Mathematics. — *Lattice points in n-dimensional star bodies II. (Reducibility Theorems.)* By K. MAHLER. (Third communication.) (Communicated by Prof. J. G. VAN DER CORPUT.)

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§ 12. *Boundedly irreducible and reducible star bodies.*

In the case of unbounded star bodies of the finite type, the following definition seems to be of interest:

Definition C: *The unbounded star body K of the finite type is called boundedly reducible if there exists a bounded star body H such that $H \prec K$, and it is called boundedly irreducible if no bounded star body H with $H \prec K$ exists.*

Theorem J: *For every dimension n, there exists a boundedly irreducible star body K in R_n.*

Proof: We choose for K the star body

$$K^*: \qquad x_1^2\, x_2^2 \ldots x_{n-1}^2\, (x_1^2 + x_2^2 + \ldots x_n^2) \leqslant 1$$

considered already in the last paragraph, and for H any bounded star body contained in K. As we saw, K^* is contained in

$$K_0: \qquad |x_1\, x_2 \ldots x_n| \leqslant 1$$

and of the same determinant $\triangle(K^*) = \triangle(K_0)$; moreover, all boundary points of K^* are *inner* points of K_0. Hence the boundary points of H are likewise *inner* points of K_0; there exists then a constant θ with $0 < \theta < 1$ such that

$$|x_1\, x_2 \ldots x_n| \leqslant \theta$$

for all points of H. But this implies that

$$\triangle(H) \leqslant \theta \triangle(K_0) < \triangle(K^*),$$

and so it is not true that $H \prec K$, whence the assertion.

If K is any star body, then, as in Part I, we denote by K^t the set of all points X of K for which $|X| \leq t$.

Theorem K: *If the star body K: $F(X) \leq 1$ is boundedly irreducible, then there exists to every $t > 0$ a critical lattice Λ of K and an infinite sequence of lattices $\Lambda_1, \Lambda_2, \Lambda_3, \ldots,$ with the following properties:*

(a): *All lattices Λ_r are K^t-admissible.*

(b): $$d(\Lambda_r) < \triangle(K) \qquad (r = 1, 2, 3, \ldots).$$

(c): *The lattices Λ_r tend to the lattice Λ.*

Proof: Denote, for $r = 1, 2, 3, \ldots$, by $\Lambda^{(r)}$ any critical lattice of K^{r+t}; all these lattices are K^t-admissible. Since K^{r+t} is a bounded subset of K, from the hypothesis,

$$d(\Lambda^{(r)}) = \triangle(K^{r+t}) < \triangle(K) \qquad (r = 1, 2, 3, \ldots).$$

Further, by the corollary to Theorem 10 of Part I,

$$\lim_{r\to\infty} d(\Lambda^{(r)}) = \lim_{r\to\infty} \triangle(K^{r+t}) = \triangle(K).$$

The lattices $\Lambda^{(1)}, \Lambda^{(2)}, \Lambda^{(3)}, \ldots$ form therefore a *bounded* sequence, and so, by Theorem 2 of Part I, there exists an infinite subsequence

$$\Lambda_1 = \Lambda^{(k_1)}, \quad \Lambda_2 = \Lambda^{(k_2)}, \quad \Lambda_3 = \Lambda^{(k_3)}, \ldots \quad (1 \leqslant k_1 < k_2 < k_3 < \ldots),$$

which converges to a limiting lattice, Λ say. It is clear that the so defined lattices Λ_r and Λ satisfy the assertions (a), (b), and (c) of the theorem; but there remains to prove that Λ is a critical lattice of K.

We show firstly that Λ is *K-admissible*. Let $P \neq O$ be any point of Λ. There is then in each lattice Λ_r a point $P_r \neq O$ such that

$$\lim_{r\to\infty} |P_r - P| = 0.$$

Further, if r is sufficiently large,

$$|P_r| < t + k_r.$$

Since Λ_r is K^{t+k_r}-admissible, this means that

$$F(P_r) \geqslant 1,$$

whence by the continuity of $F(X)$,

$$F(P) = \lim_{r\to\infty} F(P_r) \geqslant 1,$$

i.e. Λ is K-admissible.

Secondly, Λ is even critical since

$$d(\Lambda) = \lim_{r\to\infty} d(\Lambda_r) = \triangle(K).$$

This completes the proof.

Definition D: *Let K be an infinite star body of the finite type. Then a critical lattice Λ of K is called strongly critical if there exists a bounded star body K^* contained in K such that*

$$d(\Lambda^*) \geqslant d(\Lambda)$$

for every K^*-admissible lattice Λ^* sufficiently near to Λ [12]).

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12) We say that Λ^* is near to Λ if there exist reduced bases

$$Y_1, Y_2, \ldots, Y_n \text{ and } Y_1^*, Y_2^*, \ldots, Y_n^*$$

of Λ and Λ^* such that all numbers

$$|Y_g - Y_g^*| \qquad (g = 1, 2, \ldots, n)$$

are less than a prescribed constant.

It is clear from this definition and from Theorem K that if K is boundedly irreducible, then at least one critical lattice of K is not strongly critical. Hence the following theorem follows at once:

Theorem L: *Let K be an infinite star body of the finite type, and let further every critical lattice of K be strongly critical. Then K is boundedly reducible.*

Proof Assume that, on the contrary, K is boundedly irreducible, and denote by K^* any *bounded* star body contained in K. There exists then a positive number t such that $|X| \leq t$ for every point X of K^*. If Λ is now the critical lattice of K given for this value of t by Theorem K, then Λ is clearly *not* strongly critical.

Theorem L allows in many cases to decide whether a given unbounded star body is boundedly reducible. A few such cases are discussed in the next paragraphs.

§ 13. *Examples of boundedly reducible star domains in R_2.*

In his work on binary cubic forms [13], L. J. MORDELL showed that the two star domains

K_1: $$|x_1 x_2 (x_1 + x_2)| \leqslant 1$$

and

K_2: $$|x_1^3 + x_2^3| \leqslant 1$$

are of determinants

$$\Delta(K_1) = \sqrt[3]{7} \text{ and } \Delta(K_2) = \sqrt[6]{\tfrac{243}{27}}.$$

It is of interest that his proof gave, incidentally, the result that both star domains are *boundedly reducible*; they were the first non-trivial examples of this kind. I later gave an even simpler example,

K_3: $$|x_1 x_2| \leqslant 1, \qquad \text{with } \Delta(K_3) = \sqrt{5},$$

of a boundedly reducible star domain, and made some applications of this property of K_3 [14]).

By means of Theorem L, independent proofs that K_1, K_2, and K_3 are boundedly reducible, may be easily obtained. To this purpose, one uses considerations analogous to those in the next paragraphs.

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[13]) Since his latest proof has not yet appeared, I refer to two articles Journal Lond. Math. **18**, 201—210 and 210—217 (1943), where the two affine-equivalent regions

$$|x_1^3 + x_1^2 x_2 - 2 x_1 x_2^2 - x_2^3| \leqslant 1 \text{ and } |x_1^3 - x_1 x_2^2 - x_2^3| \leqslant 1$$

are considered.

[14]) Proc. Cambr. Phil. Soc. **40**, 108—116, 116—120 (1943), and Journ. Lond. Math. Soc. **18**, 233—238 (1943).

§ 14. *The star body* $|x_1 x_2 x_3| \leq 1$ *in* R_3.

By a theorem of H. DAVENPORT [15]), the star body

K: $$|x_1 x_2 x_3| \leqq 1$$

is of determinant

$$\triangle(K) = 7.$$

Let

$$\theta = 2\cos\frac{2\pi}{7}, \quad \varphi = 2\cos\frac{4\pi}{7}, \quad \psi = 2\cos\frac{6\pi}{7}$$

be the three roots of

$$t^3 + t^2 - 2t - 1 = 0.$$

Then

$$\Lambda_0: x_1 = \theta u_1 + \varphi u_2 + \psi u_3, \; x_2 = \varphi u_1 + \psi u_2 + \theta u_3, \; x_3 = \psi u_1 + \theta u_2 + \varphi u_3,$$
$$(u_1, u_2, u_3 = 0, \mp 1, \mp 2, \ldots)$$

is a critical lattice of K, and every other critical lattice of K is of the form $\Lambda = \Omega \Lambda_0$ where Ω is one of the automorphisms

Ω: $$x_1 = t_1 x'_\alpha, \quad x_2 = t_2 x'_\beta, \quad x_3 = t_3 x'_\gamma$$

of K; here t_1, t_2, t_3 are real numbers satisfying

$$t_1 t_2 t_3 = \mp 1,$$

and α, β, γ is any permutation of 1, 2, 3.

Theorem M: *The star body* K: $|x_1 x_2 x_3| \leq 1$ *in* R_3 *is boundedly reducible.*

Proof: It suffices to show that Λ_0 *is a strongly critical lattice of* K because, by affine invariance, the same is then true for all critical lattices of K, and so the assertion follows immediately from Theorem L.

By definition, the lattice Λ_0 is strongly critical if there exists a bounded star body $K^* < K$ such that

$$d(\Lambda^*) \geqq d(\Lambda_0)$$

for every K^*-admissible lattice Λ^* sufficiently near to Λ_0. Such a lattice Λ^* near to Λ_0 contains a point

$$P^* = (\theta^*, \varphi^*, \psi^*)$$

arbitrarily near to the point

$$P_0 = (\theta, \varphi, \psi)$$

of Λ_0 obtained for $u_1 = 1, \; u_2 = 0, \; u_3 = 0$. There exists then an automorphism

Ω^*: $$x_1 = t_1^* x_1^*, \quad x_2 = t_2^* x_2^*, \quad x_3 = t_3^* x_3^* \qquad (t_1^* t_2^* t_3^* = 1)$$

[15]) Proc. Lond. Math. Soc. **44**, 412—431 (1938).

of K which changes P^* into a point $\Omega^* P^*$ collinear with O and P_0:

$$t_1^* \theta^* : t_2^* \varphi^* : t_3^* \psi^* = \theta : \varphi : \psi.$$

Hence, by affine invariance, it suffices to show that

$$d(\Lambda^*) \geqslant d(\Lambda_0)$$

for every K^*-admissible lattice Λ^* which is (i) sufficiently near to Λ_0, and which (ii) contains a point

$$P^* = (\theta^*, \varphi^*, \psi^*)$$

arbitrarily near to the point

$$P_0 = (\theta, \varphi, \psi)$$

of Λ_0 such that O, P_0, P^* are collinear.

Now every lattice Λ^* near to Λ_0 can be written in the form

$$\Lambda^*: x_1 = \theta v_1 + \varphi v_2 + \psi v_3, \quad x_2 = \varphi v_1 + \psi v_2 + \theta v_3, \quad x_3 = \psi v_1 + \theta v_2 + \varphi v_3$$

with

$$\left.\begin{aligned} v_1 &= u_1 + (a_{11} u_1 + u_{12} u_2 + u_{13} u_3), \\ v_2 &= u_2 + (a_{21} u_1 + u_{22} u_2 + u_{23} u_3), \\ v_3 &= u_3 + (a_{31} u_1 + a_{32} u_3 + a_{33} u_3), \end{aligned}\right\} (u_1, u_2, u_3 = 0, \mp 1, \mp 2, \ldots),$$

where the coefficients a_{hk} are real numbers such that

$$a = \max_{h,k=1,2,3} |a_{hk}|$$

is less than any given constant. The point P^* of Λ^* corresponding to P_0 is

$$P^* = \big((1+a_{11})\theta + a_{21}\varphi + a_{31}\psi, (1+a_{11})\varphi + a_{21}\psi + a_{31}\theta, (1+a_{11})\psi + a_{21}\theta + a_{31}\varphi\big)$$

and is collinear with O and P_0 if and only if

(a): $$a_{21} = a_{31} = 0,$$

because the three points

$$P_0 = (\theta, \varphi, \psi), \quad P_1 = (\varphi, \psi, \theta), \quad P_2 = (\psi, \theta, \varphi)$$

are linearly independent. We consider from now on only lattices Λ^* for which the condition (a) is satisfied.

Put for shortness,

$$\begin{aligned} S(U) &= (\theta u_1 + \varphi u_2 + \psi u_3)(\varphi u_1 + \psi u_2 + \theta u_3)(\psi u_1 + \theta u_2 + \varphi u_3) = \\ &= (u_1^3 + u_2^3 + u_3^3) - 4(u_2 u_3^2 + u_3 u_1^2 + u_1 u_2^2) + 3(u_2^2 u_3 + u_3^2 u_1 + u_1^2 u_2) - u_1 u_2 u_3, \end{aligned}$$

so that

$$x_1 x_2 x_3 = S(U)$$

for the point of Λ_0 belonging to $U = (u_1, u_2, u_3)$. Similarly

$$x_1 x_2 x_3 = S(V)$$

for the point of Λ^* belonging to $V = (v_1, v_2, v_3)$, or, on replacing V by its value in U,

$$x_1 x_2 x_3 = S(U) + T(U);$$

here

$$T(U) = (A_1 u_1^3 + A_2 u_2^3 + A_3 u_3^3) + (B_1 u_2 u_3^2 + B_2 u_3 u_1^2 + B_3 u_1 u_2^2) + \\ + (C_1 u_2^2 u_3 + C_2 u_3^2 u_1 + C_3 u_1^2 u_2) + D u_1 u_2 u_3,$$

with the coefficients

$$\begin{aligned}
A_1 &= 3a_{11} + O(a^2),\\
A_2 &= 3a_{22} - 4a_{12} + 3a_{32} + O(a^2),\\
A_3 &= 3a_{33} - 4a_{23} + 3a_{13} + O(a^2),\\
B_1 &= -4a_{22} - 8a_{33} + 3a_{12} + 6a_{23} + 3a_{32} - a_{13} + O(a^2),\\
B_2 &= -8a_{11} - 4a_{33} + 3a_{23} + 3a_{13} + O(a^2),\\
B_3 &= -4a_{11} - 8a_{22} + 6a_{12} - a_{32} + O(a^2),\\
C_1 &= 6a_{22} + 3a_{33} - a_{12} + 3a_{23} - 8a_{32} - 4a_{13} + O(a^2),\\
C_2 &= 3a_{11} + 6a_{33} - a_{23} - 8a_{13} + O(a^2),\\
C_3 &= 6a_{11} + 3a_{22} + 3a_{12} - 4a_{32} + O(a^2),\\
D &= -a_{11} - a_{22} - a_{33} - 8a_{12} - 8a_{23} + 6a_{32} + 6a_{13} + O(a^2),
\end{aligned}$$

where, in all cases, the O-term consists of the products of two or three of the a_{hk}. These formulae imply, in particular, that when a tends to zero, then the maximum

$$A = \max(|A_1|, |A_2|, |A_3|, |B_1|, |B_2|, |B_3|, |C_1|, |C_2|, |C_3|, |D|)$$

satisfies the inequality

$$A = O(a).$$

On solving for the coefficients a_{hk}, we find further that

$$\begin{aligned}
3a_{11} &= A_1 + O(a^2),\\
105a_{22} &= -70A_1 - 15A_2 + 30A_3 + 18B_1 - 12B_2 - 27B_3 - 6D + O(a^2),\\
105a_{33} &= 65A_1 + 45A_2 - 18B_1 + 12B_2 + 27B_3 - 9D + O(a^2),\\
35a_{12} &= -25A_1 - 5A_2 + 15A_3 + 9B_1 - 6B_2 - 6B_3 - 3D + O(a^2),\\
35a_{23} &= 35A_1 + 15A_2 - 5A_3 - 6B_1 + 9B_2 + 9B_3 - 3D + O(a^2),\\
35a_{32} &= -10A_1 + 10A_2 + 10A_3 + 6B_1 - 4B_2 + B_3 - 2D + O(a^2),\\
35a_{13} &= 25A_1 + 5A_2 + 5A_3 - 2B_1 + 8B_2 + 3B_3 - D + O(a^2),
\end{aligned}$$

and we also obtain the three identities,

$$\begin{aligned}
5C_1 &= 5A_1 - 5A_2 - 10A_3 - 7B_1 + 3B_2 - 2B_3 - D + O(a^2),\\
5C_2 &= -10A_1 + 5A_2 - 5A_3 - 2B_1 - 7B_2 + 3B_3 - D + O(a^2),\\
5C_3 &= -5A_1 - 10A_2 + 5A_3 + 3B_1 - 2B_2 - 7B_3 - D + O(a^2),
\end{aligned}$$

and the inequality

$$a = O(A), \qquad O(a^2) = O(A^2).$$

So far, the star body K^* has not yet been defined; nor have we yet used that Λ^* is K^*-admissible. Let then K^* be a star body K^t where t is so large that all points of Λ_0 for which

$$S(U) = 1, \quad |u_1| \leqslant 3, \quad |u_2| \leqslant 3, \quad |u_3| \leqslant 3,$$

belong to K^t. Then the ten points of Λ_0 given by

$$U = (1,0,0),\ (0,1,0),\ (0,0,1),\ (0,1,1),\ (1,0,1),\ (1,1,0),$$
$$(0,-1,-2),\ (-2,0,-1),\ (-1,-2,0),\ (-1,-1,-1),$$

satisfy the equation,

$$S(U) = 1.$$

The points of Λ^* belonging to the same U cannot be *inner* of $K^* = K^t$ since Λ^* is K^*-admissible. The numbers

$$\alpha_1,\ \alpha_2,\ \alpha_3, \quad \beta_1,\ \beta_2,\ \beta_3, \quad \gamma_1,\ \gamma_2,\ \gamma_3, \quad \delta$$

defined by

$$T(1,0,0) = \alpha_1,\ T(0,1,1) = \beta_1,\ T(0,-1,-2) = \gamma_1,$$
$$T(0,1,0) = \alpha_2,\ T(1,0,1) = \beta_2,\ T(-2,0,-1) = \gamma_2,\ T(-1,-1,-1) = \delta,$$
$$T(0,0,1) = \alpha_3,\ T(1,1,0) = \beta_3,\ T(-1,-2,0) = \gamma_3,$$

are therefore non-negative because

$$x_1 x_2 x_3 = S(U) + T(U) = 1 + T(U) \geqslant 1$$

for these points.

Hence, on substituting in $T(U)$,

$$\begin{aligned}
\alpha_1 &= A_1, \\
\alpha_2 &= A_2, \\
\alpha_3 &= A_3, \\
\beta_1 &= A_2 + A_3 + B_1 + C_1, \\
\beta_2 &= A_1 + A_3 + B_2 + C_2, \\
\beta_3 &= A_1 + A_2 + B_3 + C_3, \\
\gamma_1 &= -A_2 - 8A_3 - 4B_1 - 2C_1, \\
\gamma_2 &= -8A_1 - A_3 - 4B_2 - 2C_2, \\
\gamma_3 &= -A_1 - 8A_2 - 4B_3 - 2C_3, \\
\delta &= -A_1 - A_2 - A_3 - B_1 - B_2 - B_3 - C_1 - C_2 - C_3 - D,
\end{aligned}$$

and conversely,

$$\begin{aligned}
A_1 &= \alpha_1,\\
A_2 &= \alpha_2,\\
A_3 &= \alpha_3,\\
B_1 &= \tfrac{1}{2}\alpha_2 - 3\alpha_3 - \beta_1 - \tfrac{1}{2}\gamma_1,\\
B_2 &= -3\alpha_1 + \tfrac{1}{2}\alpha_3 - \beta_2 - \tfrac{1}{2}\gamma_2,\\
B_3 &= \tfrac{1}{2}\alpha_1 - 3\alpha_2 - \beta_3 - \tfrac{1}{2}\gamma_3,\\
C_1 &= -\tfrac{3}{2}\alpha_2 + 2\alpha_3 + 2\beta_1 + \tfrac{1}{2}\gamma_1,\\
C_2 &= 2\alpha_1 - \tfrac{3}{2}\alpha_3 + 2\beta_2 + \tfrac{1}{2}\gamma_2,\\
C_3 &= -\tfrac{3}{2}\alpha_1 + 2\alpha_2 + 2\beta_3 + \tfrac{1}{2}\gamma_3,\\
D &= \alpha_1 + \alpha_2 + \alpha_3 - \beta_1 - \beta_2 - \beta_3 - \delta.
\end{aligned}$$

From these formulae, we deduce that identically,

$$(b):\quad \begin{aligned}
\gamma_1 &= 2\alpha_1 + \alpha_2 - 2\alpha_3 + 2\beta_2 + 2\delta + O(a^2),\\
\gamma_2 &= -2\alpha_1 + 2\alpha_2 + \alpha_3 + 2\beta_3 + 2\delta + O(a^2),\\
\gamma_3 &= \alpha_1 - 2\alpha_2 + 2\alpha_3 + 2\beta_1 + 2\delta + O(a^2).
\end{aligned}$$

If further

$$\alpha = \max(|\alpha_1|, |\alpha_2|, |\alpha_3|, |\beta_1|, |\beta_2|, |\beta_3|, |\gamma_1|, |\gamma_2|, |\gamma_3|, |\delta|),$$

then, by these formulae, all three numbers a, A, α are of the same order,

$$\alpha = O(a) = O(A),\quad O(a^2) = O(\alpha^2),\quad O(A^2) = O(\alpha^2).$$

The proof of the theorem proceeds now as follows:
The lattice Λ^* is of determinant

$$d(\Lambda^*) = d(\Lambda_0)\begin{vmatrix} 1+a_{11} & a_{12} & a_{13}\\ 0 & 1+a_{22} & a_{23}\\ 0 & a_{32} & 1+a_{33}\end{vmatrix} = d(\Lambda_0)(1+\sigma),$$

where

$$(c):\quad \begin{aligned}
\sigma &= a_{11} + a_{22} + a_{33} + O(a^2),\\
&= \tfrac{2}{7}(A_1 + A_2 + A_3) - \tfrac{1}{7}D + O(A^2),\\
&= \tfrac{1}{7}(\alpha_1 + \alpha_2 + \alpha_3) + \tfrac{1}{7}(\beta_1 + \beta_2 + \beta_3) + \tfrac{1}{7}\delta + O(\alpha^2).
\end{aligned}$$

Assume now that Λ^* is so near to Λ_0 that a, hence also A and α, are sufficiently small. Then
either

$$\sigma > 0,\quad d(\Lambda^*) > d(\Lambda_0),$$

or

$$\sigma = 0,\quad d(\Lambda^*) = d(\Lambda_0).$$

By (c), the second case cannot hold unless

$$(d):\quad \alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = \delta = 0,$$

hence also

(e): $$\gamma_1 = \gamma_2 = \gamma_3 = 0,$$

because by (b),

$$\max(|\gamma_1|, |\gamma_2|, |\gamma_3|)$$

is of at most the same order as

$$\max(|a_1|, |\alpha_2|, |\alpha_3|, |\beta_1|, |\beta_2|, |\beta_3|, |\delta|).$$

The equations (d) and (e) imply next that

$$A_1 = A_2 = A_3 = B_1 = B_2 = B_3 = C_1 = C_2 = C_3 = D = 0,$$

hence also

$$a_{11} = a_{22} = a_{33} = a_{12} = a_{23} = a_{32} = a_{13} = 0;$$

and so Λ^* coincides with Λ_0. This concludes the proof.

Although the proof just given is a pure *existence proof*, it can easily be altered so as to lead to the construction of a bounded star body K^* satisfying $K^* \prec K$.

The next theorems are all proved in a manner similar to that of Theorem M.