Mathematics. — *A generalization of* TCHEBYCHEF'*s inequality to polynomials in more than one variable.* By C. VISSER. (Communicated by Prof. J. G. VAN DER CORPUT.)

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We shall prove the following generalization of TCHEBYCHEF's inequality to polynomials in an arbitrary number of variables.

If $P(u_1, \ldots, u_h)$ is a polynomial of total degree n in h variables with real coefficients, then

$$\underset{-1 \leqq u_1 \leqq 1, \ldots; -1 \leqq u_h \leqq 1}{\mathrm{Max}} |P(u_1, \ldots, u_h)| \geqq \frac{1}{2^{n-1}} \underset{|u_1| = \ldots = |u_h| = 1}{\mathrm{Max}} |P^*(u_1, \ldots, u_h)|,$$

where P^* denotes the sum of the terms in P whose degree is n.

We put, for real $t_1, \ldots, t_h$,

$$F(t_1, \ldots, t_h) = P\left(\frac{e^{it_1} + e^{-it_1}}{2}, \ldots, \frac{e^{it_h} + e^{-it_h}}{2}\right) =$$

$$= \Sigma a_{k_1 \ldots k_h} e^{i(k_1 t_1 + \ldots + k_h t_h)}.$$

For each term in F we have $-n \leqq k_1 + \ldots + k_h \leqq n$. Further

$$\sum_{k_1 + \ldots k_h = n} a_{k_1 \ldots k_h} e^{i(k_1 t_1 + \ldots + k_h t_h)} = \frac{1}{2^n} P^*(e^{it_1}, \ldots, e^{it_h}),$$

$$\sum_{k_1 + \ldots + k_h = -n} a_{k_1 \ldots k_h} e^{i(k_1 t_1 + \ldots + k_h t_h)} = \frac{1}{2^n} P^*(e^{-it_1}, \ldots, e^{-it_h}).$$

Since

$$\frac{1}{2n} (-1)^p \sum_{p=0}^{2n-1} e^{i\left[k_1\left(t_1 + p\frac{\pi}{n}\right) + \ldots + k_h\left(t_h + p\frac{\pi}{n}\right)\right]} = \begin{matrix} 1 \text{ for } k_1 + \ldots + k_h \equiv n \pmod{2n} \\ 0 \text{ for } k_1 + \ldots + k_h \not\equiv n \pmod{2n}, \end{matrix}$$

we have

$$\frac{1}{2n} (-1)^p \sum_{p=0}^{2n-1} F\left(t_1 + p\frac{\pi}{n}, \ldots, t_h + p\frac{\pi}{n}\right) =$$

$$= \frac{1}{2^n} P^*(e^{it_1}, \ldots, e^{it_h}) + \frac{1}{2^n} P^*(e^{-it_1}, \ldots, e^{-it_h}).$$

The two terms on the right take their maximum absolute values in the same points $(t_1, \ldots, t_h)$, and among these points there is one for which both are positive. It follows that

$$\mathrm{Max}\, |F(t_1, \ldots, t_h)| \geqq \frac{1}{2^{n-1}} \mathrm{Max}\, |P^*(e^{it_1}, \ldots, e^{it_h})|,$$

and since

$$\operatorname{Max} | F(t_1, \ldots, t_h) | = \underset{-1 \leq u_1 \leq 1, \ldots, -1 \leq u_h \leq 1}{\operatorname{Max}} | P(u_1, \ldots, u_h) |,$$

the proof is complete.

There are polynomials for which there is equality. Example:

$$P(u_1, \ldots, u_h) = T_n(u_1) + \ldots + T_n(u_h),$$

where T_n denotes the n^{th} polynomial of TCHEBYCHEF.

We restricted ourselves to polynomials with real coefficients. It will be clear that it would have been sufficient if we had assumed the reality of the coefficients of P^*.