Mathematics. — *On the extension of continuous functions.* By J. DE GROOT. (Communicated by Prof. J. G. VAN DER CORPUT.)

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Well-known and important is the following theorem [1]: If R is a normal space and F is a (bounded) continuous real function, defined in all points of a closed subset $A \subset R$, then it is possible to find a (bounded) continuous function F', defined in entire R, which is identical with F in the points of A.

This result may shortly be worded as follows: Any F, defined on a closed set $A \subset R$, may continuously be extended on entire R. The closedness of A is therefore in any case a sufficient condition for the possibility of continuous extension.

As far as I know, it has up till now not yet been investigated, how far this condition is also necessary. Still, this question may in the main features be solved in a simple and elementary way.

We shall prove, that for the special case of metric spaces the closednes of A is also a necessary condition for the possibility of continuous extension of any F; that, on the other hand, this condition is not necessary for normal spaces.

So we come to the following

Theorem. *Let A be any subset of a metric space M. Then and only then any (bounded) continuous real function F, defined on A, may be extended to a (bounded) continuous real function F', defined on entire M, if A is a (in M) closed set.*

Contention. *It is possible to construct a normal space N and a not-closed subset A of N, so that any bounded continuous real function F may be continuously extended to a continuous function F', defined on entire N.*

Problem. The question, if the condition, which we just mentioned, is or is not necessary in completely normal (or in other special, but not-metric normal) spaces, has not yet been solved.

Proof of the theorem [2]).

The condition is sufficient: known.

The condition is necessary. Let m be a clusterpoint of A in M, not belonging to A. Now we consider on the set $M{-}m$ a continuous real function F by setting for every point $p \subset M{-}m$.

$$F(p) = f(\varrho_p)$$

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[1]) Compare P. ALEXANDROFF, H. HOPF, Topologie I, Berlin (1935), p. 73—78 (concerning the terminology we shall follow this book). H. HAHN, Reelle Funktionen, Leipzig (1932), p. 255 (here we only find a proof of the theorem for metric spaces). C. CARATHÉODORY, Reelle Funktionen I, Leipzig and Berlin (1939), p. 155 (a proof of the theorem only for an n-dimensional space R_n). For the case of an R_n this theorem has already been proved in the first edition of the last mentioned book (1918), p. 617 (although by means of single integrals).

[2]) A good deal of the definitive wording of this proof I owe to a communication of Dr. J. RIDDER.

whereby ϱ_p denotes the distance from p to m, while $f(\varrho)$ is a, for a positive ϱ, continuous (bounded) function, for which

$$\lim_{\varrho_a \to 0} f(\varrho_a)$$

has no (finite) limit (the ϱ_a are the distances from the points a of A to m). Such a continuous function F, defined i.a. on A, may obviously not be extended to a continuous function, which is defined on entire M.

Now there yet remains the proof of the existence of such functions F. This may, however, very simply be proved in many different ways. If we don't require the boundedness of F, then for example $F(p) = \frac{1}{\varrho_p}$ satisfies our conditions. Suppose secondly F is bounded. It is always possible to find a sequence of points $\{a_i\}$ of A, converging to m, such that $\varrho_{a_k} < \varrho_{a_l}$ for $k > l$. We then define for instance a bounded continuous function $f(\varrho_p)$, which is exactly 1 for $\varrho_{a_1}, \varrho_{a_3}, \varrho_{a_5}$ etc., and exactly 0 for $\varrho_{a_2}, \varrho_{a_4}, \varrho_{a_6}$ etc. $F(p) = f(\varrho_p)$ then gives the F we asked for.

Proof of the contention. We might construct a space N and a subset A, which satisfies the contention. We attain our end more quickly however by using the following well-known theorem [1]):

Given a completely regular space [2]) R, there exists a bicompact HAUSDORFF space $\beta(R)$, such that: 1° R is dense in $\beta(R)$, 2° any bounded continuous function φ, defined on R, may be extended to a continuous function f, defined on $\beta(R)$.

Because every bicompact HAUSDORFF space is a normal space, we attain our end by setting $N = \beta(R)$ and $A = R$.

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[1]) Compare A. TYCHONOFF, Math. Ann. **102**, 554 a.f. (1930) and E. ČECH, Ann. of Math. II, **38**, 823 a.f. (1937).

[2]) A regular space is called completely regular, if to every point a and to every closed subset A, which does not contain a, there exists a continuous function $f(x)$, defined in the whole space, such that: $f(a) = 0$ and $f(A) = 1$.