Mathematics. — *Non-homogeneous binary quadratic forms.* By H. DAVENPORT. (Communicated by Prof. J. A. SCHOUTEN.)

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1. Let α, β, γ, δ be real numbers with $\Delta = \alpha\delta - \beta\gamma \neq 0$. A famous theorem of MINKOWSKI asserts that for any real numbers λ, μ there exist integers x, y such that

$$|(\alpha x + \beta y + \lambda)(\gamma x + \delta y + \mu)| \leqslant \tfrac{1}{4}|\Delta|.$$

I shall suppose that α/β and γ/δ are irrational; it is then known that the result is true with the sign of strict inequality. If we write

$$(\alpha x + \beta y)(\gamma x + \delta y) = ax^2 + bxy + cy^2 = f(x, y),$$

we can express MINKOWSKI's theorem in the form: if $f(x, y)$ is any indefinite binary quadratic form which does not represent zero, then for any real x_0, y_0 there exist real x, y with

$$x \equiv x_0 \pmod 1, \quad y \equiv y_0 \pmod 1$$

such that [1])

$$|f(x, y)| < \tfrac{1}{4}\sqrt{d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $d = b^2 - 4ac = \Delta^2$.

Many proofs of MINKOWSKI's theorem have been given, but I believe it is still possible to add to the existing knowledge [2]). In the first place, one can easily deduce from the existing proofs slightly more than has been stated above. For any such quadratic form $f(x, y)$ there exists a number $M(f)$ satisfying

$$M(f) < \tfrac{1}{4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

such that, instead of (1), one can satisfy

$$|f(x, y)| \leqslant M(f)\sqrt{d}.$$

I define $M(f)$ to be the lower bound of all such numbers, and the present note is concerned with the investigation of some properties of $M(f)$.

In the first place, I prove an estimate for $M(f)$ in terms of any value of f which satisfies $0 < |f| < \sqrt{d}$.

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[1]) For a positive definite quadratic form, it is easily seen that no result of this type can be valid. The best possible inequality in terms of the coefficients of the equivalent reduced form was given by DIRICHLET (*Werke*, II, 29—48).

[2]) For references to literature, see KOKSMA, *Diophantische Approximationen*. See also MORDELL, *Journal London Math. Soc.*, **16** (1941), 86—88 and **18** (1943), 218—221.

Theorem 1. *Let f_1 be any value of $|f(x, y)|$ which corresponds to co-prime integral values of x, y and which satisfies*

$$0 < f_1 < \sqrt{d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Then

$$M(f) \leq \tfrac{1}{4}\,\Phi\left(\frac{f_1}{\sqrt{d}}\right),$$

where

$$\Phi(t) = \begin{cases} \sqrt{1-4t^2} \text{ for } 0 < t \leq \dfrac{1}{2\sqrt{2}}, \\ \dfrac{1}{4t} \text{ for } \dfrac{1}{2\sqrt{2}} \leq t \leq \tfrac{1}{2}, \\ t \text{ for } \tfrac{1}{2} \leq t < 1. \end{cases}$$

Since $(t) < 1$, this, incidentally, proves (2). The result is best when $|f|$ has a value f_1 which is about $\frac{1}{2}\sqrt{d}$. The existence of *some* value of $|f|$ satisfying (3) is well known from GAUSS's theory of reduction [3]).

The known results (see KOKSMA, 77—79) on non-homogeneous linear forms suggest that (2) is the best possible general inequality for $M(f)$, but this does not seem to have been proved. I give a proof in:

Theorem 2. *If $f(x, y) = x^2 + 2kxy - y^2$, where k is a positive integer, then*

$$M(f) = \frac{k}{4\sqrt{k^2+1}}.$$

Among the most interesting indefinite binary forms are those of MARKOFF's series [4]): $x^2 + xy - y^2$, $x^2 - 2y^2$, $5x^2 + 11xy - 5y^2$, ... The precise values of $M(f)$ for the first two forms are given in the following theorems.

Theorem 3. *If $f(x, y) = x^2 + xy - y^2$, then*

$$M(f) = \frac{1}{4\sqrt{5}}.$$

Theorem 4. *If $f(x, y) = x^2 - 2y^2$, then*

$$M(f) = \frac{1}{4\sqrt{2}}.$$

The second of these results is an immediate consequence of Theorem 1. For $d = 8$, and we can take $f_1 = 1$, whence $M(f) \leq \dfrac{1}{4\sqrt{2}}$. On the other

[3]) See, for example, DICKSON, *Introd. to the theory of numbers*, 101.

[4]) See, for example, BACHMANN, *Die Arithmetik der quadratischen Formen*, II, Kap. 4.

hand, if $x \equiv 0 \pmod{1}$ and $y \equiv \frac{1}{2} \pmod{1}$ then obviously

$$|x^2-2y^2| \geq \tfrac{1}{2} = \frac{1}{4\sqrt{2}}\sqrt{d}, \text{ so that } M(f) \geq \frac{1}{4\sqrt{2}}.$$

The third form of MARKOFF's series presents more difficulty and here I have not yet found the exact value of $M(f)$. A result valid for all the MARKOFF forms is:

Theorem 5. *For any form of* MARKOFF*'s series,*

$$M(f) < \tfrac{1}{4}\sqrt{\tfrac{5}{9}}.$$

2. For the proof of Theorem 1 we need two lemmas [5]).

Lemma 1. *For any real β and x_0, there exists x with $x \equiv x_0 \pmod{1}$ such that*

$$|x^2-\beta^2| \leq \begin{cases} \frac{1}{4}-\beta^2 & \text{if } \beta^2 \leq \frac{1}{8}, \\ \beta^2 & \text{if } \frac{1}{8} \leq \beta^2 \leq \frac{1}{2}, \\ \sqrt{\beta^2-\frac{1}{4}} & \text{if } \beta^2 \geq \frac{1}{2}. \end{cases}$$

P r o o f. (1) If $\beta^2 \leq \frac{1}{8}$, we choose x to satisfy $|x| \leq \frac{1}{2}$, and have $-\beta^2 \leq x^2-\beta^2 \leq \frac{1}{4}-\beta^2$, whence the result.

(2) If $\frac{1}{8} \leq \beta^2 \leq \frac{1}{2}$, we choose x to satisfy

$$\beta\sqrt{2}-1 \leq x \leq \beta\sqrt{2}.$$

Since $1-\beta\sqrt{2} \leq \beta\sqrt{2}$, we have $x^2 \leq 2\beta^2$, whence $|x^2-\beta^2| \leq \beta^2$.

(3) If $\beta^2 \geq \frac{1}{2}$ we choose x to satisfy

$$\sqrt{\beta^2-\tfrac{1}{4}}-\tfrac{1}{2} \leq x \leq \sqrt{\beta^2-\tfrac{1}{4}}+\tfrac{1}{2}.$$

Since the number on the left is positive or zero, we have

$$\beta^2-\sqrt{\beta^2-\tfrac{1}{4}} \leq x^2 \leq \beta^2+\sqrt{\beta^2-\tfrac{1}{4}},$$

whence the result.

Lemma 2. *For any $\alpha > \frac{1}{4}$, and any β with $|\beta| \leq \alpha$, and any x_0, there exists $x \equiv x_0 \pmod{1}$ such that*

$$|x^2-\beta^2| \leq \alpha\, \Phi\left(\frac{1}{4\alpha}\right),$$

where $\Phi(t)$ is the function defined in the enunciation of Theorem 1.

P r o o f. (1) If $\alpha^2 \leq \frac{1}{4}$, then $\beta^2 \leq \frac{1}{4}$, and $\dfrac{1}{4\alpha} \geq \frac{1}{2}$, and

$$\max\left(\tfrac{1}{4}-\beta^2, \beta^2\right) \leq \tfrac{1}{4} = \alpha\, \Phi\left(\frac{1}{4\alpha}\right).$$

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[5]) MORDELL, in *Journal London Math. Soc.*, **3** (1928), 19—22 gave a direct proof of MINKOWSKI's theorem, using an inequality which is a particular case of that proved here.

(2) If $\frac{1}{4} \leqslant \alpha^2 \leqslant \frac{1}{2}$ then $\beta^2 \leqslant \frac{1}{2}$, and $\frac{1}{2\sqrt{2}} \leqslant \frac{1}{4\alpha} \leqslant \frac{1}{2}$, and

$$\max\left(\tfrac{1}{4} - \beta^2, \beta^2\right) \leqslant \alpha^2 = \alpha\, \Phi\left(\frac{1}{4\alpha}\right).$$

(3) If $\alpha^2 \geqslant \frac{1}{2}$ then $\frac{1}{4\alpha} \leqslant \frac{1}{2\sqrt{2}}$, and $\alpha\, \Phi\left(\frac{1}{4\alpha}\right) = \sqrt{\alpha^2 - \frac{1}{4}}$.

If $\beta^2 \leq \frac{1}{8}$, we have

$$\tfrac{1}{4} - \beta^2 \leqslant \tfrac{1}{4} < \sqrt{\alpha^2 - \tfrac{1}{4}}.$$

If $\frac{1}{8} \leq \beta^2 \leq \frac{1}{2}$, we have

$$\beta^2 \leqslant \tfrac{1}{2} \leqslant \sqrt{\alpha^2 - \tfrac{1}{4}}.$$

Finally, if $\beta^2 \geq \frac{1}{2}$, we have

$$\sqrt{\beta^2 - \tfrac{1}{4}} \leqslant \sqrt{\alpha^2 - \tfrac{1}{4}}.$$

Thus Lemma 2 follows from Lemma 1.

Proof of Theorem 1. After an integral unimodular transformation applied to x and y, we can suppose that

$$f(x, y) = ax^2 + bxy + cy^2, \text{ where } |a| = f_1.$$

The conditions $x \equiv x_0 \pmod{1}$, $y \equiv y_0 \pmod{1}$ are transformed into similar conditions. Now

$$f(x, y) = a\left\{(x + \theta y)^2 - \frac{d}{4a^2}\, y^2\right\},$$

where $\theta = \frac{b}{2a}$. We choose y to satisfy $|y| \leq \frac{1}{2}$, then we choose x so that $x + \theta y$ satisfies the inequality of Lemma 2, where

$$\alpha^2 = \frac{d}{16a^2},\ \beta^2 = \frac{d}{4a^2}\, y^2 \leqslant \alpha^2.$$

We obtain

$$|f(x, y)| \leqslant |a|\, \frac{\sqrt{d}}{4|a|}\, \Phi\left(\frac{|a|}{\sqrt{d}}\right) = \tfrac{1}{4}\sqrt{d}\, \Phi\left(\frac{f_1}{\sqrt{d}}\right),$$

as required.

3. To prove Theorem 2 we note first that $M(f) \leq \frac{k}{4\sqrt{k^2+1}}$, on taking $f_1 = 1$ in Theorem 1. Hence it suffices (taking $x_0 = y_0 = \frac{1}{2}$) to prove that

$$\left|(x + \tfrac{1}{2})^2 + 2k\,(x + \tfrac{1}{2})(y + \tfrac{1}{2}) - (y + \tfrac{1}{2})^2\right| \geqslant \tfrac{1}{2} k$$

for all integers x, y, since the quadratic form in the theorem has

$$d = 4(k^2 + 1).$$

This is proved in the following lemma, which is of some interest in itself, though it is probably not new.

Lemma 3. *If k is any positive integer, and x, y are any odd integers, then*

$$|x^2 + 2\,kxy - y^2| \geqslant 2\,k.$$

P r o o f. The result is suggested by the fact that all the convergents to the continued fraction for $\sqrt{k^2+1} - k$ have either numerator or denominator even, so that approximations to this irrational number by fractions $\frac{x}{y}$ with x, y odd are necessarily bad. But it is easy to give a direct proof.

Suppose there exists a solution of

$$|x^2 + 2\,kxy - y^2| < 2\,k, \quad x, y \text{ odd},$$

and consider the solution for which $|y|$ is least. Without loss of generality we can suppose $y > 0$, since otherwise we change the signs of both variables. The inequality can be written

$$|(x + ky)^2 - (k^2 + 1)\,y^2| < 2\,k \quad \dots\dots \quad (4)$$

Put

$$|x + ky| = ky + z, \quad \dots\dots\dots \quad (5)$$

then z is odd. Also $|z| < y$, for if $z \geq y$ we get

$$(x + ky)^2 - (k^2 + 1)\,y^2 \geqslant (k + 1)^2\,y^2 - (k^2 + 1)\,y^2 = 2\,ky \geqslant 2\,k,$$

and if $z \leq -y$ we get

$$(x + ky)^2 - (k^2 + 1)\,y^2 \leqslant (k-1)^2\,y^2 - (k^2 + 1)\,y^2 = -2\,ky \leqslant -2\,k.$$

From (4) and (5),

$$|(-y)^2 + 2\,k\,(-y)\,z - z^2| < 2\,k,$$

and since $|z| < y$ this contradicts the hypothesis that $|y|$ was least.

4. **Lemma 4.** *If* $\frac{1}{\sqrt{5}} \leq y \leq \frac{1}{2}$ *then*

$$\sqrt{\tfrac{5}{4}\,y^2 + \tfrac{1}{4}} + \sqrt{\tfrac{5}{4}\,(y-1)^2 + \tfrac{1}{4}} \geqslant \tfrac{3}{2},$$

and

$$\sqrt{\tfrac{5}{4}\,y^2 - \tfrac{1}{4}} + \sqrt{\tfrac{5}{4}\,(y-1)^2 - \tfrac{1}{4}} \leqslant \tfrac{1}{2}.$$

P r o o f. On squaring both sides, the first inequality becomes

$$5y^2 + 5\,(1-y)^2 + 2 + 2\sqrt{(5y^2 + 1)\,(5\,(1-y)^2 + 1)} \geqslant 9,$$

or, after squaring again,

$$(5y^2 + 1)\,(5\,(1-y)^2 + 1) \geqslant (1 + 5y - 5y^2)^2.$$

The difference is $5(2\,y - 1)^2 \geq 0$.

Similarly, the second inequality is

$$5y^2 + 5(1-y)^2 - 2 + 2\sqrt{(5y^2-1)(5(1-y)^2-1)} \leqq 1,$$

or

$$(5y^2-1)(5(1-y)^2-1) \leqq (5y-5y^2-1)^2,$$

on noting that $5y - y^2$ increases as y increases, so that

$$5y - 5y^2 \geqq \sqrt{5} - 1 > 1.$$

The right hand side exceeds the left hand side by $5(2y-1)^2$.

P r o o f o f T h e o r e m 3. We shall prove that we can find $x \equiv x_0$ (mod. 1) and $y \equiv y_0$ (mod. 1) to satisfy

$$|x^2 + xy - y^2| \leqq \tfrac{1}{4}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and since for this form $d = 5$, the result follows.

We first choose y to satisfy $|y| \leq \frac{1}{2}$. We can suppose without loss of generality that $y \geqq 0$, since otherwise we can put $y = -y'$, $x = x' + y'$. Writing the inequality as

$$|(x + \tfrac{1}{2}y)^2 - \tfrac{5}{4}y^2| \leqq \tfrac{1}{4},$$

we observe first that if $\frac{5}{4}y^2 \leq \frac{1}{4}$ we can choose x so that $x + \frac{1}{2}y$ satisfies the inequality of Lemma 1, and this suffices for our purpose, since

$$\max(\tfrac{1}{4} - \tfrac{5}{4}y^2, \tfrac{5}{4}y^2) \leqq \tfrac{1}{4}.$$

Hence we can suppose $\dfrac{1}{\sqrt{5}} \leq y \leq \frac{1}{2}$. Consider the two intervals

$$\sqrt{\tfrac{5}{4}y^2 - \tfrac{1}{4}} - \tfrac{1}{2}y \leqq x \leqq \sqrt{\tfrac{5}{4}y^2 + \tfrac{1}{4}} - \tfrac{1}{2}y,$$

$$-\sqrt{\tfrac{5}{4}(y-1)^2 + \tfrac{1}{4}} - \tfrac{1}{2}(y-1) \leqq x \leqq -\sqrt{\tfrac{5}{4}(y-1)^2 - \tfrac{1}{4}} - \tfrac{1}{2}(y-1),$$

or say

$$\lambda_1 \leqq x \leqq \lambda_2,$$
$$\lambda_3 \leqq x \leqq \lambda_4.$$

By Lemma 4, we have

$$\lambda_1 \leqq \lambda_4, \quad \lambda_2 \geqq \lambda_3 + 1.$$

Hence these two intervals cover the whole of the interval

$$\lambda_3 \leqq x \leqq \lambda_3 + 1$$

and so we can find a value of $x \equiv x_0$ (mod. 1) in one of them. In the first interval,

$$\tfrac{5}{4}y^2 - \tfrac{1}{4} \leqq (x + \tfrac{1}{2}y)^2 \leqq \tfrac{5}{4}y^2 + \tfrac{1}{4}.$$

In the second interval,

$$\tfrac{5}{4}(y-1)^2 - \tfrac{1}{4} \leqq (x + \tfrac{1}{2}(y-1))^2 \leqq \tfrac{5}{4}(y-1)^2 + \tfrac{1}{4}.$$

Thus in the former case the pair x, y and in the latter case the pair $x, y-1$ satisfy (6).

That (6) is the best possible inequality is obvious on taking $x \equiv \frac{1}{2}$ (mod. 1) and $y \equiv \frac{1}{2}$ (mod. 1).

5. Finally, Theorem 5 is a simple deduction from Theorem 1. Any form of MARKOFF's series has [6]) minimum Q and discriminant $d = 9Q^2 - 4$, where Q is one of the MARKOFF numbers

$$1, 2, 5, 13, 29, 34, 89, \ldots .$$

Hence we can take $f_1 = Q$ and have

$$\frac{f_1}{\sqrt{d}} = \frac{1}{\sqrt{9 - \frac{4}{Q^2}}},$$

so that

$$\tfrac{1}{3} < \frac{f_1}{\sqrt{d}} \leq \frac{1}{\sqrt{5}}.$$

By Theorem 1, since $\Phi(t)$ decreases as t increases for $t < \frac{1}{2}$,

$$M(f) < \tfrac{1}{4}\sqrt{1 - \tfrac{4}{9}},$$

whence the result.

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[6]) BACHMANN, loc. cit. 123.

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