Mathematics. — *Over reeksen en bepaalde integralen, waarbij functies van* BESSEL *optreden.* II. Door J. G. RUTGERS. (Communicated by Prof. J. A. SCHOUTEN.)

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3. Voor $\mu = \nu - m$, waarin m geheel $\geqq 0$ is, gaat I over in een eindige reeks; vervangen we tevens ν door ϱ, dan krijgen we:

$$\sum_{n=0}^{m} \frac{(-1)^n \binom{m}{n}}{\Gamma(\varrho+n-m+1)} \left(\frac{x}{2}\right)^{\varrho+n} I_{\varrho+n}(x) = \frac{\left(\frac{x}{2}\right)^{\varrho+m}}{\Gamma(\varrho+1)} I_{\varrho-m}(x), \quad . \quad (V)$$

ϱ willekeurig, m geheel $\geqq 0$.

Door in (1) tot en met (8) ϱ te vervangen door $\varrho+n$ en daarna beide leden te vermenigvuldigen met $\dfrac{(-)^n \binom{m}{n}}{2^{\varrho+n}\,\Gamma(\varrho+n-m+1)}$, volgen na sommatie over n van 0 tot m, onder toepassing van V in het linkerlid onder het integraalteeken, voor $R(\nu) > -\frac{1}{2}$, $R(\varrho) > -\frac{1}{2}$ en $m \geqq 0$:

$$\left.\begin{aligned} &\int_0^x I_\nu(x-\alpha)\, I_{\varrho-m}(\alpha)\,(x-\alpha)^\nu\, \alpha^{\varrho+m}\, d\alpha = \\ &= \frac{\Gamma(\nu+\frac{1}{2})\, m!\, 2^{\nu+\varrho+m}}{\sqrt{\pi}} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho}{m-n}}{n!} \cdot \frac{\Gamma(\varrho+n+\frac{1}{2})}{\Gamma(\nu+\varrho+n+1)} \left(\frac{x}{2}\right)^{\nu+\varrho+n+\frac{1}{2}} I_{\nu+\varrho+n+\frac{1}{2}}(x), \end{aligned}\right\} \quad (16)$$

$$\left.\begin{aligned} &\int_0^x I_{\nu-\frac{1}{2}}(x-\alpha)\, I_{\varrho-m}(\alpha)\,(x-\alpha)^{\nu+\frac{1}{2}}\, \alpha^{\varrho+m}\, d\alpha = \\ &= \frac{\Gamma(\nu+1)\, m!\, 2^{\nu+\varrho+m+\frac{1}{2}}}{\sqrt{\pi}} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho}{m-n}}{n!} \cdot \frac{\Gamma(\varrho+n+\frac{1}{2})}{\Gamma(\nu+\varrho+n+\frac{3}{2})} \left(\frac{x}{2}\right)^{\nu+\varrho+n+1} I_{\nu+\varrho+n}(x), \end{aligned}\right\} \quad (17)$$

$$\int_0^x I_{\varrho-m}(\alpha)\, \alpha^{\varrho+m} \sin(x-\alpha)\, d\alpha = m!\, 2^{\varrho+m+1} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{\varrho+n+1}}{2\varrho+2n+1} I_{\varrho+n+1}(x), \quad (18)$$

$$\int_0^x I_{\varrho-m}(\alpha)\, \alpha^{\varrho+m} \cos(x-\alpha)\, d\alpha = m!\, 2^{\varrho+m+1} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{\varrho+n+1}}{2\varrho+2n+1} I_{\varrho+n}(x), \quad . \quad (19)$$

$$\left.\begin{aligned} &\int_0^x I_{\varrho-m}(\alpha)\, \alpha^{\varrho+m} \sin\alpha\, d\alpha = \\ &= m!\, 2^{\varrho+m+1} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{\varrho+n+1}}{2\varrho+2n+1} \{\sin x\, I_{\varrho+n}(x) - \cos x\, I_{\varrho+n+1}(x)\}, \end{aligned}\right\} \quad (20)$$

$$\left.\begin{aligned} &\int_0^x I_{\varrho-m}(\alpha)\, \alpha^{\varrho+m} \cos\alpha\, d\alpha = \\ &= m!\, 2^{\varrho+m+1} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{\varrho+n+1}}{2\varrho+2n+1} \{\cos x\, I_{\varrho+n}(x) + \sin x\, I_{\varrho+n+1}(x)\}, \end{aligned}\right\} \quad (21)$$

$$\left.\begin{aligned} &\int_0^x I_{\varrho-m}(\alpha)\, \alpha^{\varrho+m} \sin(2x-\alpha)\, d\alpha = \\ &= m!\, 2^{\varrho+m+1} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{\varrho+n+1}}{2\varrho+2n+1} \{\sin x\, I_{\varrho+n}(x) + \cos x\, I_{\varrho+n+1}(x)\}, \end{aligned}\right\} \quad (22)$$

$$\left.\begin{aligned} &\int_0^x I_{\varrho-m}(\alpha)\, \alpha^{\varrho+m} \cos(2x-\alpha)\, d\alpha = \\ &= m!\, 2^{\varrho+m+1} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{\varrho+n+1}}{2\varrho+2n+1} \{\cos x\, I_{\varrho+n}(x) - \sin x\, I_{\varrho+n+1}(x)\}. \end{aligned}\right\} \quad (23)$$

Vervangen we in (9′) tot en met (15) ϱ door $\varrho+n$ en vermenigvuldigen we daarna beide leden met $\dfrac{(-1)^n \binom{m}{n}}{2^{\varrho+n-\frac{1}{2}}\,\Gamma(\varrho+n-m+\frac{1}{2})}$, dan volgen, na sommatie over n van 0 tot m, onder toepassing van V, zoo hierin vooraf ϱ vervangen wordt door $\varrho-\frac{1}{2}$, in het linkerlid onder het integraalteeken, voor $R(\nu) > -\frac{1}{2}$, $R(\varrho) > -\frac{1}{2}$ en $m \geqq 0$:

$$\left.\begin{aligned} &\int_0^x I_{\nu-\frac{1}{2}}(x-\alpha)\, I_{\varrho-m-\frac{1}{2}}(\alpha)\,(x-\alpha)^{\nu+\frac{1}{2}}\, \alpha^{\varrho+m+\frac{1}{2}}\, d\alpha = \\ &= \frac{\Gamma(\nu+1)\, m!\, 2^{\nu+\varrho+m}}{\sqrt{\pi}} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho-\frac{1}{2}}{m-n}}{n!} \cdot \frac{\Gamma(\varrho+n+1)}{\Gamma(\nu+\varrho+n+2)} \left(\frac{x}{2}\right)^{\nu+\varrho+n+\frac{1}{2}} \{x I_{\nu+\varrho+n-\frac{1}{2}}(x) + I_{\nu+\varrho+n+\frac{1}{2}}(x)\}, \end{aligned}\right\} \quad (24)$$

$$\left.\begin{aligned} &\int_0^x I_{\varrho-m-\frac{1}{2}}(\alpha)\, \alpha^{\varrho+m+\frac{1}{2}} \cos(x-\alpha)\, d\alpha = \\ &= m!\, 2^{\varrho+m-\frac{1}{2}} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho-\frac{1}{2}}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{\varrho+n+\frac{1}{2}}}{\varrho+n+1} \{x\, I_{\varrho+n-\frac{1}{2}}(x) + I_{\varrho+n+\frac{1}{2}}(x)\}, \end{aligned}\right\} \quad (25)$$

$$\int_0^x I_{\varrho-m-\frac{1}{2}}(\alpha)\, \alpha^{\varrho+m+\frac{1}{2}} \sin(x-\alpha)\, d\alpha = m!\, 2^{\varrho+m+\frac{1}{2}} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho-\frac{1}{2}}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{\varrho+n+\frac{3}{2}}}{\varrho+n+1} I_{\varrho+n+\frac{1}{2}}(x). \quad (26)$$

$$\int_0^x I_{\varrho-m-\frac{1}{2}}(\alpha)\, \alpha^{\varrho+m+\frac{1}{2}} \sin \alpha\, d\alpha = m!\, 2^{\varrho+m-\frac{1}{2}} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho-\frac{1}{2}}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{\varrho+n+\frac{1}{2}}}{\varrho+n+1} \{x \sin x\, I_{\varrho+n-\frac{1}{2}}(x) + (\sin x - x \cos x)\, I_{\varrho+n+\frac{1}{2}}(x)\}, \qquad (27)$$

$$\int_0^x I_{\varrho-m-\frac{1}{2}}(\alpha)\, \alpha^{\varrho+m+\frac{1}{2}} \cos \alpha\, d\alpha = m!\, 2^{\varrho+m-\frac{1}{2}} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho-\frac{1}{2}}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{\varrho+n+\frac{1}{2}}}{\varrho+n+1} \{x \cos x\, I_{\varrho+n-\frac{1}{2}}(x) + (\cos x + x \sin x)\, I_{\varrho+n+\frac{1}{2}}(x)\}, \qquad (28)$$

$$\int_0^x I_{\varrho-m-\frac{1}{2}}(x)\, \alpha^{\varrho+m+\frac{1}{2}} \sin (2x-\alpha)\, d\alpha = m!\, 2^{\varrho+m-\frac{1}{2}} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho-\frac{1}{2}}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{\varrho+n+\frac{1}{2}}}{\varrho+n+1} \{x \sin x\, I_{\varrho+n-\frac{1}{2}}(x) + (\sin x + x \cos x)\, I_{\varrho+n+\frac{1}{2}}(x)\}, \qquad (29)$$

$$\int_0^x I_{\varrho-m-\frac{1}{2}}(x)\, \alpha^{\varrho+m+\frac{1}{2}} \cos (2x-\alpha)\, d\alpha = m!\, 2^{\varrho+m-\frac{1}{2}} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho-\frac{1}{2}}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{\varrho+n+\frac{1}{2}}}{\varrho+n+1} \{x \cos x\, I_{\varrho+n-\frac{1}{2}}(x) + (\cos x - x \sin x)\, I_{\varrho+n+\frac{1}{2}}(x)\}. \qquad (30)$$

Substitutie van $\varrho = m + \frac{1}{2}$ in (16) tot en met (23) geeft na eenige herleiding de volgende formules, geldig voor $R(\nu) > -\frac{1}{2}$ en $m \geq 0$:

$$\int_0^x I_\nu(x-\alpha)\,(x-\alpha)^\nu\, \alpha^{2m} \sin \alpha\, d\alpha = m!\, \Gamma(\nu+\tfrac{1}{2})\, 2^{\nu+2m} \sum_{n=0}^{m} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n}}{n!} \cdot \frac{(m+n)!}{\Gamma(\nu+m+n+\frac{3}{2})} \left(\frac{x}{2}\right)^{\nu+m+n+1} I_{\nu+m+n+1}(x), \qquad (31)$$

$$\int_0^x I_{\nu-\frac{1}{2}}(x-\alpha)\,(x-\alpha)^{\nu+\frac{1}{2}}\, \alpha^{2m} \sin \alpha\, d\alpha = m!\, \Gamma(\nu+\tfrac{1}{2})\, 2^{\nu+2m+\frac{1}{2}} \sum_{n=0}^{m} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n}}{n!} \cdot \frac{(m+n)!}{\Gamma(\nu+m+n+2)} \left(\frac{x}{2}\right)^{\nu+m+n+\frac{3}{2}} I_{\nu+m+n+\frac{3}{2}}(x), \qquad (32)$$

$$\int_0^x \alpha^{2m} \cos (2\alpha - x)\, d\alpha = \frac{x^{2m+1}}{2m+1} \cos x + m!\, 2^{2m+1} \sqrt{\pi} \sum_{n=0}^{m} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} I_{m+n+\frac{3}{2}}(x), \qquad (33)$$

$$\int_0^x \alpha^{2m} \sin (2\alpha - x)\, d\alpha = -\frac{x^{2m+1}}{2m+1} \sin x + m!\, 2^{2m+1} \sqrt{\pi} \sum_{n=0}^{m} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} I_{m+n+\frac{1}{2}}(x), \qquad (34)$$

$$\int_0^x \alpha^{2m} \cos 2\alpha\, d\alpha = \frac{x^{2m+1}}{2m+1} - m!\, 2^{2m+1} \sqrt{\pi} \sum_{n=0}^{m} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} \{\sin x\, I_{m+n+\frac{1}{2}}(x) - \cos x\, I_{m+n+\frac{3}{2}}(x)\}, \qquad (35)$$

$$\int_0^x \alpha^{2m} \sin 2\alpha\, d\alpha = m!\, 2^{2m+1} \sqrt{\pi} \sum_{n=0}^{m} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} \{\cos x\, I_{m+n+\frac{1}{2}}(x) + \sin x\, I_{m+n+\frac{3}{2}}(x)\}, \qquad (36)$$

$$\int_0^x \alpha^{2m} \cos 2(x-\alpha)\, d\alpha = \frac{x^{2m+1}}{2m+1} \cos 2x + m!\, 2^{2m+1} \sqrt{\pi} \sum_{n=0}^{m} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} \{\sin x\, I_{m+n+\frac{1}{2}}(x) + \cos x\, I_{m+n+\frac{3}{2}}(x)\}, \qquad (37)$$

$$\int_0^x \alpha^{2m} \sin 2(x-\alpha)\, d\alpha = \frac{x^{2m+1}}{2m+1} \sin 2x - m!\, 2^{2m+1} \sqrt{\pi} \sum_{n=0}^{m} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} \{\cos x\, I_{m+n+\frac{1}{2}}(x) - \sin x\, I_{m+n+\frac{3}{2}}(x)\}. \qquad (38)$$

Substitutie van $\varrho = m - \frac{1}{2}$ in (16) tot en met (23) geeft, na vervanging van m door $m+1$ en eenige herleiding, de volgende formules, geldig voor $R(\nu) > -\frac{1}{2}$ en $m \geqq 0$:

$$\left.\begin{aligned}\int_0^x I_\nu(x-\alpha)(x-\alpha)^\nu \alpha^{2m+1} \cos\alpha \, d\alpha = (m+1)!\,\Gamma(\nu+\tfrac{1}{2})\, 2^{\nu+2m+1} \sum_{n=0}^{m+1} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n+1}}{n!} \,.\\ \cdot \frac{(m+n)!}{\Gamma(\nu+m+n+\frac{3}{2})} \left(\frac{x}{2}\right)^{\nu+m+n+1} I_{\nu+m+n+1}(x),\end{aligned}\right\} \quad (39)$$

$$\left.\begin{aligned}\int_0^x I_{\nu-\frac{1}{2}}(x-\alpha)(x-\alpha)^{\nu+\frac{1}{2}} \alpha^{2m+1} \cos\alpha \, d\alpha = (m+1)!\,\Gamma(\nu+\tfrac{1}{2})\, 2^{\nu+2m+\frac{3}{2}} \sum_{n=0}^{m+1} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n+1}}{n!} \,.\\ \cdot \frac{(m+n)!}{\Gamma(\nu+m+n+2)} \left(\frac{x}{2}\right)^{\nu+m+n+\frac{3}{2}} I_{\nu+m+n+\frac{1}{2}}(x),\end{aligned}\right\} \quad (40)$$

$$\left.\begin{aligned}&\int_0^x \alpha^{2m+1} \sin(2\alpha - x)\, d\alpha = \\ &= \frac{x^{2m+2}}{2(m+1)} \sin x - (m+1)!\, 2^{2m+2} \sqrt{\pi} \sum_{n=0}^{m+1} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n+1}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} I_{m+n+\frac{3}{2}}(x),\end{aligned}\right\} \quad (41)$$

$$\left.\begin{aligned}&\int_0^x \alpha^{2m+1} \cos(2\alpha - x)\, d\alpha = \\ &= -\frac{x^{2m+2}}{2(m+1)} \cos x + (m+1)!\, 2^{2m+2} \sqrt{\pi} \sum_{n=0}^{m+1} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n+1}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} I_{m+n+\frac{1}{2}}(x),\end{aligned}\right\} \quad (42)$$

$$\left.\begin{aligned}\int_0^x \alpha^{2m+1} \sin 2\alpha \, d\alpha = (m+1)!\, 2^{2m+2} \sqrt{\pi} \sum_{n=0}^{m+1} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n+1}}{n!} \,.\\ \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} \{\sin x\, I_{m+n+\frac{1}{2}}(x) - \cos x\, I_{m+n+\frac{3}{2}}(x)\},\end{aligned}\right\} \quad (43)$$

$$\left.\begin{aligned}\int_0^x \alpha^{2m+1} \cos 2\alpha \, d\alpha = -\frac{x^{2m+2}}{2(m+1)} + (m+1)!\, 2^{2m+2} \sqrt{\pi} \sum_{n=0}^{m+1} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n+1}}{n!} \,.\\ \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} \{\cos x\, I_{m+n+\frac{1}{2}}(x) + \sin x\, I_{m+n+\frac{3}{2}}(x)\},\end{aligned}\right\} \quad (44)$$

$$\left.\begin{aligned}\int_0^x \alpha^{2m+1} \sin 2(x-\alpha)\, d\alpha = \frac{x^{2m+2}}{2(m+1)} \sin 2x - (m+1)!\, 2^{2m+2} \sqrt{\pi} \sum_{n=0}^{m+1} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n+1}}{n!} \,.\\ \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} \{\sin x\, I_{m+n+\frac{1}{2}}(x) + \cos x\, I_{m+n+\frac{3}{2}}(x)\},\end{aligned}\right\} \quad (45)$$

$$\left.\begin{aligned}\int_0^x \alpha^{2m+1} \cos 2(x-\alpha)\, d\alpha = -\frac{x^{2m+2}}{2(m+1)} \cos 2x + (m+1)!\, 2^{2m+2} \sqrt{\pi} \sum_{n=0}^{m+1} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n+1}}{n!} \,.\\ \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} \{\cos x\, I_{m+n+\frac{1}{2}}(x) - \sin x\, I_{m+n+\frac{3}{2}}(x)\}.\end{aligned}\right\} \quad (46)$$

Substitutie van $\varrho = m+1$ resp. $\varrho = m$ in (24) geeft voor $R(\nu) > -\frac{1}{2}$ en $m \geqq 0$:

$$\left.\begin{aligned}\int_0^x I_{\nu-\frac{1}{2}}(x-\alpha)(x-\alpha)^{\nu+\frac{1}{2}} \alpha^{2m+1} \sin\alpha \, d\alpha = m!\,\Gamma(\nu+1)\, 2^{\nu+2m+\frac{1}{2}} \sum_{n=0}^{m} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n}}{n!} \,.\\ \cdot \frac{(m+n+1)!}{\Gamma(\nu+m+n+3)} \left(\frac{x}{2}\right)^{\nu+m+n+\frac{3}{2}} \{x\, I_{\nu+m+n+\frac{1}{2}}(x) + I_{\nu+m+n+\frac{3}{2}}(x)\},\end{aligned}\right\} \quad (47)$$

$$\left.\begin{aligned}\int_0^x I_{\nu-\frac{1}{2}}(x-\alpha)(x-\alpha)^{\nu+\frac{1}{2}} \alpha^{2m} \cos\alpha \, d\alpha = m!\,\Gamma(\nu+1)\, 2^{\nu+2m-\frac{1}{2}} \sum_{n=0}^{m} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n}}{n!} \,.\\ \cdot \frac{(m+n)!}{\Gamma(\nu+m+n+2)} \left(\frac{x}{2}\right)^{\nu+m+n+\frac{1}{2}} \{x I_{\nu+m+n-\frac{1}{2}}(x) + I_{\nu+m+n+\frac{1}{2}}(x)\},\end{aligned}\right\} \quad (48)$$

terwijl dezelfde substituties in (25) en (26) voeren tot de integralen, voorkomende in (41), (42) en (33), (34), wel is waar met uitkomsten van eenigszins anderen vorm. Door echter die verschillende uitkomsten voor dezelfde integralen aan elkaar gelijk te stellen, komen na eenige herleiding betrekkingen voor den dag, die uit V blijken te volgen voor $\varrho = m + \frac{1}{2}$ en $\varrho = m - \frac{1}{2}$.