Mathematics. — *On the theory of linear integral equations.* IV. By A. C. ZAANEN. (Communicated by Prof. W. VAN DER WOUDE.)

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§ 1. *Introduction.*

We suppose the reader to be acquainted with the contents of the first two papers bearing the same title, to which we shall refer with I and II [1]). In this paper we shall consider linear integral equations with kernel $K(x, y) \epsilon L_2^{(2m)}(\triangle)$, having the property that there exists a positive Hermitean kernel $H(x, y) \epsilon L_2^{(2m)}(\triangle)$, such that

$$P(x, y) = \int_{\triangle} H(x, z)\, K(z, y)\, dz$$

(then also belonging to $L_2^{(2m)}(\triangle)$) is Hermitean. Defining the completely continuous, linear transformations K, H and $P = HK$ in the space $L_2^{(m)}(\triangle)$ by

$$Kf = \int_{\triangle} K(x, y)\, f(y)\, dy,$$

$$Hf = \int_{\triangle} H(x, y)\, f(y)\, dy,$$

$$Pf = \int_{\triangle} P(x, y)\, f(y)\, dy,$$

H is therefore positive self-adjoint and $P = HK$ is self-adjoint, so that K is symmetrisable relative to H. Making now the additional assumption that every $f(x) \epsilon L_2$, satisfying $Hf = 0$, satisfies also $Kf = 0$, we shall call the kernel $K(x, y)$ a *Marty-kernel.* We observe that this last condition is certainly satisfied if $H(x, y)$ is *definite*, that is, if

$$(Hf, f) = \int_{\triangle \times \triangle} H(x, y)\, \overline{f(x)}\, f(y)\, dx\, dy = 0$$

implies $f(x) = 0$, since then $Hf = 0$ implies $f = 0$, so that also $Kf = 0$. It was this case that was considered for the first time by J. MARTY [2]) for bounded kernels.

To terminate this paragraph we mention that the condition of H-orthogonality for two functions $f(x)$ and $g(x)$, belonging to L_2, takes the form

$$(Hf, g) = \int_{\triangle \times \triangle} H(x, y)\, \overline{g(x)}\, f(y)\, dx\, dy = 0,$$

[1]) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49** (1946).

[2]) J. MARTY, Valeurs singulières d'une équation de FREDHOLM, C. R. Acad. sc. Paris **150**, 1499—1502 (1910).

and that $f(x)$ is H-normal if

$$(Hf, f) = \int_{\Delta\times\Delta} H(x, y)\, \overline{f(x)}\, f(y)\, dx\, dy = 1.$$

§ 2. *Integral equation with Marty-kernel.*

We consider now the linear integral equation $Kf - \lambda f = g$ or

$$\int_{\Delta} K(x, y)\, f(y)\, dy - \lambda f(x) = g(x), \quad \ldots\ldots \quad (1)$$

where $f(x)$, $g(x)$ belong to L_2, and $K(x, y)$ is a Marty-kernel. Supposing that

$$\| H(x, y) \|_{2m}^2 = \int_{\Delta\times\Delta} | H(x, y) |^2\, dx\, dy \neq 0,$$

so that $H \neq O$, all theorems proved in I may therefore be applied to this equation.

Theorem 1. *The characteristic values of (1) are real and characteristic functions belonging to different characteristic values are H-orthogonal.*

Proof. Follows from I, Theorem 5.

Theorem 2. *If $\lambda \neq 0$ is a characteristic value of (1), this equation has, for a given function $g(x) \in L_2$, a solution $f(x) \in L_2$ for those and only those functions $g(x)$ that are H-orthogonal to all characteristic functions, belonging to the characteristic value λ. If $\lambda \neq 0$ is no characteristic value, so that it is a regular value, the equation has a uniquely determined solution for every $g(x) \in L_2$.*

Proof. Follows from I, Theorem 12 and Theorem 3.

Theorem 3. *If*

$$\| P(x, y) \|_{2m}^2 = \int_{\Delta\times\Delta} | P(x, y) |^2\, dx\, dy \neq 0, \quad \ldots\ldots \quad (2)$$

where

$$P(x, y) = \int_{\Delta} H(x, z)\, K(z, y)\, dz,$$

the equation (1) has a characteristic value $\neq 0$.

Proof. Since, by (2), $P = HK \neq O$, the result follows from I, Theorem 6.

Let now $\lambda_i (| \lambda_1 | \geqq | \lambda_2 | \geqq \ldots)$ be the sequence of all characteristic values $\neq 0$, each of them occurring as many times as denoted by its multiplicity, and $\psi_i(x)$ a corresponding H-orthonormal sequence of characteristic functions. These functions satisfy therefore the relations

$$\int_{\Delta\times\Delta} H(x,y)\,\overline{\psi_i(x)}\,\psi_i(y)\,dx\,dy = 1,$$

$$\int_{\Delta\times\Delta} H(x,y)\,\overline{\psi_i(x)}\,\psi_j(y)\,dx\,dy = 0 \text{ for } i \neq j,$$

or, writing

$$\int_{\Delta} H(x,y)\,\psi_j(y)\,dy = \chi_j(x),$$

$$\int_{\Delta} \psi_i(x)\,\overline{\chi_j(x)}\,dx = \begin{cases} 1 \text{ for } i = j, \\ 0 \text{ for } i \neq j, \end{cases}$$

so that the sequences $\psi_i(x)$ and $\chi_i(x)$ are biorthogonal.

Theorem 4. (*Expansion Theorem*). *Writing*

$$a_i = (f, \chi_i) = \int_{\Delta} f(x)\,\overline{\chi_i(x)}\,dx,$$

and introducing the notation

$$N(f) = (Hf, f)^{1/2} = \left(\int_{\Delta\times\Delta} H(x,y)\,\overline{f(x)}\,f(y)\,dx\,dy\right)^{1/2},$$

we have

$$\lim_{k\to\infty} N\left(Kf - \sum_{i=1}^{k} \lambda_i\,a_i\,\psi_i\right) = 0;$$

in other words

$$\lim_{k\to\infty} \int_{\Delta\times\Delta} H(x,y) \left[\int_{\Delta} K(x,z)\,f(z)\,dz - \sum_{i=1}^{k} \lambda_i\,a_i\,\psi_i(x)\right] \overline{\left[\int_{\Delta} K(y,z)\,f(z)\,dz - \sum_{i=1}^{k} \lambda_i\,a_i\,\psi_i(y)\right]}\,dx\,dy = 0$$

for any $f(x) \in L_2$. Furthermore

$$\int_{\Delta\times\Delta} P(x,y)\,\overline{f(x)}\,f(y)\,dx\,dy = \Sigma\,\lambda_i\,|a_i|^2.$$

Proof. Follows from I, Theorem 9.

Theorem 5. *Let λ_{n_i} $(i = 1, 2, \ldots)$ be the subsequence of all positive characteristic values where $\lambda_{n_1} \geq \lambda_{n_2} \geq \ldots$, let the functions $p_1(x), \ldots, p_{i-1}(x) \in L_2$ be arbitrarily given, and let*

$$\mu_i = \text{upper bound} \int_{\Delta\times\Delta} P(x,y)\,\overline{f(x)}\,f(y)\,dx\,dy \Big/ \int_{\Delta\times\Delta} H(x,y)\,\overline{f(x)}\,f(y)\,dx\,dy$$

for all $f(x) \in L_2$ *satisfying* $\int\limits_{\Delta\times\Delta} H(x,y)\overline{f(x)}f(y)dx\,dy \neq 0$ *and*

$$\int\limits_{\Delta\times\Delta} H(x,y)\,\overline{p_1(x)}\,f(y)\,dx\,dy = \ldots = \int\limits_{\Delta\times\Delta} H(x,y)\,\overline{p_{i-1}(x)}\,f(y)\,dx\,dy = 0.$$

The number μ_i *depends on* $p_1(x), \ldots, p_{i-1}(x)$. *Letting now these functions run through the whole space* L_2, *we have* $\lambda_{n_i} = \min \mu_i$.

A similar statement holds for the negative characteristic values.

Proof. Follows from I, Theorem 11.

Theorem 6. *Let* $\lambda \neq 0$, *and let* $g(x) \in L_2$ *be H-orthogonal to all characteristic functions of* (1), *belonging to the characteristic value* λ (*If* λ *is no characteristic value,* $g(x)$ *is therefore arbitrary*). *Then every solution* $f(x)$ *of* (1) *satisfies a relation of the form*

$$\lim_{k\to\infty} N\left(f + \frac{g}{\lambda} + \sum_{i=1}^{k}{}' \frac{\lambda_i}{\lambda(\lambda-\lambda_i)} a_i \psi_i\right) = 0$$

or

$$\lim_{k\to\infty} \int\limits_{\Delta\times\Delta} H(x,y)\, \overline{\left[f(x) + \frac{g(x)}{\lambda} + \sum_{i=1}^{k}{}' \frac{\lambda_i}{\lambda(\lambda-\lambda_i)} a_i \psi_i(x)\right]}$$

$$\left[f(y) + \frac{g(y)}{\lambda} + \sum_{i=1}^{k}{}' \frac{\lambda_i}{\lambda(\lambda-\lambda_i)} a_i \psi_i(y)\right] dx\,dy = 0,$$

where $a_i = \int\limits_{\Delta} g(x)\overline{\chi_i(x)}dx$ *for* $\lambda_i \neq \lambda$, *and where* Σ' *denotes that for those values of* i *for which* $\lambda_i = \lambda$ *the coefficient of* ψ_i *has the value* $-\int\limits_{\Delta} f(x)\overline{\chi_i(x)}dx$. *For every set of arbitrarily prescribed values of the latter coefficients there exists a solution of* (1).

Proof. Follows from I, Theorem 13.

Theorem 7. (*Expansion Theorem*).

$$\lim_{k\to\infty} \int\limits_{\Delta\times\Delta\times\Delta} H(x,y)\, \overline{\left[K(x,z) - \sum_{i=1}^{k} \lambda_i \psi_i(x)\, \overline{\chi_i(z)}\right]}$$

$$\left[K(y,z) - \sum_{i=1}^{k} \lambda_i \psi_i(y)\, \overline{\chi_i(z)}\right] dx\,dy\,dz = 0.$$

Proof. This theorem is the analogue of II, Theorem 8, (4), and the proof, though more complicated, is also analogous. We begin by introducing a HILBERT space Z, the elements $[f]$ of which are classes of elements of L_2. We define the element $[f]$ to contain f and all elements g for which $Hg = Hf$ (equivalent with $N(f-g) = 0$). Furthermore we define $[f] + [g] = [f + g]$, $a[f] = [af]$ for every complex a, and

$([f], [g]) = (Hf, g)$, so that $\|[f]\| = (Hf, f)^{1/2} = N(f)$. It is not difficult to prove that these definitions are without contradiction, and it is clear that, since $(Hf, g) = 0$ is equivalent with $([f], [g]) = 0$, two functions $f(x)$ and $g(x) \in L_2$ are H-orthogonal if and only if $[f]$ and $[g]$ are orthogonal in Z. In the same way, since $(Hf, f)^{1/2} = 1$ is equivalent with $\|[f]\| = 1$, the function $f(x)$ is H-normal if and only if the element $[f]$ is normal. The space Z is not necessarily complete, that is, $\lim \|[f]_m - [f]_n\| = 0$ as $m, n \to \infty$ will not always imply the existence of an element $[f]$ satisfying $\lim_{n \to \infty} \|[f] - [f]_n\| = 0$. By adjunction of ideal elements, however, we can obtain from Z the complete HILBERT space $\overline{Z}$, the closure of Z. Evidently every bounded, linear transformation in Z may be continued on $\overline{Z}$ as a linear transformation with the same bound. A certain class of linear transformations C in the space L_2 corresponds now with linear transformations in the space Z. We define $[C][f] = [Cf]$; this definition however is only then without contradiction if $[f] = [g]$ implies $[Cf] = [Cg]$, or, in other words, if $Hf = 0$ implies $HCf = 0$. We shall therefore consider only transformations C satisfying this condition. Amongst them is our symmetrisable transformation K, since $Hf = 0$ implies $Kf = 0$, hence certainly $HKf = 0$. Furthermore we observe that the linear transformation $[K]$ in the space Z is bounded and self-adjoint. The boundedness follows from I, Theorem 8, by which

$$|\lambda_1| = \max N(Kf)/N(f)$$

for all $f(x) \in L_2$ satisfying $N(f) \neq 0$, hence

$$|\lambda_1| = \max \|[Kf]\|/\|[f]\| = \max \|[K][f]\|/\|[f]\|$$

for all $[f] \in Z$ satisfying $[f] \neq [0]$ or

$$\|[K][f]\| \leq |\lambda_1| \cdot \|[f]\|;$$

the self-adjointness follows from

$$([K][f], [g]) = (HKf, g) = (f, HKg) = (Hf, Kg) = ([f], [K][g]).$$

The transformation $[K]$ has the characteristic values λ_i with the orthonormal characteristic elements $[\psi_i]$, since $K\psi_i = \lambda_i \psi_i$ implies $[K][\psi_i] = \lambda_i [\psi_i]$. Moreover, if the element $[g] \in Z$ is orthogonal to all $[\psi_i]$, we have $[K][g] = [0]$. To prove this statement, we observe that, if $([g], [\psi_i]) = (Hg, \psi_i) = (g, \chi_i) = 0$ $(i = 1, 2, \ldots)$, we find by Theorem 4 that $N(Kg) = 0$, hence also $HKg = 0$ or $[K][g] = [Kg] = [0]$. Since $[K]$ is bounded, it may be continued on the closure $\overline{Z}$, and it is not difficult to prove that for an element $[g] \in \overline{Z}$, orthogonal to all $[\psi_i]$, we have also $[K][g] = [0]$.

After these preliminaries we observe that the relation $K\psi_i = \lambda_i \psi_i$ implies, for every $f \in L_2$,

$$(K^* H\psi_i, f) = (H\psi_i, Kf) = (\psi_i, HKf) = (HK\psi_i, f) = (\lambda_i H\psi_i, f),$$

hence $K^*H\psi_i = \lambda_i H\psi_i$ or $K^*\chi_i = \lambda_i\chi_i$, or, since the adjoint transformation K^* is given by

$$K^* f = \int_\triangle \overline{K(y, x)}\, f(y)\, dy,$$

the relation

$$\int_\triangle \overline{K(x, y)}\, \chi_i(x)\, dx = \lambda_i\, \chi_i(y) \quad \ldots\ldots \quad (3)$$

for almost every $y \in \triangle$. Writing now $f(x) = K(x, z)$, the integral $\int_\triangle |f(x)|^2\, dx = \int_\triangle |K(x, z)|^2\, dx$ is finite for almost every $z \in \triangle$; hence $f(x) \in L_2$ for almost every $z \in \triangle$. With $f(x) \in L_2$ corresponds the element $[f]$ in the space Z, and we have, by (3),

$$([f], [\psi_i]) = (Hf, \psi_i) = (f, H\psi_i) = (f, \chi_i) = \int_\triangle K(x, z)\, \overline{\chi_i(x)}\, dx = \lambda_i\, \overline{\chi_i(z)}$$

for almost every $z \in \triangle$.

Considering now an element $[g] \in \overline{Z}$, orthogonal to all $[\psi_i]$, there exists a sequence of elements $[g_n] \in Z$ such that $\lim\, [g_n] = [g]$. Since $[K][g] = [0]$, we have

$$\lim\, [K]\, [g_n] = \lim\, [K g_n] = [0]$$

or

$$\lim\, \|\, [K g_n]\, \| = \lim N(K g_n) = \lim\, (HK g_n, K g_n)^{1/2} = 0,$$

so that certainly $\lim\, \|\, HKg_n\, \| = 0$. Writing

$$HK g_n = P g_n = \int_\triangle P(z, x)\, g_n(x)\, dx = p_n(z),$$

the relation $\lim \|HKg_n\| = 0$ is equivalent with $\lim \int_\triangle |p_n(z)|^2 dz = 0$; the sequence of functions $p_n(z)$ converges therefore in mean to 0. Then, as well-known, there exists a subsequence $n_k (k = 1, 2, \ldots)$ of indices such that $p_{n_k}(z)$ converges point-wise to 0 as $k \to \infty$ for almost every $z \in \triangle$. Consequently, since

$$([f], [g]) = \lim\, ([f], [g_n]) = \lim\, (Hf, g_n) =$$

$$= \lim \int_{\triangle \times \triangle} H(x, y)\, K(y, z)\, \overline{g_n(x)}\, dx\, dy = \lim \int_\triangle P(x, z)\, g_n(x)\, dx =$$

$$= \lim\, \overline{\int_\triangle P(z, x)\, \overline{g_n(x)}\, dx} = \lim\, \overline{p_n(z)}$$

for almost every $z \in \triangle$, and therefore certainly

$$([f], [g]) = \lim_{k \to \infty} \overline{p_{n_k}(z)}$$

for these values of z, we have $([f], [g]) = 0$ for almost every $z \in \triangle$.

From the relations $([f], [\psi_i]) = \lambda_i \overline{\chi_i(z)}$ and $([f], [g]) = 0$ for any $[g] \in \overline{Z}$ orthogonal to all $[\psi_i]$, both holding for almost every $z \in \triangle$, we deduce now that $[f] = \Sigma \lambda_i \overline{\chi_i(z)} [\psi_i]$ for almost every $z \in \triangle$, hence

$$\| [f] - \sum_{i=1}^{k} \lambda_i \overline{\chi_i(z)} [\psi_i] \|^2 = \sum_{i=k+1} \lambda_i^2 \, | \chi_i(z) |^2$$

or

$$N^2 \left(K(x, z) - \sum_{i=1}^{k} \lambda_i \overline{\chi_i(z)} \psi_i(x) \right) = \sum_{i=k+1} \lambda_i^2 \, | \chi_i(z) |^2$$

or

$$\int_{\triangle \times \triangle} H(x, y) \overline{\left[K(x, z) - \sum_{i=1}^{k} \lambda_i \psi_i(x) \overline{\chi_i(z)} \right]} \left[K(y, z) - \sum_{i=1}^{k} \lambda_i \psi_i(y) \overline{\chi_i(z)} \right] dx\, dy = \sum_{i=k+1} \lambda_i^2 \, | \chi_i(z) |^2$$

for almost every $z \in \triangle$. Taking $k = 0$ in this relation, we find

$$\left.\begin{aligned} &\sum_{i=1} \lambda_i^2 \, | \chi_i(z) |^2 = \int_{\triangle \times \triangle} H(x, y) \overline{K(x, z)} K(y, z) \, dx\, dy = \\ &= \int_{\triangle \times \triangle} \overline{H(y, x) K(x, z)} \, K(y, z) \, dx\, dy = \int_{\triangle} \overline{P(y, z)} \, K(y, z) \, dy = \\ &= \int_{\triangle} P(z, y) K(y, z) \, dy \leqslant \left(\int_{\triangle} | P(z, y) |^2 \, dy \right)^{1/2} \cdot \left(\int_{\triangle} | K(y, z) |^2 \, dy \right)^{1/2} \end{aligned} \right\} \quad (4)$$

for almost every $z \in \triangle$, so that

$$\int_{\triangle} \sum_{i=1} \lambda_i^2 \, | \chi_i(z) |^2 \, dz = \sum_{i=1} \lambda_i^2 \int_{\triangle} | \chi_i(z) |^2 \, dz$$

is finite. This shows that

$$\int_{\triangle} \sum_{i=k+1} \lambda_i^2 \, | \chi_i(z) |^2 \, dz = \sum_{i=k+1} \lambda_i^2 \int_{\triangle} | \chi_i(z) |^2 \, dz$$

tends to 0 as $k \to \infty$, hence also

$$\lim_{k \to \infty} \int_{\triangle \times \triangle \times \triangle} H(x, y) \overline{\left[K(x, z) - \sum_{i=1}^{k} \lambda_i \psi_i(x) \overline{\chi_i(z)} \right]} \left[K(y, z) - \sum_{i=1}^{k} \lambda_i \psi_i(y) \overline{\chi_i(z)} \right] dx\, dy\, dz = 0.$$

The completely continuous transformations $K^n (n = 2, 3, \dots)$, defined by

$$K^n f = \int\limits_{\triangle} K_n(x, y) f(y)\, dy,$$

where $K_n(x, y) \in L_2^{(2m)}$ is the n-th iterated kernel of $K(x, y)$, are also symmetrisable relative to H. Indeed, if it has already been proved that HK^{n-1} is self-adjoint, we have

$$(HK^n f, g) = (HK^{n-1} Kf, g) = (Kf, HK^{n-1} g) =$$
$$= (HKf, K^{n-1} g) = (f, HKK^{n-1} g) = (f, HK^n g)$$

for every pair of functions $f(x), g(x) \in L_2$; the transformation HK^n is therefore self-adjoint, in other words, K^n is symmetrisable.

Theorem 8. *The sequence $\lambda_i^n (i = 1, 2, \dots)$ is the sequence of all characteristic values $\neq 0$ of the Marty-kernel $K_n(x, y)$ and $\psi_i(x)$ is corresponding sequence of characteristic functions.*

Proof. Since $K\psi_i = \lambda_i \psi_i$ implies $K^n \psi_i = \lambda_i^n \psi_i$, the functions $\psi_i(x)$ are characteristic functions of $K_n(x, y)$, belonging to the characteristic values $\lambda_{i_1}^n$. Any further characteristic function $\psi(x)$ of $K_n(x, y)$, linearly independent of $\psi_1(x), \psi_2(x), \dots$, may be chosen so as to be H-orthogonal to all $\psi_i(x)$; hence, $(H\psi, \psi_i) = (\psi, \chi_i) = 0\ (i = 1, 2, \dots)$ implying $HK\psi = 0$ by Theorem 4, we have $K^2\psi = KK\psi = 0$, so that

$$\int\limits_{\triangle} K_n(x, y)\, \psi(y)\, dy = K^n \psi = 0$$

for $n \geq 2$, which shows that $\psi(x)$ belongs to the characteristic value 0. The sequence λ_i^n $(i = 1, 2, \dots)$ is therefore the sequence of all characteristic values $\neq 0$ of $K_n(x, y)$.

It follows from this theorem that the Theorems 1—7 hold for the integral equation with kernel $K_n(x, y)$, replacing everywhere λ_i by λ_i^n.