Mathematics. — *A P-adic Analogue of a Theorem of* LEBESGUE *in the Theory of Measure.* By J. POPKEN and H. TURKSTRA. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of June 29, 1946.)

1. We consider on the real axis a measurable set of points A and an arbitrary interval I [1]. The ratio

$$\frac{m\,(A\,I)}{m\,(I)}$$

is said to be the *mean density* of A in I.

Now let α be an arbitrary point on the real axis. Let I denote an interval containing α and let this interval "contract itself on α", i.e. let its length $m(I)$ tend to zero. Whenever the mean density of A in I always tends to the same limit $d = d(\alpha)$, then d is said to be the *density* of A at α.

The following fundamental theorem in the theory of density is due to LEBESGUE [2]:

If A denotes an arbitrary measurable set on the real axis, then there exists a set Z of measure zero, such that at all points outside [3] *Z the density exists, and is equal to unity at points of A and equal to zero at points outside A.*

This theorem also is of special importance in the metric theory of Diophantic approximations [4].

In this note we will prove the analogue of this theorem if we take the field of P-adic numbers in stead of the field of real numbers (theorem I). It is clear that we have to use a theory of measure in the field of all P-adic numbers. Such a theory was established by TURKSTRA in his dissertation [5].

---

[1]) In this note we use the general symbolism of the theory of sets. Let A, B be two sets, then $A + B$, the *sum* of A and B, denotes the set of all elements in A or in B; A—B, the *complement* of B with respect to A, denotes the set of all elements in A but not in B; AB, the *intersection* of A and B, consists of the elements in A and also in B. Further $A \subset B$ or $B \supset A$ means that A is a sub-set of B, the case that A coincides with B not being excluded. A set is said to be *empty* if it has no elements. The measure of a measurable set A is denoted by $m\,(A)$.

[2]) H. LEBESGUE, Sur l'intégration des fonctions discontinues, Ann. Scient. de l'École Normale Supérieure (3) 27, p. 405—407 (1910).

[3]) "Outside Z" means here: "belonging to the complement of Z".

[4]) Cf. e.g. J. F. KOKSMA, Diophantische Approximationen (Ergebnisse der Mathematik IV 4), Kap. III § 5.

[5]) H. TURKSTRA, Metrische bijdragen tot de theorie der Diophantische approximaties in het lichaam der P-adische getallen. Dissertation of the „Vrije Universiteit" at Amsterdam, Groningen 1936.

We shall refer to this book as "T".

See also: W. FELLER and E. TORNIER, Mass- und Inhaltstheorie des Baireschen Nullraumes, Math. Ann. **107**, 165—187 (1933).

In the next section of this note we give an extract of this theory as far as we use it in our present investigation.

2. From now on we confine ourselves to the field $K = K(P)$ of all P-adic numbers, P denoting an arbitrary prime.

A P-adic number α can be written in the form

$$\alpha = \sum_{n=-\infty}^{\infty} a_n P^n, \quad \ldots \ldots \ldots \quad (1)$$

where the coefficients a_n are integers taken from the interval $0 \leqq a \leqq P-1$. such that only a finite number of the coefficients $a_{-1}, a_{-2}, a_{-3}, \ldots$ do not vanish.

Suppose $\alpha \neq 0$, and let a_{-t} be the first coefficient in (1) different from zero. Then

$$\alpha = \sum_{n=-t}^{\infty} a_n P^n \quad , \quad a_{-t} \neq 0.$$

The *P-adic value* $|\alpha|_P$ of α is defined to be

$$|\alpha|_P = P^t \qquad (|0|_P = 0).$$

Let α, β be two P-adic numbers or "points", then $|\alpha - \beta|_P$ is said to be the P-adic *distance* between α and β; we shall denote it by $\overline{\alpha\beta}$. Clearly $\overline{\alpha\beta} \geqq 0$ and $\overline{\alpha\beta} = 0$ if and only if α and β coincide; also $\overline{\alpha\beta} = \overline{\beta\alpha}$ [6]. Finally the important "inequality of the triangle" is satisfied: If α, β, γ are three arbitrary P-adic points, then

$$\overline{\alpha\beta} \leqq \overline{\alpha\gamma} + \overline{\gamma\beta};$$

even a sharper inequality holds:

$$\overline{\alpha\beta} \leqq \max(\overline{\alpha\gamma}, \overline{\gamma\beta}) \text{ [7]};$$

in other words this conception of distance is "non-Archimedic", and this property is responsible for some peculiarities in the theory of P-adic point sets.

Evidently the set $K(P)$ of all P-adic points is a metric space.

Now we are in a position to define a *P-adic interval* of *order n*: Let α be an arbitrary P-adic number and let n be a fixed integer, then the set of all P-adic numbers ξ, satisfying

$$\overline{\alpha\xi} \leqq P^{-n}$$

is said to be a P-adic interval of order n [8].

Open and closed sets can be defined in the ordinary manner, but it is convenient to enlarge these definitions by the convention that an empty set

---

6) T. p. 39 (Proof of II Stelling 11).

7) T. p. 39 (Proof of II Stelling 11) and p. 30 (II Stelling 4).

8) For an equivalent definition see T. p. 72 (V Definitie 1); compare p. 74 (V Stelling 1).

is considered as open and also as closed. Then the following theorems are true [9]):

The complement of an open set is closed and the complement of a closed set is open.

The sum of a finite number or of an infinity of open sets is open.

The sum of a finite number of closed sets is closed.

The intersection of a finite number of open sets is open.

The intersection of a finite number or of an infinity of closed sets is closed.

Now we state some properties of P-adic intervals:

Theorem 1 [10]): *The set of all P-adic intervals is enumerable.*

Theorem 2 [11]): *If I_1 and I_2 denote intervals, such that they have at least one point in common; if the order of I_1 is not less than the order of I_2, then*

$$I_1 \subset I_2.$$

It follows, that *two intervals of the same order either coincide, or do not overlap.*

Let α be an arbitrary P-adic number. Now

$$\overline{\alpha\xi} \leqq P^{-n}$$

defines an interval $I^{(n)}$ of order n containing α. Hence:

Theorem 3: *Let α be an arbitrary P-adic number. For every integer n there exists one and only one interval $I^{(n)}$ of order n, enclosing α. Moreover*

$$\ldots \subset I^{(n+1)} \subset I^{(n)} \subset I^{(n-1)} \subset I^{(n-2)} \subset \ldots.$$

Let $I^{(n)}$ denote an interval of order n. Take an arbitrary P-adic number α of $I^{(n)}$. By theorem 3 there exists for every integer m one and only one interval $I^{(m)}$ of order m containing α. Moreover

$$I^{(n)} \subset I^{(n-1)} \subset I^{(n-2)} \subset \ldots.$$

Hence we have:

Theorem 4: *Let $I^{(n)}$ denote an interval of order n. For every integer $m \leqq n$ there exists one and only one interval $I^{(m)}$ of order m enclosing $I^{(n)}$. Moreover*

$$I^{(n)} \subset I^{(n-1)} \subset I^{(n-2)} \subset \ldots.$$

It follows that all intervals, containing $I^{(n)}$ as a sub-set, belong to the sequence $I^{(n)}, I^{(n-1)}, I^{(n-2)}, \ldots$.

Other fundamental properties of intervals are given by:

Theorem 5 [12]): *Any interval is an open and also a closed set.*

---

[9]) T. p. 40—41.

[10]) T. p. 78 (V Stelling 3).

[11]) T. p. 79 (V Stelling 4).

[12]) T. p. 78 (V Stelling 2).

Theorem 6 [13]: *An open set* [14] *consists of a finite number or an enumerable infinity of non-overlapping intervals.*

Now the theory of measure in the field of all P-adic numbers can be developed in a manner similar to the ordinary theory of measure.

First the *measure of an interval $I^{(n)}$ of order n is defined to be the number*

$$m\,(I^{(n)}) = P^{-n} \text{ }^{15)}.$$

Then the measure of a *bounded open set* O is introduced. If O is an empty set, then by definition its measure $m(O)$ shall be equal to zero. If O is not an empty set, then by theorem 6 there exists at least one decomposition of O in a set of non-overlapping intervals $I_1, I_2, I_3, \ldots$, such that

$$O = I_1 + I_2 + I_3 + \ldots$$

Now it can be shown, that the sum

$$m\,(I_1) + m\,(I_2) + m\,(I_3) + \ldots$$

does not depend on the particular decomposition of O we choose; it is said to be the measure $m(O)$ of O [16].

Next we consider the so-called *bounded* sets:

Let t be an arbitrary integer; then the inequality

$$|\,\xi\,|_P \leqq P^t$$

defines the interval of order $-t$ containing zero; we shall denote this interval by K_t. All intervals K_t enclose zero, hence by theorem 3

$$\ldots \subset K_1 \subset K_0 \subset K_{-1} \subset \ldots$$

A set B is said to be *bounded* if, for a suitably chosen integer t, B is contained in the set K_t; in other words if it is possible to choose a positive number T, such, that all points β of B satisfy the inequality

$$|\,\beta\,|_P \leqq T.$$

It is easily shown, that every interval is a bounded set.

The *exterior measure* $\overline{m}(B)$ of a bounded set B is defined to be the lower bound of the measures of all bounded open sets O, which contain B [17].

B is a bounded set, hence by definition it is contained in an interval K_t. The quantity

$$\underline{m}\,(B) = m\,(K_t) - \overline{m}\,(K_t - B)$$

can be shown not to depend on the particular interval K_t we choose; it defines the *interior measure* of B [18]. Always $\underline{m}(B) \leqq \overline{m}(B)$ [19].

13) T. p. 80 (V Stelling 6, Gevolg 2).

14) Different from an empty set.

15) T. p. 86 (V Definitie 5a).

16) T. p. 96, V Definitie 9, but here the author uses a particular decomposition $I_1, I_2, \ldots$ of O, where $I_1, I_2, \ldots$ are the "largest" intervals of O (see p. 79 and 80, V Definitie 3 and Gevolg 2). Afterwards (p. 97, 98, V Stelling 18) it is shown, that we may substitute it by an arbitrary decomposition of O.

17) T. p. 101 (V Definitie 10; it is clear that in this definition O denotes a bounded open set).

18) T. p. 103 (V Definitie 11).

19) T. p. 104 (V Stelling 24).

If for a bounded set B the exterior and interior measures are equal, then B is said to be a *measurable* set with its *measure* $m(B) = \overline{m}(B) = \underline{m}(B)$ [20].

Finally the conception of measure is extended to *unbounded* sets: An arbitrary set A is said to be *measurable* if for any integer t the intersection AK_t is measurable in the above sense [21]; its *measure* is defined by

$$m(A) = \lim_{t \to \infty} m(A K_t) \text{ [22]}.$$

This limit always exists, but may be infinite. Evidently $m(A) \geqq 0$.

These extended definitions of a measurable set and of its measure are in accordance with the previous definitions with respect to open and bounded sets [23].

The following theorems are true:

Theorem 7 [24]: *The complement $K - A$ of a measurable set is measurable again.*

Theorem 8 [25]: *If A and B denote measurable sets, such that*

$$A \supset B,$$

then $m(A) \geqq m(B)$; moreover $m(A - B)$ is measurable, and

$$m(A - B) = m(A) - m(B).$$

Theorem 9 [26]: *The sum $A_1 + A_2 + A_3 + \ldots$ and the intersection $A_1 A_2 A_3 \ldots$ of a finite number or of an enumerable infinity of measurable sets $A_1, A_2, A_3, \ldots$ are measurable again. Moreover*

$$m(A_1 + A_2 + A_3 + \ldots) \leqq m(A_1) + m(A_2) + m(A_3) + \ldots,$$

and, if $A_1, A_2, A_3, \ldots$ do not overlap, even

$$m(A_1 + A_2 + A_3 + \ldots) = m(A_1) + m(A_2) + m(A_3) + \ldots$$

3. Let I be an arbitrary interval in $K(P)$. The *mean density* of a measurable set A in I is defined to be

$$\frac{m(A I)}{m(I)}.$$

Now let α be an arbitrary P-adic number. By theorem 3 there exists for any integer n exactly one interval $I^{(n)}$ of order n containing α. The quantity

$$d = \overline{\lim_{n \to \infty}} \frac{m(A I^{(n)})}{m(I^{(n)})}$$

is said to be the *upper density* of A at the point α. Evidently $0 \leq \overline{d} \leq 1$.

---

[20] T. 108 (V Definitie 12).
[21] T. p. 112 (V Definitie 13).
[22] T. p. 112 (V Definitie 14).
[23] T. p. 112 (Opmerking 1 and 2).
[24] T. p. 115 (V Stelling 35).
[25] T. p. 115 (V Stelling 34).
[26] T. p. 116 (V Stelling 37), p. 117 (V Stelling 38) and p. 113 (V Stelling 33).

If

$$d = \lim_{n \to \infty} \frac{m(A\,I^{(n)})}{m(I^{(n)})}$$

exists, then d is said to be the *density* of A at the point α. If $\bar{d} = 0$ then it follows $d = 0$.

The analogue of the theorem of LEBESGUE we want to prove is:

Theorem I: *If A denotes an arbitrary measurable set in the field of all P-adic numbers, then there exists a set Z of measure zero, such that at all points outside Z the density exists, and is equal to unity at points of A and equal to zero at points outside A.*

This theorem was stated without proof in the dissertation of TURKSTRA [27]) and there it was used to obtain certain results in the theory of Diophantic approximations in the field of P-adic numbers. The authors of this note proved the theorem some years before the war, but the publication was delayed on account of several circumstances and the war.

It is easily shown, that it is sufficient to prove only the second part of theorem I:

Theorem II: *If A denotes a measurable set, then there exists a set Z of measure zero, such that at all points outside A and outside Z the density of A is equal to zero.*

For let us suppose that this last theorem has been proved, then we shall show that also theorem I is true: Let A be a measurable set. Now the complement $K - A$ of A also is measurable. Applying theorem II with $K - A$ in stead of A we obtain the following result: There exists a set Z' of measure zero, such that at all points α outside $K - A$ and outside Z' the density of $K - A$ is equal to zero, or

$$\lim_{n \to \infty} \frac{m\{(K - A)\,I^{(n)}\}}{m(I^{(n)})} = 0$$

at all points α of A outside Z'; here $I^{(n)}$ is the interval of order n containing α. Now $A\,I^{(n)}$ and $(K - A)I^{(n)}$ do not overlap, hence $m(A\,I^{(n)}) + m\{(K - A)I^{(n)}\} = m(I^{(n)})$.

It follows

$$\lim_{n \to \infty} \frac{m(A\,I^{(n)})}{m(I^{(n)})} = 1$$

at all points α of A outside Z'.

The set $Z + Z'$ evidently is of measure zero; outside $Z + Z'$ the density is equal to unity at points of A and on account of theorem II equal to zero at points outside A.

Hence theorem I is a corollary of theorem II.

In the next section of this paper we first prove the fundamental lemma 1,

---

[27]) T. p. 137 (VII Stelling 1*).

then we show that the assertion of theorem II is true if we substitute a bounded open set O for the measurable set A (lemma 2), next we prove the theorem for a bounded set B (lemma 3) and lastly we establish theorem II in its general form.

4. In this section we often consider a sequence of intervals $i_1, i_2, i_3, \dots$, satisfying certain conditions, then $i_1 + i_2 + i_3 + \dots$ denotes the sum of these intervals. But it may happen, that there are no intervals satisfying the conditions. In this case we still write formally $i_1, i_2, i_3, \dots$ in order to avoid the consideration of several special cases. Then $i_1 + i_2 + i_3 + \dots$ denotes an empty set, and $m(i_1) + m(i_2) + m(i_3) + \dots$ by definition will be equal to zero.

Now let O be a bounded open set. By theorem 6 the set O consists of a finite number or an enumerable infinity of non-overlapping intervals $I_1, I_2, I_3, \dots$, or

$$O = I_1 + I_2 + I_3 + \dots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It follows

$$m(O) = m(I_1) + m(I_2) + m(I_3) + \dots,$$

where the series at the right-hand side is either finite or convergent.

For every positive integer r we define the open set O_r by

$$O_r = I_r + I_{r+1} + I_{r+2} + \dots, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where O_r may be an empty set. Clearly

$$\lim_{r \to \infty} m(O_r) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Lemma 1: *Let O be a bounded open set and let (2) be a decomposition of O in non-overlapping intervals. Let r and k denote arbitrary positive integers. Then the set O_r, defined in (3), is enclosed in an open set $\overline{O}_{r,k} = \overline{O}_r$ with the following two properties:*

1° $m(\overline{O}_r) \leqq kP\, m(O_r)$;

2° *if the point a is outside O and outside $\overline{O}_r$, then the upper density of O at a is at most $\frac{1}{k}$.*

P r o o f: We denote the *mean density of O_r* in an arbitrary interval i by

$$\delta(i) = \frac{m(O_r\, i)}{m(i)}.$$

Let $I = I_{r+\varrho}$ ($\varrho = 0, 1, 2, \dots$) be an arbitrary interval of O_r and let its order be n. If ν is an arbitrary integer $< n$, then by theorem 4 there exists exactly one interval $I^{(\nu)}$ of order ν, containing $I = I^{(n)}$ as a sub-set. Moreover

$$I = I^{(n)} \subset I^{(n-1)} \subset I^{(n-2)} \subset \dots, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and all intervals containing I as a sub-set belong to the sequence $I^{(n)}$, $I^{(n-1)}$, $I^{(n-2)}$, ...

Now $m(I^{(\nu)}) = P^{-\nu}$, hence the measure of the intervals in this sequence increases indefinitely.

By definition

$$\delta(I^{(\nu)}) = \frac{m(O_r I^{(\nu)})}{m(I^{(\nu)})}, \text{ thus } \delta(I^{(\nu)}) \leqq \frac{m(O)}{m(I^{(\nu)})}.$$

Hence the mean density of O_r in $I = I^{(n)}$ is equal to unity, but the mean density in $I^{(\nu)}$ tends to zero if ν decreases indefinitely.

Let $I^{(\mu)}$ be the *last* interval of the sequence (5) with mean density $> \frac{1}{Pk}$. It follows

$$\delta(I^{(\mu)}) > \frac{1}{kP}, \quad \text{. . . . . (6)}$$

and, for the next interval $I^{(\mu-1)}$ in the sequence,

$$\delta(I^{(\mu-1)}) = \frac{m(O_r I^{(\mu-1)})}{m(I^{(\mu-1)})} \leqq \frac{1}{kP}. \quad \text{. . . . . . . (7)}$$

But

$$m(I^{(\mu-1)}) = P^{-\mu+1} = P\, m(I^{(\mu)}) \quad \text{. . . . . . . (8)}$$

and $I^{(\mu)} \subset I^{(\mu-1)}$, hence

$$m(O_r I^{(\mu)}) \leqq m(O_r I^{(\mu-1)}). \quad \text{. . . . . (9)}$$

It follows from (7), (8) and (9)

$$\frac{m(O_r I^{(\mu)})}{P\, m(I^{(\mu)})} \leqq \frac{1}{kP}. \quad \text{. . . . . (10)}$$

Hence, if we denote $I^{(\mu)}$ by $\overline{I}$, we derive from (6) and (10):

An arbitrary interval $I_{r+\varrho}$ of O_r can be enclosed in an interval $\overline{I}_{r+\varrho}$, such that the mean density of O_r in $\overline{I}_{r+\varrho}$ satisfies the inequalities

$$\frac{1}{kP} < \delta(\overline{I}_{r+\varrho}) \leqq \frac{1}{k}. \quad \text{. . . . . (11)}$$

Evidently O_r is a sub-set of the open set

$$\overline{O}_r = \overline{I}_r + \overline{I}_{r+1} + \overline{I}_{r+2} + \dots \quad \text{. . . . . . . (12)}$$

and we shall prove that $\overline{O}_r$ also has the other properties stated in the lemma [28]).

First we shall show that two intervals $\overline{I}_{r+\sigma}$ and $\overline{I}_{r+\tau}$ in the right-hand side of (12) either coincide or do not overlap.

Let us suppose, that $\overline{I}_{r+\sigma}$ and $\overline{I}_{r+\tau}$ at least have one point in common

[28]) If O_r is an empty set, then, by the conventions assumed in the beginning of section 4, the set $\overline{O}_r$ is empty also.

and that the order of, say, $\overline{I}_{r+\sigma}$ is not less than the order of $\overline{I}_{r+\tau}$. It follows from theorem 2

$$\overline{I}_{r+\sigma} \subset \overline{I}_{r+\tau},$$

hence

$$I_{r+\sigma} \subset \overline{I}_{r+\sigma} \subset \overline{I}_{r+\tau}.$$

These intervals belong to the sequence (compare (5))

$$I_{r+\sigma} = I^{(n)}_{r+\sigma} \subset I^{(n-1)}_{r+\sigma} \subset I^{(n-2)}_{r+\sigma} \subset \ldots$$

and $\overline{I}_{r+\sigma}$ by definition is the last interval in this sequence with mean density $> \frac{1}{kP}$. But it follows from (11), that $\overline{I}_{r+\tau}$ also has a mean density $> \frac{1}{kP}$. We obtain a contradiction unless $\overline{I}_{r+\sigma}$ and $\overline{I}_{r+\tau}$ coincide.

If we denote the different intervals of (12) by $\overline{I}_{r_1}, \overline{I}_{r_2}, \overline{I}_{r_3}, \ldots$, then these intervals do not overlap and

$$\overline{O}_r = \overline{I}_{r_1} + \overline{I}_{r_2} + \overline{I}_{r_3} + \ldots$$

Hence

$$m(\overline{O}_r) = m(\overline{I}_{r_1}) + m(\overline{I}_{r_2}) + m(\overline{I}_{r_3}) + \ldots \quad (13)$$

The mean density of O_r in $\overline{I}_{r_\nu}$ ($\nu = 1, 2, 3, \ldots$) is $> \frac{1}{Pk}$, or

$$m(O_r \overline{I}_{r_\nu}) > \frac{1}{Pk} m(\overline{I}_{r_\nu});$$

we derive from (13), remembering that O_r and $\overline{O}_r$ may be empty,

$$\frac{1}{Pk} m(\overline{O}_r) = \frac{1}{Pk} m(\overline{I}_{r_1}) + \frac{1}{Pk} m(\overline{I}_{r_2}) + \ldots$$
$$\leqq m(O_r \overline{I}_{r_1}) + m(O_r \overline{I}_{r_2}) + \ldots = m(O_r \overline{O}_r) = m(O_r),$$

hence $\overline{O}_r$ has the property 1° of lemma 1.

Let a be a point outside O and outside $\overline{O}_r$ (the conditions in lemma 1, 2°). In particular a is outside $I_1 + I_2 + \ldots + I_{r-1}$; this set is closed, for every interval by theorem 5 is a closed set. Hence the complement is open. Therefore we can include a in an interval $E^{(p)}$ of order p, outside $I_1 + I_2 + \ldots + I_{r-1}$.

Let n denote an integer $\geqq p$. By theorem 3 there exists exactly one interval $E^{(n)} = E$ of order n containing a; moreover $E^{(n)} \subset E^{(p)}$. Hence E is outside $I_1 + I_2 + \ldots + I_{r-1}$, or

$$EO = EO_r. \quad (14)$$

We have to show, that the upper density of O at a is $\frac{1}{k}$ at most, or

$$\overline{\lim_{n \to \infty}} \frac{m(E^{(n)} O)}{m(E^{(n)})} \leqq \frac{1}{k}.$$

Evidently it is sufficient to prove, that the mean density of O in $E^{(n)} = E$ is $\frac{1}{k}$ at most, i.e.

$$\delta(E) = \frac{m(EO)}{m(E)} \leqq \frac{1}{k},$$

for every integer $n \geqq p$.

By (14)

$$\delta(E) = \frac{m(EO_r)}{m(E)}.$$

We may suppose $m(E\overline{O}_r) \neq 0$, for otherwise $m(EO_r) = 0$, hence $\delta(E) = 0$. It follows

$$\delta(E) \leqq \frac{m(EO_r)}{m(E\overline{O}_r)}. \quad \ldots \ldots \ldots \quad (15)$$

Now

$$E\overline{O}_r = E\overline{I}_{r_1} + E\overline{I}_{r_2} + E\overline{I}_{r_3} + \ldots \ldots \ldots \quad (16)$$

If $\overline{I}_{r_\nu}$ is an arbitrary interval of the sequence $\overline{I}_{r_1}, \overline{I}_{r_2}, \overline{I}_{r_3}, \ldots$, and if E and $\overline{I}_{r_\nu}$ have at least one point in common, then by theorem 2 either $\overline{I}_{r_\nu} \subset E$ or $E \subset \overline{I}_{r_\nu}$. But this latter possibility would involve, that a was a point of $\overline{O}_r$, contrary to hypothesis, hence $\overline{I}_{r_\nu} \subset E$. Hence for an arbitrary interval $\overline{I}_{r_\nu}$ either $E\overline{I}_{r_\nu}$ is empty or $E\overline{I}_{r_\nu}$ coincides with $\overline{I}_{r_\nu}$.

Therefore we deduce from (16) the existence of a sequence $\overline{I}_{p_1}, \overline{I}_{p_2}, \overline{I}_{p_3}, \ldots$ of non-overlapping intervals, such that

$$E\overline{O}_r = \overline{I}_{p_1} + \overline{I}_{p_2} + \overline{I}_{p_3} + \ldots$$

Hence, taking account of $O_r \subset \overline{O}_r$,

$$EO_r = O_r\overline{I}_{p_1} + O_r\overline{I}_{p_2} + O_r\overline{I}_{p_3} + \ldots,$$

where $O_r\overline{I}_{p_1}, O_r\overline{I}_{p_2}, O_r\overline{I}_{p_3}, \ldots$ are non-overlapping measurable sets.

From (15) it follows therefore

$$\delta(E) \leqq \frac{m(O_r\overline{I}_{p_1}) + m(O_r\overline{I}_{p_2}) + \ldots}{m(\overline{I}_{p_1}) + m(\overline{I}_{p_2}) + \ldots}.$$

We know by (11) that the mean density of O_r in $\overline{I}_{p_\nu}$

$$\frac{m(O_r\overline{I}_{p_\nu})}{m(\overline{I}_{p_\nu})} \leqq \frac{1}{k} \qquad (\nu = 1, 2, \ldots).$$

It follows $\delta(E) \leqq \frac{1}{k}$ and this proves the lemma.

Lemma 2: *If O denotes an open bounded set, then there exists a set Z of measure zero, such that at all points outside O and outside Z the density of O is equal to zero.*

P r o o f : 1. Let k be an integer > 1, further let β be a P-adic number, such that the upper density of O at β is $> \frac{1}{k}$.

We introduce for $r = 1, 2, \ldots$ the open sets $\overline{O}_r$ considered in lemma 1. Then — applying this lemma — we deduce that outside O and outside $\overline{O}_r$ the upper density of O at every point is $\leqq \frac{1}{k}$.

It follows that β belongs to every set $O + \overline{O}_r$, hence it is contained in the intersection S_k of these sets. All sets $O + \overline{O}_r$ are open, hence S_k is measurable. Moreover

$$O \subset S_k \subset O + \overline{O}_r \qquad (r = 1, 2, \ldots),$$

hence

$$m(O) \leqq m(S_k) \leqq m(O) + m(\overline{O}_r).$$

By lemma 1

$$m(\overline{O}_r) \leqq k P\, m(O_r)$$

and by (4)

$$\lim_{r \to \infty} m(O_r) = 0;$$

it follows

$$m(S_k) = m(O).$$

Introducing the set $Z_k = S_k - O$, we obtain: Every point β with upper density $> \frac{1}{k}$ either belongs to O or it is contained in a set Z_k of measure zero ($k = 2, 3, \ldots$).

2. The set

$$Z = Z_2 + Z_3 + Z_4 + \ldots$$

clearly is of measure zero.

We consider a point β outside O and outside Z. We shall show that the density at β is equal to zero. For otherwise the upper density $\overline{d}$ at β was positive. Now take an integer k, such that $\overline{d} > \frac{1}{k}$, then $k > 1$, and by the previous result β belongs either to O or to the set Z_k, contrary to hypothesis.

Lemma 3: *Let B be a bounded measurable set. Then there exists a set Z of measure zero, such that at all points outside B and outside Z the density is equal to zero.*

P r o o f : B is a bounded set; hence for a suitably chosen integer t the set K_t of all P-adic numbers ξ with

$$|\xi|_P \leqq P^t$$

contains the set B. The measure of B is equal to the exterior measure, hence it is the lower bound of the measures of all open bounded sets O which contain B.

It follows the existence of a sequence

$$O_1, O_2, O_3, \ldots$$

of bounded open sets, such that

$$B \subset O_n \text{ and } \lim_{n\to\infty} m(O_n) = m(B).$$

Now every set

$$B_n = O_n - B$$

is measurable and

$$\lim_{n\to\infty} m(B_n) = \lim_{n\to\infty} \{m(O_n) - m(B)\} = 0.$$

Hence the intersection Z_0 of these sets $B_1, B_2, B_3, \ldots$,

$$Z_0 = B_1 B_2 B_3 \ldots,$$

is of measure zero.

By lemma 2 there exists for every (bounded open) set O_n a set Z_n of measure zero, such that at all points outside O_n and outside Z_n the density of O_n is equal to zero ($n = 1, 2, \ldots$).

We will show that the set

$$Z = Z_0 + Z_1 + Z_2 + \ldots$$

of measure zero has the property stated in lemma 3.

Therefore let a denote a point outside B and outside Z, then clearly a is outside $Z_0 = B_1B_2B_3 \ldots$, hence it is not contained in at least one of the sets $B_1, B_2, B_3, \ldots$; say a is outside $B_n = O_n - B$. But a is not contained in B either, hence a is outside O_n. Moreover a is outside Z_n. Applying lemma 2 we find that the density of O_n at a is zero, i.e.

$$\lim_{\nu\to\infty} \frac{m(O_n I^{(\nu)})}{m(I^{(\nu)})} = 0;$$

here $I^{(\nu)}$ denotes the interval of order ν containing a. Now $B \subset O_n$, hence $m(B I^{(\nu)}) \leq m(O_n I^{(\nu)})$, hence the density of B at a

$$\lim_{\nu\to\infty} \frac{m(B I^{(\nu)})}{m(I^{(\nu)})} = 0.$$

Proof of theorem II: The set of all intervals is enumerable (theorem 1). Let $I_1, I_2, I_3, \ldots$ denote the set of all intervals of order zero. Now A is an arbitrary measurable set, hence every set

$$B_n = A I_n \qquad (n = 1, 2, \ldots)$$

is bounded and measurable; moreover

$$A = B_1 + B_2 + B_3 + \ldots$$

Applying lemma 3 on the sets $B_1, B_2, B_3, \ldots$ we derive the existence of sets $Z_1, Z_2, Z_3, \ldots$ of measure zero, such that at all points outside B_n and outside Z_n the density of B_n is equal to zero ($n = 1, 2, 3, \ldots$). The set

$$Z_1 + Z_2 + Z_3 + \ldots = Z$$

is of measure zero. We shall show that at a point α outside Z and outside A the density of A is equal to zero.

The point α belongs to the interval

$$|\xi \alpha|_P \leqq 1$$

of order zero. This interval belongs to the sequence $I_1, I_2, I_3, \ldots$; we shall denote it by I_N.

Now α is outside B_N and also outside Z_N, hence by lemma 3 the density of B_N at α is equal to zero:

$$\lim_{\nu \to \infty} \frac{m(B_N I^{(\nu)})}{m(I^{(\nu)})} = 0,$$

where $I^{(\nu)}$ denotes the interval of order ν containing α. Both intervals $I^{(0)}$ and I_N contain α and are of order zero, hence $I^{(0)}$ coincides with I_N and all intervals $I^{(\nu)}$ with $\nu > 0$ are enclosed in I_N, hence

$$A I^{(\nu)} \subset A I_N = B_N.$$

It follows

$$A I^{(\nu)} \subset B_N I^{(\nu)};$$

clearly

$$B_N I^{(\nu)} \subset A I^{(\nu)},$$

so that $A I^{(\nu)}$ coincides with $B_N I^{(\nu)}$ for $\nu \geqq 0$, hence

$$\lim_{\nu \to \infty} \frac{m(A I^{(\nu)})}{m(I^{(\nu)})} = 0$$

and this shows that the density of A at α in fact is equal to zero.

This proves the theorem.