Mathematics. — *Conformal differential geometry. III. Surfaces in three-dimensional space.* By J. HAANTJES. (Communicated by Prof. W. VAN DER WOUDE.)

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Introduction.

In dealing with conformal differential geometry of surfaces in conformal euclidean spaces various methods can be used. In most of the textbooks about this subject the theory is based upon the isomorphism between the three-dimensional conformal group and a subgroup of the projective group in four dimensions [1]). The coordinates employed are the so called pentaspherical coordinates and a surface is considered as an envelope of a two-parameter family of spheres.

The purpose of this paper is to develop the differential geometry by another method based upon the theorem [2]) that the conformal invariant properties in a flat space are those properties, which are unaffected by a conformal transformation of the fundamental tensor

$$g'_{hi} = \sigma^2 g_{hi}, \qquad \qquad (1)$$

σ satisfying the equation

$$\partial_j s_i - s_j s_i + \tfrac{1}{2} g_{ji} s_h s^h = 0, \qquad (s_i = \partial_i \log \sigma). \qquad (2)$$

In using this theorem we avoid the introduction of pentaspherical coordinates. Moreover it appears to be unnecessary to look upon a surface as an envelope of a system of ∞^2 spheres.

§ 1. *The conformal invariant fundamental tensors.*

Let x^h $(h, i, j, \ldots = 1, 2, 3)$ be rectangular cartesian coordinates in a three-dimensional flat space R_3, in which the fundamental tensor is denoted by g_{hi}. The equations of a surface S may be written

$$x^h = x^h(u^\alpha); \qquad (\alpha, \beta, \ldots = 1, 2). \qquad (3)$$

By means of the unit vector n^h, normal to the surface, and the quantity $B^h_\alpha = \partial_\alpha x^h$ the ordinary first and second fundamental tensor are defined as follows:

$$a_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta = \sum_i B^i_\alpha B^i_\beta, \qquad (4)$$

$$h_{\alpha\beta} = n_h \partial_\alpha B^h_\beta = - B^h_\beta \partial_\alpha n_h. \qquad (5)$$

These tensors are not conformal invariant. From (1) and (4) it follows immediately that

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[1]) Comp. W. BLASCHKE, Vorlesungen über Differentialgeometrie III, Springer, Berlin, 1929.

T. TAKASU, Differentialgeometrien in den Kugelräumen I, Tokyo, 1938.

[2]) Comp. f. i. J. HAANTJES, Conformal differential geometry I, II, Proc. Ned. Akad. v. Wetensch., Amsterdam, **44**, 814—824 (1941); **45**, 249—255 (1942), referred to as C.D.G. I, II.

the transformation of the first fundamental tensor under conformal transformations (1) is given by

$$a'_{\alpha\beta} = \sigma^2 a_{\alpha\beta}. \qquad (6)$$

In order to find the transformation of $h_{\alpha\beta}$ we have to take into consideration that the coordinate system (h) is not a cartesian one with respect to the fundamental tensor g'_{hi}. But if in (5) the ordinary derivative is replaced by the covariant derivative belonging to the fundamental tensor considered the equation will hold for any coordinate system. The parameters of the covariant derivative belonging to g'_{hi} are given by the CHRISTOFFEL symbols constructed with this tensor:

$$\left\{ {h \atop j\,i} \right\}' = A_i^h s_j + A_j^h s_i - g_{ij} s^h, \qquad (7)$$

where the unit affinor A_i^h and s_i are defined by

$$A_i^h = 0 \ (h \neq i); \ A_i^h = 1 \ (h = i); \ s_i = \partial_i \log \sigma. \qquad (8)$$

Thus if g'_{hi} is considered as fundamental tensor in R_3, the second fundamental tensor of the surface is given by

$$\left. \begin{aligned} h'_{\alpha\beta} &= -B_\beta^h \delta'_\alpha n'_h = -B_\beta^h \delta'_\alpha \sigma n_h = -\sigma B_\beta^h \left(\partial_\alpha n_h - B_\alpha^j \left\{ {i \atop j\,h} \right\}' n_i \right) \\ &= \sigma (h_{\alpha\beta} - a_{\alpha\beta} s); \qquad s = s_i n^i. \end{aligned} \right\} \qquad (9)$$

From (6) and (9) it follows that the normal curvature $\varkappa$ of a given direction transforms under conformal transformations as follows

$$\varkappa' = \sigma^{-1} (\varkappa - s). \qquad (10)$$

But since the principal directions at a point are conformal invariant as a consequence of (6) and (9) the transformation of the principal curvatures too is given by (10). Hence the transformations of the mean curvature l and the scalar ϱ, defined by

$$l = \tfrac{1}{2} (\varkappa_1 + \varkappa_2); \quad \varrho = \tfrac{1}{2} | \varkappa_1 - \varkappa_2 | \qquad (11)$$

are given by

$$l' = \sigma^{-1} (l - s); \quad \varrho' = \sigma^{-1} \varrho.\text{[3]} \qquad (12)$$

The quantity ϱ vanishes if $\varkappa_1$ and $\varkappa_2$ are equal, which happens only at an umbilical point. In the following umbilical points will be excluded from our considerations. As a consequence of this we have to exclude the case that the surface S is a sphere.

The scalar ϱ enables us to define on the surface a conformal invariant fundamental tensor G_{hi} together with its induced tensor $A_{\alpha\beta}$

$$G_{hi} = \varrho^2 g_{hi}; \ A_{\alpha\beta} = G_{hi} B_\alpha^h B_\beta^i = \varrho^2 a_{\alpha\beta}. \qquad (13)$$

The tensor $A_{\alpha\beta}$ is called the *first conformal invariant fundamental tensor* of S.

We shall adopt the convention that a conformal invariant quantity will be denoted by a *capital letter*. The raising and lowering of a suffix of a conformal invariant quantity will be done by means of G_{hi}, $A_{\alpha\beta}$ and the inverse tensors G^{hi} and $A^{\alpha\beta}$. For other

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[3]) In the following σ is supposed to be positive.

quantities the raising or lowering of a suffix proceeds by means of g_{hi} and $a_{\alpha\beta}$ [4]).

The tensors G_{hi} and $A_{\alpha\beta}$ may be used to introduce several conformal invariant notions. We have for example

a) *the conformal invariant "length" of a curve* $u^\alpha = u^\alpha(t)$ upon the surface, defined by

$$\tau(t) = \int_{t_0}^{t} \sqrt{A_{\alpha\beta} \frac{du^\alpha}{dt} \frac{du^\beta}{dt}}\, dt; \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

b) *the conformal invariant normal vector* N^h, unit vector with respect to G_{hi} (compare footnote [4])).

c) *the conformal invariant derivative* of affinors of the surface. This covariant derivative is defined by the equation

$$\nabla_\gamma A_{\alpha\beta} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Its parameters $\Gamma^\alpha_{\gamma\beta}$ are therefore the CHRISTOFFEL symbols constructed with the tensor $A_{\alpha\beta}$. From (13) it follows that

$$\Gamma^\alpha_{\gamma\beta} = \left\{ {\alpha \atop \gamma\beta} \right\} + q_\gamma A^\alpha_\beta + q_\beta A^\alpha_\gamma - a_{\beta\gamma} q^\alpha; \qquad (q_\beta = \partial_\beta \log \sigma), \quad . \quad (16)$$

where $\left\{ {\alpha \atop \gamma\beta} \right\}$ are the CHRISTOFFEL symbols belonging to $a_{\alpha\beta}$.

Because of (15) the process of raising and lowering of suffixes by means of $A_{\alpha\beta}$ is commutative with the process of covariant differentiation.

The covariant derivative in R_3. A conformal invariant derivative of affinors in R_3 can be defined if we have at our disposal a quantity q_i with the transformation

$$q'_i = q_i - s_i. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Then the parameters of this covariant derivative are given by

$$\Gamma^h_{ji} = \left\{ {h \atop ji} \right\} + q_j A^h_i + q_i A^h_j - g_{ij} q^h, \quad . \quad . \quad . \quad . \quad (18)$$

where $\left\{ {h \atop ji} \right\}$ are the CHRISTOFFEL symbols constructed with g_{hi}. These CHRISTOFFEL symbols vanish with respect to the system (h), but in writting $\left\{ {h \atop ji} \right\}$ on the right hand side of (18), the equation (18) will hold for any other coordinate system and in this form the conformal invariance of the parameters Γ^h_{ji} is at once clear.

From (12) it follows that the q_i, defined by

$$q_i = l\, n_i + q_\alpha B^\alpha_i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

transforms in the right way. It should be remarked that q_i is a function of u^α, which

4) Is the unit normal vector with respect to G_{hi} denoted by N^h we have f.i.

$$N_i = G_{hi} N^h; \; n_i = g_{ih} n^h; \; N_i = \varrho\, n_i; \; N^h = \varrho^{-1} n^h.$$

Other conformal invariant quantities are B^h_α and $B^\alpha_i = a^{\alpha\beta} B^h_\beta g_{hi} = A^{\alpha\beta} B^h_\beta G_{hi}$.

means that Γ^h_{ji} is only defined on the surface S. In the following we use the covariant derivative defined by (18) and (19). It will be denoted by the symbol ∇.

The covariant derivative of G_{hi} along the surface appears to be

$$B^j_\alpha \nabla_j G_{hi} \equiv \partial_\alpha G_{hi} - B^j_\alpha \Gamma^l_{jh} G_{li} - B^j_\alpha \Gamma^l_{ji} G_{hl} = 0 \quad . \quad . \quad (20)$$

as follows at once from (18). Hence it is immaterial whether a suffix is raised or lowered by means of G_{hi} before of after the covariant differentiation.

The covariant derivative in R_3 induces in S a covariant derivative with the parameters [5])

$$\Gamma^h_{ji} B^j_\gamma B^i_\beta B^\alpha_h + B^\alpha_h \partial_\gamma B^h_\beta . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

These parameters are however identical with $\Gamma^\alpha_{\gamma\beta}$ as may be seen from (16) and (18). This is the reason why we use the same symbol ∇ for both the covariant derivatives in R_3 and on S.

The conformal geodesics. A curve, for which

$$\int d\tau = \int \sqrt{A_{\alpha\beta}\, du^\alpha\, du^\beta} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

is stationary, is called a conformal geodesic. The differential equations for these curves are

$$\frac{d^2 u^\alpha}{d\tau^2} + \Gamma^\alpha_{\gamma\beta} \frac{du^\gamma}{d\tau} \frac{du^\beta}{d\tau} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

In a following paper we shall define the geodesics geometrically.

The second conformal invariant fundamental tensor. This tensor $H_{\alpha\beta}$ is defined by the following equation

$$H_{\alpha\beta} = N_h \nabla_\alpha B^h_\beta = - B^h_\beta \nabla_\alpha N_h = - B^h_\beta B^i_\alpha \nabla_i N_h . \quad . \quad . \quad (24)$$

In comparing this definition with that of $h_{\beta\alpha}$ (formel (5)) it may be proved, that $H_{\alpha\beta}$ is equal to

$$H_{\alpha\beta} = \varrho\, (h_{\alpha\beta} - l\, a_{\alpha\beta}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

From equation (24) we obtain the following:

$$\nabla_\beta B^h_\alpha = H_{\beta\alpha} N^h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$\nabla_\alpha N^h = - H_\alpha^{\cdot\beta} B^h_\beta . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

In consequence of (25) the tensor $H_{\alpha\beta}$ satisfies a few algebraic equations. We have

$$\left.\begin{aligned} &a) \quad H_\alpha^{\cdot\alpha} = H_{\alpha\beta} A^{\alpha\beta} = 0 \\ &b) \quad Det\,(H_{\alpha\beta}) = - Det\,(A_{\alpha\beta}) \equiv - \mathfrak{A} \\ &c) \quad H_{\alpha\gamma} H^\gamma_{\cdot\beta} = A_{\alpha\beta}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (28)$$

Since the determinant of $H_{\alpha\beta}$ is negative the two directions defined by

$$H_{\alpha\beta}\, du^\alpha\, du^\beta = 0 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

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5) Comp. J. A. SCHOUTEN and D. J. STRUIK, Einführung in die neueren Methoden der Differentialgeometrie I, Noordhoff, Groningen 1935, p. 93.

are distinct and real for real surfaces. It is seen from (28*a*) that these directions are orthogonal.

The lines of curvature. The principal directions at a point are given by the well-known equation

$$h_{\alpha[1}\, a_{2]\beta}\, du^{\alpha}\, du^{\beta} = 0. \quad \ldots \ldots \quad (30)$$

Because of (13) and (25) this equation may be written in the form

$$H_{\alpha[1}\, A_{2]\beta}\, du^{\alpha}\, du^{\beta} = 0, \quad \ldots \ldots \quad (31)$$

from which the conformal invariant character of the principal directions and therefore of the lines of curvature is evident. The equation for the principal directions may be put in still another form, using the bivector $I^{\alpha\beta}$ defined by

$$I^{12} = -I^{21} = \mathfrak{A}^{-\frac{1}{2}}. \quad \ldots \ldots \quad (32)$$

Equation (31) is namely identical with

$$C_{\alpha\beta}\, du^{\alpha}\, du^{\beta} = 0 \quad \ldots \ldots \quad (33)$$

where $C_{\alpha\beta}$ is defined by

$$C_{\alpha\beta} = H_{\alpha\gamma}\, I^{\gamma\delta}\, A_{\delta\beta} = H_{\alpha}^{\cdot\gamma}\, I_{\gamma\beta}. \quad \ldots \ldots \quad (34)$$

Identities. The components of the quantities $A_{\alpha\beta}$, $H_{\alpha\beta}$, $C_{\alpha\beta}$ and $I_{\alpha\beta}$ satisfy a few algebraic equations, which follow immediately from the definitions of the quantities involved. We give here the equations without proof.

$$\left.\begin{array}{llll} a) & C_{\alpha\beta} = C_{\beta\alpha} & d) & H_{\alpha\gamma}\, C^{\gamma}_{\cdot\beta} = I_{\alpha\beta} \\ b) & C_{\alpha\beta}\, A^{\alpha\beta} = 0 & e) & H_{\alpha\beta}\, C^{\alpha\beta} = 0 \\ c) & Det\,(C_{\alpha\beta}) = -\mathfrak{A} & f) & C_{\alpha\gamma}\, C^{\gamma}_{\cdot\beta} = A_{\alpha\beta}. \end{array}\right\} \quad \ldots \quad (35)$$

The equation (*b*) expresses that the lines of curvature form an orthogonal system.

If the unit vectors (with respect to $A_{\alpha\beta}$) in the principal directions are denoted by P^{α} and Q^{α} respectively, the quantities $A_{\alpha\beta}$, $H_{\alpha\beta}$, $C_{\alpha\beta}$ and $I_{\alpha\beta}$ may be expressed in terms of P^{α} and Q^{α} We have as a consequence of (32) and (41)

$$\left.\begin{array}{ll} A_{\alpha\beta} = P_{\alpha}\, P_{\beta} + Q_{\alpha}\, Q_{\beta}; & C_{\alpha\beta} = P_{\alpha}\, Q_{\beta} + Q_{\alpha}\, P_{\beta} \\ H_{\alpha\beta} = P_{\alpha}\, P_{\beta} - Q_{\alpha}\, Q_{\beta}; & I_{\alpha\beta} = P_{\alpha}\, Q_{\beta} - Q_{\alpha}\, P_{\beta} \end{array}\right\} \quad \ldots \quad (36)$$

As the quantities (36) are linear independent every affinor of order 2 can be expressed as a linear form in $A_{\alpha\beta}$, $H_{\alpha\beta}$, $C_{\alpha\beta}$ and $I_{\alpha\beta}$.

§ 2. *Geometrical interpretations of $A_{\alpha\beta}$ and $H_{\alpha\beta}$.*

The null-directions of $H_{\alpha\beta}$. From the equations (28*c*) and (35*e*) it follows that

$$\overset{-1}{H}{}^{\alpha\beta}\, C_{\alpha\beta} = H^{\alpha\beta}\, C_{\alpha\beta} = 0, \quad \ldots \ldots \quad (37)$$

which means that the null-directions of $H_{\alpha\beta}$ defined by (35) are conjugate with respect to the principal directions. But we know already that they are orthogonal. So the null-directions of $H_{\alpha\beta}$ are bisecting the angle formed by the lines of curvature [6]).

[6]) From this property it follows that $H_{\alpha\beta}$ is proportional to the tensor c_{ij} used by BLASCHKE, l.c. p. 313.

Another geometrical property of the null-directions of $H_{\alpha\beta}$ at the point P is obtained by considering the spheres, which are at P tangent to the surface. The curve of intersection of such a sphere and the surface has a double point at P. It can be shown that the directions of the tangents at P to this curve are given by

$$(h_{\alpha\beta} - \varkappa\, a_{\alpha\beta})\, du^\alpha\, du^\beta = 0, \qquad (38)$$

where $\varkappa$ is the curvature of the sphere considered. The directions (38) are perpendicular if the curvature $\varkappa$ is equal to the mean curvature of the surface. The sphere, which has this property, is called the *central sphere*. From (25) and (38) it is seen that the directions at P of the curve of intersection of the central sphere and the surface are identical with the null-directions of $H_{\alpha\beta}$ [7]).

It is easily seen that the directions determined by equation (38) coincide (principal directions), if $\varkappa$ is equal to one of the principal curvatures of the surface. The corresponding spheres are called the *curvature spheres*. To each principal direction belongs one curvature sphere.

The quadratic form $A_{\alpha\beta}\, du^\alpha\, du^\beta$. Consider the angle between the central spheres at the points u^α and $u^\alpha + du^\alpha$. If the coordinates of the center of the central sphere are denoted by y^h, we get the following expression for the angle $d\varphi$

$$(d\varphi)^2 = l^2 \Sigma \left(\frac{\partial y^h}{\partial u^\alpha} \frac{\partial y^h}{\partial u^\beta} - \frac{\partial l^{-1}}{\partial u^\alpha} \frac{\partial l^{-1}}{\partial u^\beta} \right) du^\alpha\, du^\beta. \qquad (39)$$

which leads to

$$(d\varphi)^2 = H_{\alpha\delta}\, H^{\delta}_{.\beta}\, du^\alpha\, du^\beta = A_{\alpha\beta}\, du^\alpha\, du^\beta. \qquad (40)$$

This formel gives a geometrical meaning of the quadratic form $A_{\alpha\beta}\, du^\alpha\, du^\beta$ [8]).

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[7]) BLASCHKE, l.c. p. 313.

[8]) It may be seen from this result that $A_{\alpha\beta}$ is identical with the tensor g_{ij} used by BLASCHKE, l.c. p. 312.