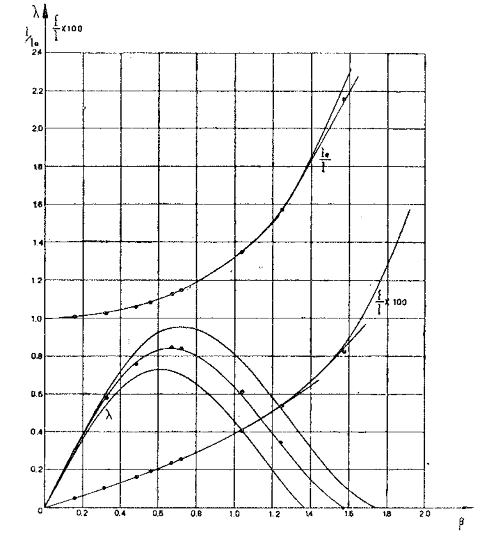

Fig. 4.

3. *Additional remarks.* Some generalizing remarks can be made.

a. The construction given in this paper can be extended without any difficulty to symmetrical girders of varying flexural rigidity.

b. It can as well be extended to cases where friction becomes into being during the gliding motion of the girder over its supports. According to the two possible directions of gliding frictional forces of the magnitude $\pm \mu N$ (μ = coefficient of friction) have to be introduced. The value of N being assumed, the influence of these frictional forces can be brought into account in the very same manner as that of the reaction N, and the whole construction can therefore be performed as in fig. 3. The required loading force P is given by $P = 2N\,(\cos\beta \pm \mu \sin\beta)$. It has been indicated in the left upper corner of fig. 3, how (for $\mu = 0.2$) P could be found as soon as the central line would be known. If the direction of P would be given by n_8, $P_{7,8}$ would be equal either to

S_1 — (7,8) or to S_2 — (7,8). From the figures 4 and 5, in which for both gliding directions the graphs for λ corresponding to $\mu = 0{,}2$ are represented, it is seen that the frictional influence is considerable, and therefore it does not surprise that SONNTAG's experiments in which friction was not sufficiently eliminated did not very well agree with his theoretical results, which no doubt are very accurate.

c. Whereas SONNTAG's deductions only hold true as far as the central line of the girder can be approximated in a sufficient way by a suitable chosen circle-arch, the method proposed in this paper allows an important extension in that the construction of fig. 3 can be continued as far as we wish. If we proceed in the prescribed way a line arises as represented in fig. 6, the obvious properties of symmetry of which can be easily proved.

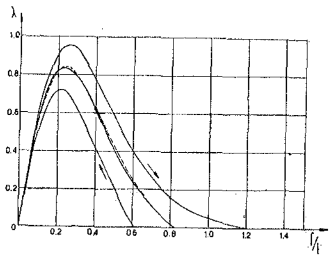

Fig. 5.

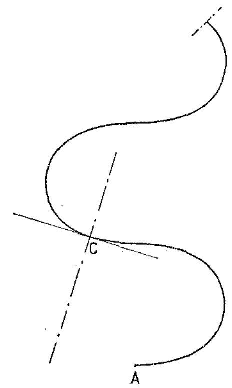

Fig. 6.

The normal of any point C of this curve can be looked upon as a line of symmetry of a possible central line of the girder, one half of which is represented by the arch AC. It is worth while to draw the different central lines, corresponding to a great number of points C; for this will give an insight in the numerous ways in which the girder can deform. This, however, must be left to the reader.

4. *Acknowledgment.* The author wishes to thank his former assistant, Mr. A. D. DE PATER to whom he is indebted for valuable help in drawing the figures, and in discussing the matters, mentioned under 3*b* and *c*.

Mathematics — *Die repräsentierende Menge der stetigen Funktionen des Einheitskontinuums.* Von Prof. L. E. J. BROUWER.

(Communicated at the meeting of April 25, 1942.)

Wenn eine unbegrenzte Folge von Operationen F_ν ausgeführt wird, deren jede darin besteht, dass jedem Punktkerne des Einheitskontinuums ein λ-Intervall [1]) zugeordnet wird, und zwar in solcher Weise, dass für jeden Punktkern das von $F_{\nu+1}$ erzeugte Intervall jedesmal im von F_ν erzeugten Intervall im engern Sinne enthalten ist, so soll diese Operationsfolge eine *volle Freifunktion des Einheitskontinuums* heissen. Offenbar ordnet sie jedem Punktkern des Einheitskontinuums einen Punktkern des Linearkontinuums zu. Demgemäss bilden die früher eingeführten *vollen Funktionen des Einheitskontinuums* [2]) einen Spezialfall der vollen Freifunktionen des Einheitskontinuums. Der früher für die vollen Funktionen geführte Beweis der gleichmässigen Stetigkeit [3]) lässt sich aber ungeändert für die vollen Freifunktionen übernehmen.

Wenn für gegebenes m und n jedem $\varkappa^{(m)}$-Intervall [1]) K des Einheitskontinuums ein $\lambda^{(\nu)}$-Intervall [1]) (ν variabel mit dem Minimum n) $\varphi(K)$ so zugeordnet ist, dass aneinander grenzenden K entweder identische oder ein gemeinsames Segment besitzende $\varphi(K)$ entsprechen, so nennen wir diese Zuordnung ein *m n-Treppenpolygon.* Als Spezialfall definieren wir einen *m n-Treppenblock,* indem wir weiter fordern dass jedes $\nu = n$ ist.

Für $m_2 > m_1$ und $n_2 > n_1$ soll das $m_2 n_2$-Treppenpolygon T_2 im $m_1 n_1$-Treppenpolygon T_1 *eingelagert* heissen, wenn für ein beliebiges $\varkappa$-Intervall K_1 von T_1 und ein beliebiges $\varkappa$-Intervall K_2 von T_2 die Beziehung $K_2 \subset K_1$ nach sich zieht, dass $\varphi_2(K_2)$ einen echten Teil von $\varphi_1(K_1)$ bildet.

Unter einer *Treppenfunktion* (bzw. *Blockfunktion*) verstehen wir eine unbegrenzte Folge von Treppenpolygonen (bzw. Treppenblöcken), deren jedes in dem ihm vorangehenden eingelagert ist. Offenbar ordnet eine Treppenfunktion (bzw. Blockfunktion) jedem Punktkern des Einheitskontinuums einen Punktkern des Linearkontinuums zu.

Wir nennen eine volle Freifunktion des Einheitskontinuums und eine Treppenfunktion oder Blockfunktion einander *gleich* und sagen, dass sie einander *repräsentieren,* wenn sie jedem Punktkern des Einheitskontinuums denselben Punktkern des Linearkontinuums zuordnen.

Man beweist unschwer, dass *erstens* jede Treppenfunktion bzw. Blockfunktion einer vollen Freifunktion des Einheitskontinuums gleich ist, *zweitens* jede volle Freifunktion des Einheitskontinuums sich durch eine Treppenfunktion und sogar durch eine Blockfunktion repräsentieren lässt.

Nun sind aber die Spezies aller Treppenblöcke (ebenso wie die Spezies aller Treppenpolygone) und die Spezies der Treppenblöcke, welche in einem gegebenen Treppenblock eingelagert werden können (ebenso wie die Spezies der Treppenpolygone, welche in einem gegebenen Treppenpolygon eingelagert werden können) abzählbar unendlich. Nach dem vorhergehenden folgt hieraus unmittelbar, dass *die Spezies der vollen Freifunktionen des Einheitskontinuums sich durch eine non-finite Menge repräsentieren lässt,* nämlich durch die Menge welche entsteht, wenn zunächst ein beliebiger Treppenblock T_1 gewählt wird, und sodann der Reihe nach für jede natürliche Zahl ν ein beliebiger in T_ν eingelagerter Treppenblock $T_{\nu+1}$.

1) Mathem. Annalen, Bd. **93**, S. 253 (1925); Bd. **97**, S. 60 (1926).
2) Mathem. Annalen, Bd. **97**, S. 62 (1926).
3) Mathem. Annalen, Bd. **97**, S. 66, 67 (1926).