Physical Geography. — *Projective-geometric treatment of* O. LEHMANN'*s theory of the transformation of steep mountain slopes.* By J. P. BAKKER and J. W. N. LE HEUX. (Communicated by Prof. A. PANNEKOEK.)

(Communicated at the meeting of April 27, 1946.)

Geomorphologic science, at its present stage of development, suffers from a lack of quantitative exactness. All too often discussions do not rise above qualitative aspects, with the result that communications, although in themselves most interesting and emanating from prominent authors, frequently remain quantitatively undefined, so that no decision in favour of one view or another can be taken without, in its turn, being to a great extent subjective.

WILLIAM MORRIS DAVIS has enriched geomorphology, in a large number of publications (Nos. 1 and 2) [1], by his brilliant classification of co-existing types of landscapes. As he himself was already aware, the basis of his argument was always partly inductive and partly deductive. To the extent, however, that DAVIS applied the deductive method, it was certainly not pure as such. For DAVIS' purpose was to explain present landscape-types descriptively. He knew in his own mind, from the start, the type of landscape his deduction ought to produce. His argument accordingly directed itself to a definite aim, which endowed it in the first place with a final, not a deductive character. Such a method, whatever its merits, is hardly justified in natural science.

WALTHER PENCK, in his "Morphologische Analyse" (No. 3), attempted further to elaborate the deductive method. His premature death unfortunately prevented him from carrying out this work to its full extent. The idea before his mind's eye, namely the mathematical treatment of the theory of the development of mountain slopes, was never realized. His deductions are largely unacceptable, a fact which has been pointed out before, amongst others by J. P. BAKKER (*vide* Nos. 4, 5 and 6). We shall return to this later, in connection with LEHMANN's theory.

We are, therefore, still faced with the question: is it possible to trace a connection, in a way justified from a quantitatively exact standpoint, between unequal mountain-shapes existing side by side, such as valley- and mountain slopes, longitudinal profiles of rivers, etc.? Only in a few isolated cases, as in the study of the conditions of the formation of meander belts, longitudinal profiles, and deposits in flood plains does the science of physiologic morphology succeed in lifting a tip of the veil, and carrying the statement of the problem a little further (HJULLSTRÖM, LEIGHLY and others).

---

[1]) These numbers refer to the list of literature at the close of this article.

When, however, it is a question of determining alterations in the shape of mountain- and valley-profiles in firm rocks, we are forced to admit that the processes of change take place at such an infinitely slow rate that it is only in very rare exceptions — as, for example, in the case of landslides — that the transition from one mountain-form to another, differing considerably from the first, can be observed directly. In the science of physiognomic morphology, therefore, in which direct evidence *ad oculos* is generally lacking, both the problem itself and the argumentation will necessarily be of a different character from that in physiologic morphology. The question, however, still remains: Is it possible, in physiognomic morphology (leaving aside morpho-tectonic problems), to arrive at a more exact argumentation on a mathematical basis?

In 1932, BAKKER published an article in the Dutch language in which he expressed the opinion that, if the signs were not deceptive, the development of a more theoretical morphology, based upon physico-mathematical treatment, was imminent (Cyclus-theorie en Morphologische Analyse, part 2, p. 17/18). Although doubt was expressed in some quarters as to the correctness of the author's view, a statement confirming it appeared the very next year in the form of OTTO LEHMANN's "Morphologische Theorie der Verwitterung von Steinschlagwänden" (No. 7).

As far as we are aware, OTTO LEHMANN's theory, which bases itself upon a law discovered as early as 1866 by the Rev. OSMOND FISHER (No. 8), and starts from rigorously defined premisses, is the only existing deduction in physiognomic morphology that is quantitatively accounted for. It is, however, merely a beginning; and many problems in which a rigorous deduction would seem possible, were left untouched by LEHMANN.

Apart from the above theory however, another and quite different way to arrive at greater exactness in physiognomic geomorphology might be followed. With the steady increase in the publication of photogrammetrically produced topographical detail-maps the possibility is created of determining with greater exactness the shapes of longitudinal and cross-profiles of valleys and mountain slopes than was hitherto possible. With the aid of nomographic methods it should be possible to open up an entirely new branch of this science, namely, geomorphologic curve-analysis, which may well prove to be able to throw new light upon many of the problems confronting us, such as that of the equilibrium-curves of rivers, about which discussion appears to have come to a deadlock. To this question, too, we shall return later. We shall now present LEHMANN's theory in the following new guise.

In LEHMANN's theory, the example is given of an evenly parallel receding steep mountain slope or cliff FS of known height h, with a great slope-angle β (fig. 1 and 2), bordered at the top by a horizontal plateau SR, and at the foot by an also approximately horizontal form FR', upon which falling angular debris may accumulate.

It is further assumed that the sheer wall FS is exposed exclusively to the free play of weathering, any direct effect of lateral erosion, by rivers, glaciers, etc. being left out of consideration.

Numerous authors, both European and American, have described such steep rock-slopes, in respect of which a parallel recession as a result of weathering approximately may be assumed to have placed during some phase of their development. We know them from the upper mountain ranges of the Alpine type; from the cuesta- and mesa-landscapes of Western Europe and North America; from some "inselbergs" in the arid and semi-arid regions of Africa, South- and East-Asia, America, etc. Observations have also shown that, in those cases where the screes of angular debris are not deposited on the terrace *R'F* suddenly, by a landslide, but little by little, these screes protect a rocky nucleus with a convex cross-profile (*FABCR* in fig. 1); of this nucleus, of which we shall assume that its constitution remains unchanged, therefore, the exact shape has to be determined.

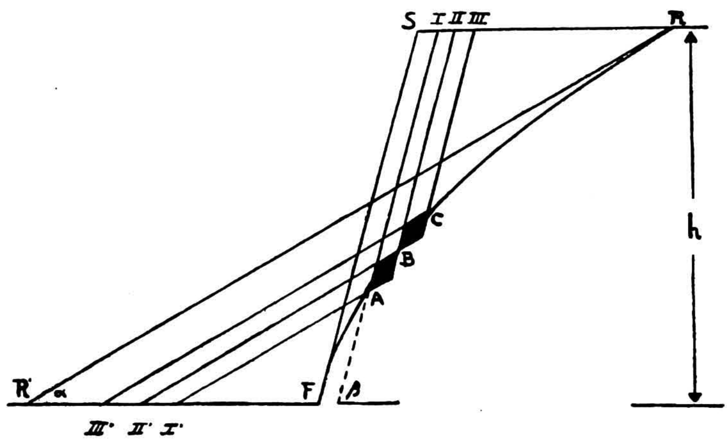

Fig. 1. Parallel recession of the steep slope *FS* and the simultaneous formation of screes [*I'A*, *II'B* etc.]. After the disappearance of the steep slope with angle β, the cross profile *R'R* of the screes also forms the tangent on the curve *FABCR* at the intersection point *R* with the plateau (after OTTO LEHMANN).

As early as 1866, OSMOND FISHER concluded that, if the volumes of ruptured rock per unit of time (i.e., in fig. 1, *SFAI*, *SFBII*, *SFCIII*, etc.) are equal to those of the fragments deposited on *R'F* (i.e., in fig. 1, *I'AF*, *II'FB*, *III'FC*, etc.), the curve *FABCR* will assume a parabolic shape. This is, in fact, quite correct; but it is a special case of a general theory put forward by LEHMANN, in which, e.g., the slope-angle of the cliff was introduced as a variable. Moreover, attention should be given before anything to the ratio between the volume of solid rock removed per unit of time from the steep mountain slope, and the screes volume accumulated at the base during the same time-unit. It is evident that FISHER's case is possible only when part of the fallen fragments is removed from the terrace *R'F* owing to outside influences e.g. by avalanches, since otherwise the respective volumes of rock removed and debris deposited per unit of time, because of the greater pores-volume of the screes, can never be equal. However this may be, in order to arrive at an exact derivation, we shall begin by introducing the formula

$$\frac{\text{rock-volume}}{\text{screes-volume}} = \frac{1-c}{1},$$

in which c is constant.

We further take, following LEHMANN, the basic point of the steep slope F' as zero-point in our co-ordination system, while imagining the stretches of rock and screes, respectively, I, II, III and I', II', III', to be infinitely thin. We may then neglect the black triangles at AB, BC, etc., in fig. 1 and regard the corresponding stretches of fallen rock and deposited debris as parallelograms. We further imagine the wall to be perpendicular to the plane of the drawing, so that two-dimensional figures may suffice, which enables us to replace, in our exposition, the volumes of the corresponding quantities of solid rock and debris by the areas (base $\times$ height) of the respective parallelograms. With the aid of fig. 2 we then get

$$\text{rock-volume} = (1-c) \times \text{screes-volume}$$

$$(dx - dy \cot \beta)(h-y) = (1-c)\left(dy - \frac{dx}{\cot a}\right) y \cot a$$

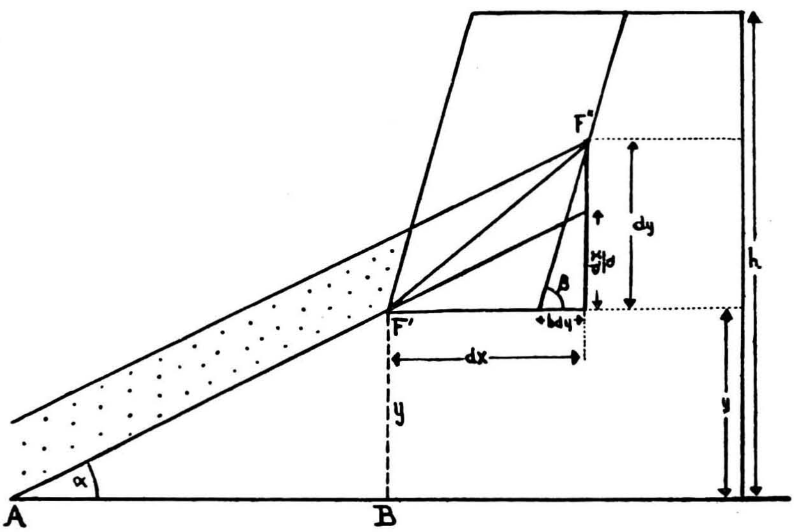

Fig. 2. Diagram from which the differential equation (1) may be obtained. $F'F''$ represents an infinitely small increase in the convex nucleus (after OTTO LEHMANN).

Now putting $\cot \beta = b$, and $\cot a = a$ (co-tangent of the slope-angle of the screes) we get the following differential equation: —

$$(dx - bdy)(h-y) = (1-c)\, ay \left(dy - \frac{dx}{a}\right) \quad . \quad . \quad . \quad . \quad (1)$$

or: —

$$dx = \frac{bh + (a - ac - b)\,y}{h - cy}\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By introducing the new constants

$$k = \frac{a - ac - b}{c}$$

$$l = h\,\frac{b}{a - ac - b}$$

$$m = \frac{h}{c},$$

we get

$$dx = k\,\frac{l + y}{m - y}\,dy = k\left(\frac{l + m}{m - y} - 1\right)dy. \quad . \quad . \quad . \quad . \quad (2a)$$

By integrating this formula we find the relation between x and y in the curve sought: —

$$x = k\,(l + m)\int\frac{dy}{m - y} - k\int dy + A, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which A is the integration constant.

Further, for $x = y = 0$, $A = k(l + m)\,{}^{e}\!\log m$,

$$x = k\,(l + m)\,{}^{e}\!\log\frac{m}{m - y} - ky \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This is a logarithmic curve of a higher order: —

If $\beta = 90^\circ$, or if $\cot\beta = 0$, then $k = a\frac{1 - c}{c}$; $l = 0$, and $m = \frac{h}{c}$

$$x = ah\,\frac{1 - c}{c^2}\,{}^{e}\!\log\frac{h}{h - cy} - a\,\frac{1 - c}{c}\,y \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It follows from this that it is *not* the angle of the wall β (90°) which has changed FISHER's parabola into a logarithmic curve, but the introduction of the constant c. In cases in which, owing to secondary factors, so many fragments have been removed that notwithstanding the greater pores-volume the screes volume is smaller than that of solid rock detached during the same period, it is better to take, for c, a negative number. Formula (5) for $\beta = 90^\circ$ then becomes

$$x = ah\,\frac{1 + c}{c^2}\,{}^{e}\!\log\frac{h}{h + cy} + a\,\frac{1 + c}{c}\,y \quad . \quad . \quad . \quad . \quad (5a)$$

The derivation, for FISHER's parabola, for $c = 0$ and $\beta = 90^\circ$ follows immediately from formula (2): —

$$dx = \frac{a\,y\,dy}{h}$$

$$y^2 = 2\,h\,x\tan\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Another special case arises when $c = -\infty$; from the differential equation (1) it then follows that

$$a\,y\,dy - a\,y\,\frac{dx}{a} = 0$$

$$y = \frac{1}{a}\,x \quad \ldots\ldots\ldots\ldots \quad (7)$$

This equation refers to a straight line having the same slope-angle as the maximal slope of the screes in Nature in the rock-formation in question. This straight line exists already when $x = -100$, which means that in the complete or nearly complete absence of regular accumulation of debris at the base of the disintegrating steep mountain the latter must adopt a less steep slope with a straight-line profile and slope-angle α. We here meet with the phenomenon described as early as 1900 by E. RICHTER (No. 9) under the name of "schiefe Denudationsebene" (slanting denudation-surface), from the crests of the Alpine mountain ranges, and which also occurs frequently in the neighbourhood of the "Schliffgrenze" above the trough-shoulders of former glacier valleys. Such straight-line forms also occur, as is well known in the "island-mount"-landscapes of arid and semi-arid regions.

Since, according to the investigations by PIWOWAR and STINY (Nos. 10 and 11), the slope-angles of the screes, in Nature, may reach a maximum, according to the petrologic conditions of the rock, of between 26° and 43°, we may say that this determines more or less the slope-angle of RICHTER's straight-line denudation slope. We shall return to this point in a further publication.

It is finally possible to prove, that when the upstanding part of the mountain with slope-angle β has completely disappeared and that the screes have thereby reached the level of the top-plateau, their straight-line cross profile forms the tangent on the logarithmic curve at the intersecting point with the plateau.

In this case, $y = h$. According to the differential equation (2), we then get

$$\frac{dy}{dx} = \frac{h - cy}{bh + (a - ac - b)\,y} = \frac{h\,(1-c)}{ah\,(1-c)} = \frac{1}{a} \quad \ldots\ldots \quad (8)$$

This is the tangent of the maximum slope-angle of the screes in Nature, as was already derived by FISHER for his own special case; but we are here dealing with a general law, valid for any slope-angle $\beta > \alpha$ of the steep wall.

Bearing in mind that the phenomena coming within the scope of LEHMANN's theory are examples taken from the upper region of the Alpine mountain range, where, besides the striking straightness of line in many slopes of from 26° to 43°, in crests and peaks, we may also find finely curved lines underneath some of the screes, as in the case of "Drei Zinnen" (fig. 3*a* and 3*b*); and further, that kindred phenomena have been observed in Saxon Switzerland, in cuesta- and mesa-landscapes [2]), in the chalk cliff regions in the South of England and the "island-mounts" land-

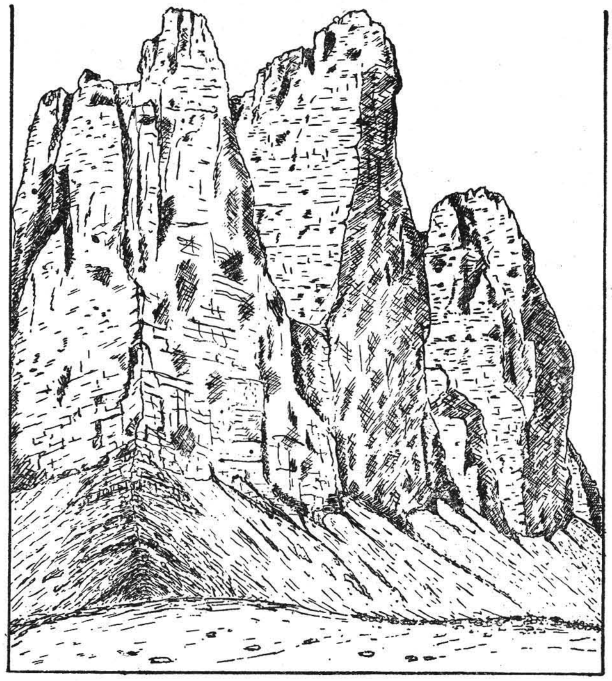

Fig. 3*a*. "Drei Zinnen" in the Ampezza Dolomites near the former Austrian frontier. In the foreground the Paternsattel, where the convex slopes meet in a manner, resembling the theoretic interpretation shown in fig. 3*b*.

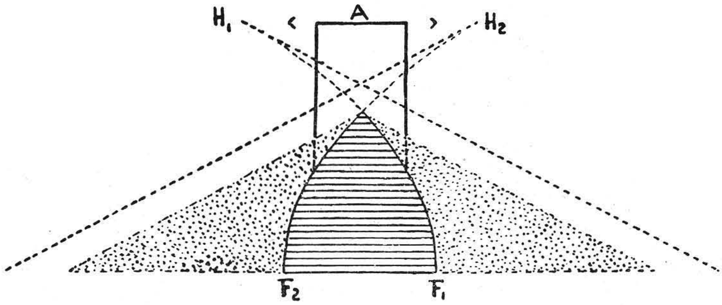

Fig. 3*b*. A narrow plateau *A*, which at first had the width F_1F_2, reduced on both sides by parallel recession. If the plateau had been wider, the screes would have reached the heights H_1 and H_2 (after OTTO LEHMANN).

[2]) Cf. also, with respect to this, the typical but more complicated forms of the Downs in South-England, shown in fig. 38 of ALBERT DE LAPPARENT's „Leçons de Géographie physique", p. 91, Paris (1896).

scapes of the arid and semi-arid territories, then it may not be too bold an assertion to say that all these phenomena comply to the same *general law*, which we propose henceforth to call *the steep mountain transformation law of* FISHER–LEHMANN, and which we shall formulate as follows:

In the case of slow and regular parallel recession of a steep mountain slope protruding above a horizontal form, a rocky nucleus is formed underneath the screes, of which nucleus the cross profile is a logarithmic curve of a higher order, whose form is dependent upon the slope-angle β of the steep mountain, of the slope-angle α of the screes, and of the ratio between the volumes of rock removed and debris deposited (fig. 4 and 5).

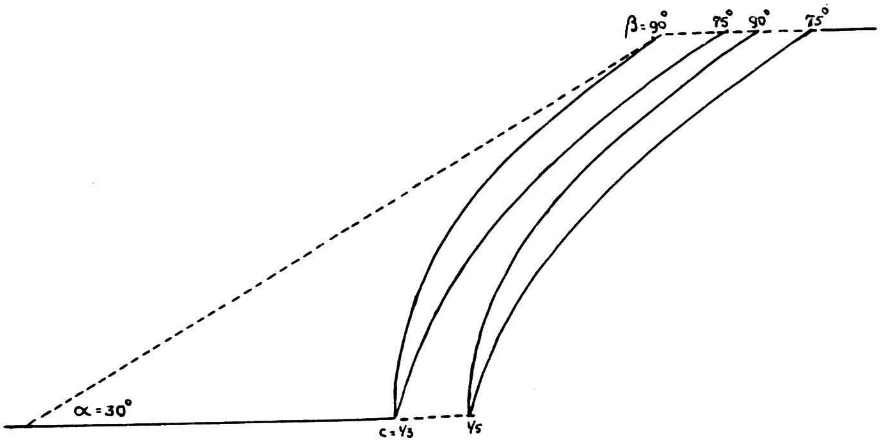

Fig. 4. The influence of the slope-angle β of the initial steep mountain on the curvature of the convex nucleus underneath the screes resp. for $c = \frac{1}{3}$ and $\frac{1}{5}$ and $\alpha = 30°$. As zeropoint of the system of coordinates we invariable took the footpoint of the convex curve (after LEHMANN).

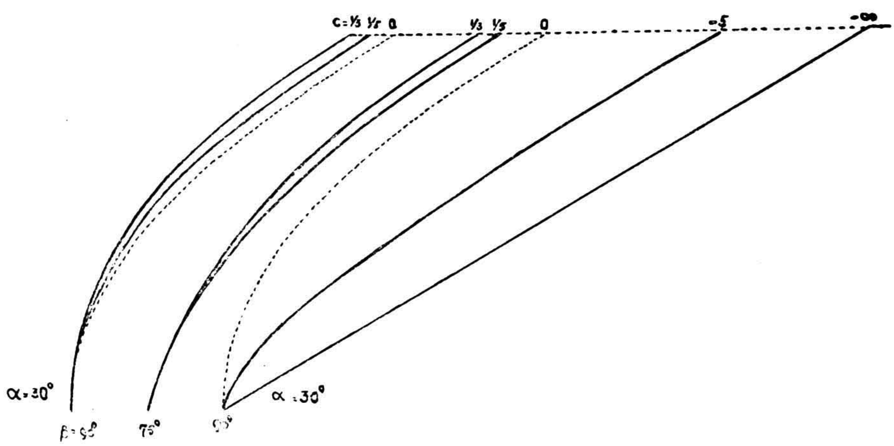

Fig. 5. The influence of the constant c on the form of the convex curves for $\alpha = 30°$ and $\beta = 75°$, $\beta = 90°$. For $c = -\infty$ we get RICHTER's "Denudationsböschung"; for $c = 0$ FISHER's parabola (after LEHMANN).

Of this law, the formulae (6), (7) and (8) are special cases.

Many geomorphologists, little accustomed as they are to an entirely exact treatment of their problems, will be loth to acknowledge the great significance of LEHMANN's theory. They should reflect, however, that the aim of all science is to reduce what is apparently incoherent to one common root-principle, which enables us to survey a vast field, with all its complications, from a central point of view. In the realization of this lay the great merit both of DAVIS and of LEHMANN's theory.

Those who are inclined to doubt the actual value of LEHMANN's theory will probably point out that, in many landscapes, there exist, side by side with forms apparently lending themselves to explanation with the aid of this theory, relief-types which do fit in with it at all. But how variable the premisses may be, even at very small distances! And the last thing we wish to argue is that, with LEHMANN's theory,the last word upon this question has been spoken. On the contrary, it will have to be sounded and elaborated at many points, for LEHMANN's theory is a special application of the more general theory of non-parallel recession of steep mountains; but it constitutes at least a first beginning of a truly exact treatment of physiognomic geomorphology.

When deviating forms are found in Nature, therefore, the theory should not be thrown overboard; in such cases it should be inquired to what extent the premisses have to be modified in order to find an explanation of the deviation found. This point will be made the subject of a number of future publications.

In attempting to explain such deviations in the way indicated, however, we shall find that the form in which LEHMANN has cast his theory does not readily lend itself to this purpose. LEHMANN himself mentions more than once the lengthy and painfully laborious task of constructing the curves [3]), in which he had to use logarithms with as may as 7 decimals so as to obtain smoothly running lines. The fact that he himself produced only very few such curves also points to the same thing.

The science of morphology, however, will in the future have need of a method allowing of a rapid construction of the curves required, thus rendering possible a ready comparison with the phenomena observable in Nature. This may be realized by making use of a diagrammatic method. When, for example, a formula has the general form

$$v = \mu f(x) = \mu \frac{ax + b}{cx + d}$$

---

[3]) With the aid of formula (4) or the equations derived from it, LEHMANN determined 10 points of each curve. He further introduced y as the known factor in tenths of the wall height. Putting $h = 100$, and $y = 10$, 20, etc., we get x expressed in % for any given height of the cliff etc.

As long as $y < 50$, it appeared to be impossible to miss any of the points. For $y > 50$, computation of 2 or 3 points could in some cases be omitted.

in which μ is the modulus, and a, b, c and d, the constants, then it will lend itself to the use of a projective scale (*vide* No. 12).

It will be clear that the form of our differential equation (2),

$$\frac{dx}{dy} = \frac{bh + \{(1-c)\,a - b\}\,y}{h - cy} = w$$

satisfies this condition. This means that the co-tangents of the angles formed by the tangents on the profile sought, with the X-axis, are obtained by projective transformation of the regular Y-scale.

To this end three points of the projective scale sought are first fixed on the horizontal W-axis (fig. 6). After this the Y-axis, to be divided up regularly, is drawn through one of these points in an arbitrary direction, after which the projection-centre C is determined, from which the dividing points of the Y-axis are projected upon the W-axis.

We shall choose, as given points, B, $H(\infty)$ and $W(\infty)$. The points B and H are found by plotting, from an arbitrarily chosen starting point E, the distances $EB = b$ (for $y = 0$, hence numbered 0) and $EH = \frac{(1-c)\,a-b}{-c}$ (for $y = \infty$, hence numbered ∞).

We now draw the y-axis through the zero point B. In this case this is done perpendicular upon the W-axis. The point K is then determined, so that $BK = \frac{h}{c}$ (for $W = \infty$).

The projection centre C is now found by connecting the similarly numbered points of the W- and the Y-axis, which is easily done in this case by drawing, from H and K, lines parallel to the axes.

In this instance we have selected the following:

$$a = \cot 30^\circ = 1{,}73205;$$
$$b = \cot 75^\circ = 0{,}26795;$$
$$c = \tfrac{1}{3}.$$

We then find: $EH = -2{,}66$, and $\frac{h}{c} = 3h$.

The Y-scale is now projected from C upon the W-scale. This also implies that the co-tangents of the angles of the tangents on the profile sought with the W-axis, are found by measuring the distances from the points of the projective scale BA to the point E.

The angles themselves are found by drawing a circle with the radius 1, of which EA is the line of the co-tangents. This may be done, for example, by giving to $\frac{dx}{dy}$, in the formula (2), the value 1, so that LB obtains the value $0{,}60 \ldots h$. By projecting L on the W-axis we get the point N, numbered 1, while the centre point of the circle with radius 1 is found by plotting a perpendicular line in E, and giving to EM the value EN.

M may also be determined, of course, as the intersecting point of MB

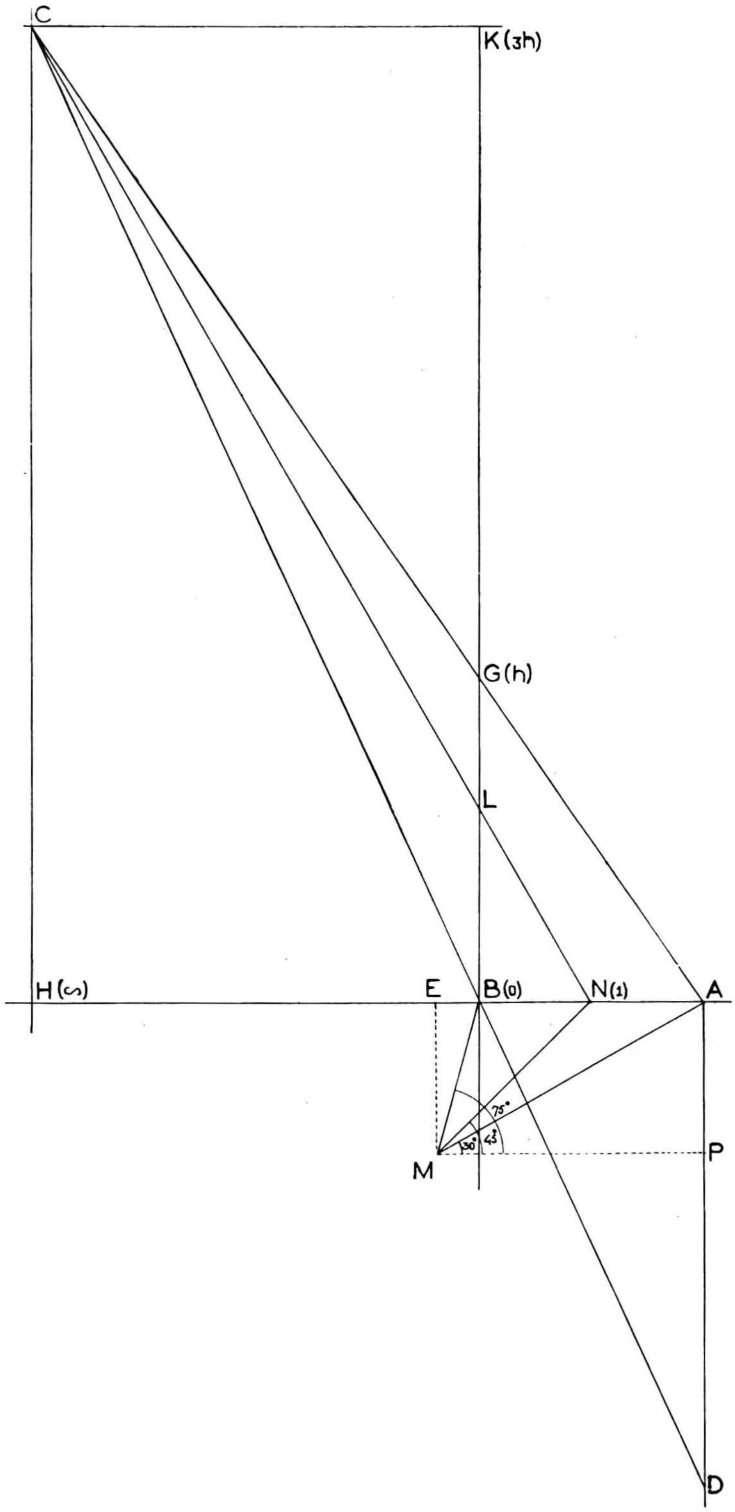

Fig. 6. See text.

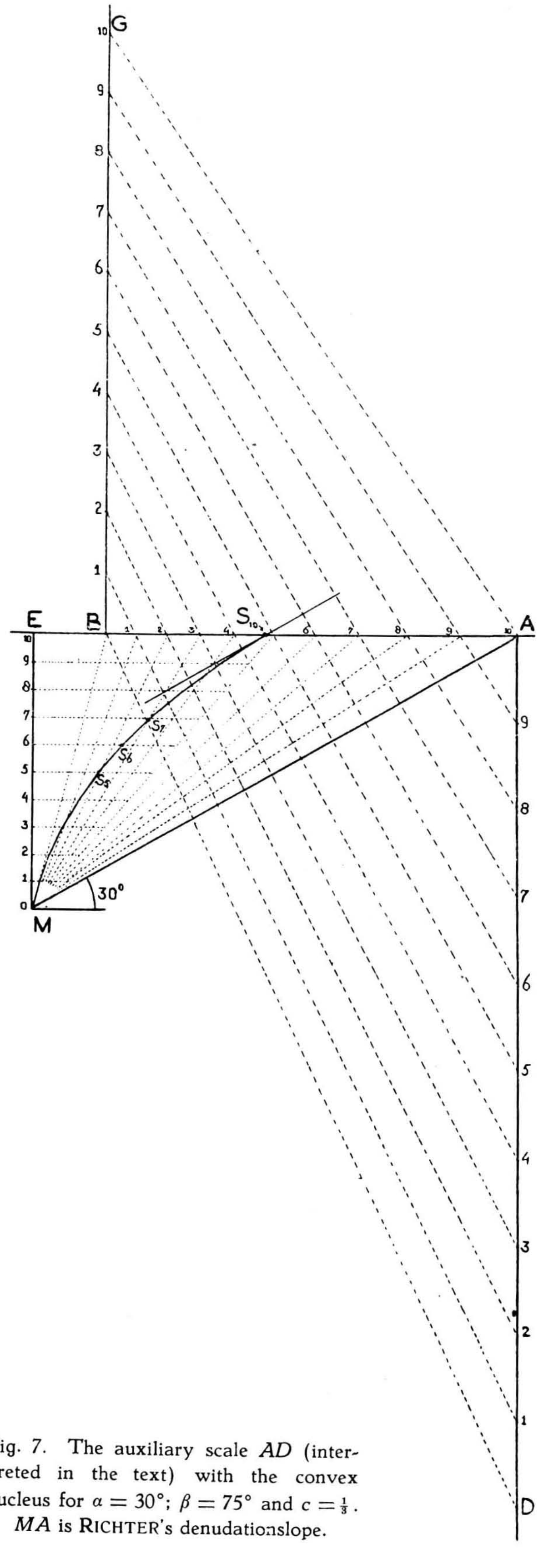

Fig. 7. The auxiliary scale AD (interpreted in the text) with the convex nucleus for $\alpha = 30°$; $\beta = 75°$ and $c = \frac{1}{3}$. MA is RICHTER's denudationslope.

and MA, which must respectively form angles of 75° and 30° with the W-axis.

Between MB and MA lie the directions of all tangents on the profile sought. *This profile, therefore, must turn its convex side to the W-axis, intersecting the plateau above (in fig. 7 and 8, this plateau is, therefore, $S_{10}A$), at an angle α.*

The solution of the differential equations (1) and (2) is not required in order to arrive at this conclusion. Neither is this solution necessary to enable us to sketch the profile with sufficient exactness. For, the differential equation (2) gives the direction of the tangent at a point whose ordinate is known. As the starting point of our co-ordination system coincides with the basic point of the steep mountain slope with its straight-line profile, from which we started, we also know that to an ordinate zero belongs a direction angle $\beta = 75°$ of the new curve.

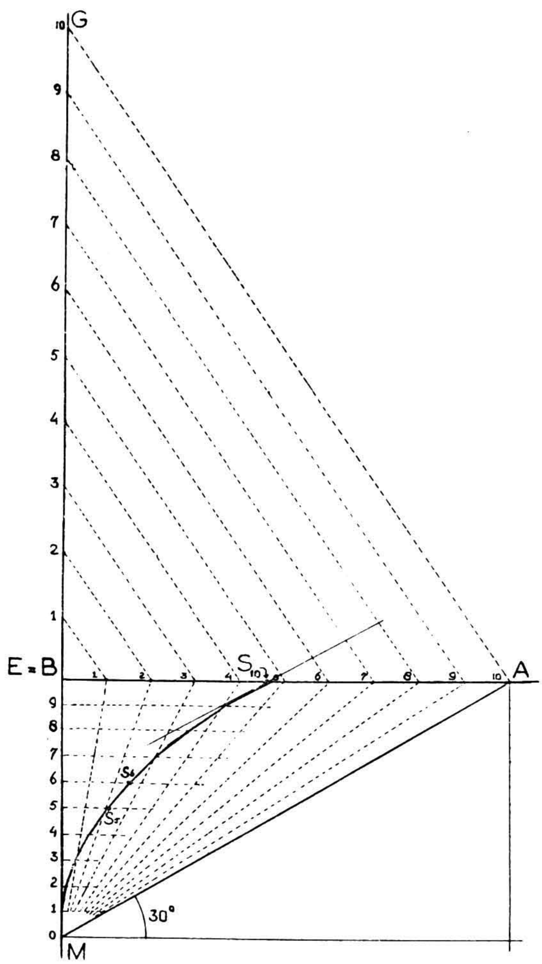

Fig. 8. The auxiliary scale used to obtain FISHER's parabola for $\alpha = 30°$, $\beta = 90°$, $c = 0$.

Now in order further to construct the curve, we must bear in mind that, in fig. 7, showing the (enlarged) projective scale BA, the intersecting point of the above-mentioned first tangent with the line $y = 1$ will not deviate much from the point with the ordinate 1 on the curve sought; the less so according as the parts of the y-axis are smaller. We may, therefore, draw a straight line through this intersecting point S_1, parallel to the direction $M - 1$; at the intersecting point S_2, thus obtained, a line parallel to the direction $M - 2$, and so on [1, 2 etc. are the points of the scale BA].

If this degree of exactitude is not considered sufficient, one may, instead of dividing the auxiliary scale AD into ten parts, divide it into twenty. Since, however, the irregularities occurring in Nature, will probably be of a greater order of magnitude, this will hardly be necessary in most cases.

From fig. 6 it follows at once, with the aid of similar triangles, that the auxiliary scale AD is equal to $1\frac{1}{2}$ GB, or put in a more general form,

$$AD = \frac{1}{1-c} GB.$$

The advantages of the method followed, apart even from the fact that it absolves one from the repeated solution of the differential equation, do not require much further comment.

With the aid of figures such as 6, 7 and 8 it becomes an easy matter to ascertain the influence which any change in α, β and c will have on the shape of the profile curve.

MA is, naturally, a case of RICHTER's "schiefe Denudationsebene", for $c = -\infty$ (i.e. $AD = 0$).

When $\beta = 90^\circ$, and, therefore, $b = 0$, the points B and E will coincide, which, as we have seen, was one of the conditions of the appearance of FISHER's parabola sensu strictu; whilst $AD = GB$, so that the distance BA may be divided regularly in the ordinary way (fig. 8).

Further details concerning this subject will be discussed in the course of future publications.

We are indebted to Mr. A. J. WIGGERS, assistant for phys. Geography at the University of Amsterdam, for his kindness in preparing the figures for the press.

LITERATURE.

1. DAVIS, W. M., Geographical essays (Edition Douglas Johnson) Boston (1909).
2. ———————, Die erklärende Beschreibung der Landformen (Ed. A. Rühl) Leipzig (1912 and 1924).
3. PENCK, W., Die Morphologische Analyse. Stuttgart (1924).
4. BAKKER, J. P., Over Piedmonttrappen (a critical essay on the piedmontbenchland theories of W. Penck and H. Spreitzer). Tijdschrift Koninklijk Nederlandsch Aardrijkskundig Genootschap, Leiden **51**, 303—329 (1934).
5. ———————, Morphologische Untersuchungen im zentralen und nordöstlichen Morvan (Zentral Frankreich). Zeitschr. d. Gesellschaft f. Erdkunde, Berlin, p. 89—132 (especially p. 91—95 and p. 99—100) (1937).
6. MAYER, ROBERT, Ueber Piedmonttreppen, Zeitschrift f. Geomorphologie, **10**, 113—115 (1937).

7. LEHMANN, OTTO, Morphologische Theorie der Verwitterung von Steinschlagwänden. Vierteljahrsschrift der Naturforschende Gesellschaft in Zürich, p. 83—126 (1933).
8. FISHER, OSMOND, On the disintegration of a chalk cliff. Geolog. Mag. **3**, 354—356 (1866).
9. RICHTER, ED., Geomorphologische Untersuchungen in den Hochalpen. Petermanns Mitteilungen. Gotha, Ergänzungsband **24**, 1—103 (1901).
10. PIWOWAR, A., Ueber Maximalböschungen trockener Schuttkegel und Schutthalden. Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich, p. 335—339 (1903).
11. STINY, J., Neigungswinkel von Schutthalden, Zeitschrift f. Geomorphologie, **1**, 60—61 (1925—'26).
12. LE HEUX, J. W. N., Beginselen der Nomografie. Deventer (1942).