Applied Mechanics. — *The generalized buckling problem of the circular ring.* By C. B. BIEZENO and J. J. KOCH.

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1. *Introduction.* It is well known, that a circular ring, subjected to a uniform radial pressure q per unit of circumferential length is apt to buckle under the action of one of the so-called "critical" loads $q = (n^2—1) EI/r^3$ (n integer and $\geqq 2$), EI representing the flexural rigidity of the ring and r its radius. This case of buckling is analogous to the buckling of a straight rod under the action of two compressive forces, as far as the cross-sections of both ring and rod are loaded by a normal force of constant magnitude. If a straight rod is loaded by a prescribed system of axial forces, so that the normal force of the cross-section varies with its coordinate, proportional increase of the loadsystem, say to the multiple λ, leads as well to critical buckling loads. The (positive or negative) value of the factor of magnification λ can best be found by a method of iteration [1]). It is obvious, that for the circular ring the analogous problem exists, if only it is subjected to an external loadsystem such that in every crossection of the ring the bending moment M and the shearing force D are zero, whereas the normal force of the section varies with its coordinate. Evidently the first of these conditions will not be fulfilled if the ring is loaded by an arbitrary system, of radial and tangential forces; but is will be shown in section **2** that every loadsystem can be split up into two components A and B, the first of which will be called the "compressive" system, because it is characterised by $M = D = 0$, whereas the second one will be called the "bending" system, characterized as it is by $N = 0$.

The first system A, if suitably magnified, leads to the generalized buckling problem of the circular ring, which will be treated in this paper, but it is seen at once, that an arbitrary loadsystem, consisting of both components A and B, gives rise to a problem, which again is analogous to a well-known problem, viz. the straight rod subjected to axial thrust and transverse bending loads. Just as well as with the straight rod the axial forces tend to increase the deflections caused by the transverse loads, the A loadsystem of the circular ring will affect the deflections due to the B-system. In a subsequent paper this latter question will be treated in connection with a particular problem, which led the authors to the present investigation, and which itself will be treated in a third communication.

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[1]) Comp. f. i. C. B. BIEZENO und R. GRAMMEL, Technische Dynamik, Chapt. VII, 7, 509 (Springer 1939).

2. *The A and B-system.* If a ring is subjected to radial and tangential loads q and t per unit of circumferential length the equilibrium of a ringelement requires

$$\left.\begin{array}{lll} N\,d\varphi - qr d\varphi - dD = 0 & & N - D' = qr \\ dN + D d\varphi + tr d\varphi = 0 & \text{resp.:} & N' + D = -tr \\ dM + r dN + tr^2\,d\varphi = 0 & & M' + rN' = -tr^2 \end{array}\right\} \quad . \quad . \quad (1)$$

(φ denoting the angular coordinate of the ringelement under consideration). The requirement $D = M = 0$ leads to

$$N = qr \quad , \quad N' = -tr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

from which it follows, that a A-loadsystem is characterized by

$$t = -q'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

At the other hand the condition $N = 0$ requires

$$-D' = qr \qquad D = -tr \qquad M' = -tr^2 \quad . \quad . \quad . \quad . \quad (4)$$

from which it is seen, that a B-loadsystem is characterized by

$$q = t' \; ^{2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It can easily be shown that any arbitrary equilibrium loadsystem (q, t) can be decomposed in a unique way into a A-system (q^*, t^*) and a B-system (q^{**}, t^{**}). Let all quantities be expanded into their Fourier-series, so that:

$$q = a_0 + \sum_1^\infty a_k \cos k\varphi + \sum_1^\infty b_k \sin k\varphi,\; t = c + \sum_1^\infty c_k \cos k\varphi + \sum_1^\infty d_k \sin k\varphi \quad (6)$$

$$q^* = a_0^* + \sum_1^\infty a_k^* \cos k\varphi + \sum_1^\infty b_k^* \sin k\varphi,\; t^* = c_0^* + \sum_1^\infty c_k^* \cos k\varphi + \sum_1^\infty d_k^* \sin k\varphi \quad (7)$$

$$q^{**} = a_0^{**} + \sum_1^\infty a_k^{**} \cos k\varphi + \sum_1^\infty b_k^{**} \sin k\varphi,\; t^{**} = c_0^{**} + \sum_1^\infty c_k^{**} \cos k\varphi + \sum_1^\infty d_k^{**} \sin k\varphi(\quad (8)$$

then it must be remarked beforehand, that in consequence of the equilibrium-conditions

$$\int_0^{2\pi} (q \sin\varphi + t\cos\varphi)\, r\,d\varphi = 0,\; \int_0^{2\pi} (q\cos\varphi - t\sin\varphi)\, rd = 0,\; \int_0^{2\pi} tr^2\,d\varphi = 0 \quad (9)$$

the following relations hold

$$b_1 + c_1 = 0 \quad a_1 - d_1 = 0 \quad c_0 = 0. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Analogously the conditions

$$b_1^* + c_1^* = 0 \qquad a_1^* - d_1^* = 0 \qquad c_0^* = 0 \quad . \quad . \quad . \quad . \quad (11)$$

$$b_1^{**} + c_1^{**} = 0 \qquad a_1^{**} - d_1^{**} = 0 \qquad c_0^{**} = 0 \quad . \quad . \quad . \quad . \quad (12)$$

must be fulfilled.

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[2]) Strictly spoken it must be proved that inversely (3) leads to $D = M = 0$, and (4) to $N = 0$. This may be left to the reader.

Furthermore we have, in consequence of $(q, t) \equiv (q^*, t^*) + (q^{**}, t^{**})$

$$a_0^* + a_0^{**} = a_0 \qquad a_k^* + a_k^{**} = a_k \qquad b_k^* + b_k^{**} = b_k \quad . \quad . \quad (13)$$

$$c_0^* + c_0^{**} = c_0 \qquad c_k^* + c_k^{**} = c_k \qquad d_k^* + d_k^{**} = d_k \quad . \quad . \quad (14)$$

in consequence of $t^* = -q^*$:

$$c_0^* = 0 \qquad c_k^* = -k\, b_k^* \qquad d_k^* = k\, a_k^* \quad . \quad . \quad . \quad . \quad . \quad (15)$$

in consequence of $q^{**} = t^{**}$:

$$a_0^{**} = 0 \qquad -k\, c_k^{**} = b_k^{**} \qquad k\, d_k^{**} = a_k^{**}. \quad . \quad . \quad . \quad . \quad (16)$$

From these equations we deduce, having due regard to (10), (11) and (12)

$$\left.\begin{aligned}
& a_0^* = a_0,\ a_0^{**} = c_0^* = c_0^{**} = 0 \\
& a_1^* = a_1^{**} = \tfrac{1}{2} a_1,\ b_1^* = b_1^{**} = \tfrac{1}{2} b_1,\ c_1^* = c_1^{**} = -\tfrac{1}{2} b_1,\ d_1^* = d_1^{**} = \tfrac{1}{2} a_1 \\
& a_k^* = -\frac{1}{k^2-1} a_k + \frac{k}{k^2-1} d_k \qquad b_k^* = -\frac{1}{k^2-1} b_k + \frac{k}{k^2-1} c_k \\
& a_k^{**} = \frac{k^2}{k^2-1} a_k - \frac{k}{k^2-1} d_k \qquad b_k^{**} = \frac{k^2}{k^2-1} b_k + \frac{k}{k^2-1} c_k \\
& d_k^* = -\frac{k}{k^2-1} a_k + \frac{k^2}{k^2-1} d_k \qquad c_k^* = \frac{k}{k^2-1} b_k + \frac{k^2}{k^2-1} c_k \\
& d_k^{**} = \frac{k}{k^2-1} a_k - \frac{1}{k^2-1} d_k \qquad c_k^{**} = -\frac{k}{k^2-1} b_k - \frac{1}{k^2-1} c_k
\end{aligned}\right\} \quad (17)$$

Therefore the A and B components of the loadsystem (q, t) are represented by

$$\left.\begin{aligned}
q^* &= a_0 + \tfrac{1}{2} a_1 \cos\varphi + \sum_2^\infty \left[\frac{-1}{k^2-1} a_k + \frac{k}{k^2-1} d_k\right] \cos k\varphi + \tfrac{1}{2} b_1 \sin\varphi + \\
&\qquad + \sum_2^\infty \left[-\frac{1}{k^2-1} b_k - \frac{k}{k^2-1} c_k\right] \sin k\varphi \\
t^* &= \tfrac{1}{2} b_1 \cos\varphi + \sum_2^\infty \left[\frac{k}{k^2-1} b_k + \frac{k^2}{k^2-1} c_k\right] \cos k\varphi + \tfrac{1}{2} a_1 \sin\varphi + \\
&\qquad + \sum_2^\infty \left[-\frac{k}{k^2-1} a_k + \frac{k^2}{k^2-1} d_k\right] \sin k\varphi
\end{aligned}\right\} \quad (18a)$$

$$\left.\begin{aligned}
q^{**} &= \tfrac{1}{2} a_1 \cos\varphi + \sum_2^\infty \left[\frac{k^2}{k^2-1} a_k - \frac{k}{k^2-1} d_k\right] \cos k\varphi + \tfrac{1}{2} b_1 \sin\varphi + \\
&\qquad + \sum_2^\infty \left[\frac{k^2}{k^2-1} b_k + \frac{k}{k^2-1} c_k\right] \sin k\varphi \\
t^{**} &= -\tfrac{1}{2} b_1 \cos\varphi + \sum_2^\infty \left[-\frac{k}{k^2-1} b_k - \frac{1}{k^2-1} c_k\right] \cos k\varphi + \tfrac{1}{2} a_1 \sin\varphi + \\
&\qquad + \sum_2^\infty \left[\frac{k}{k^2-1} a_k - \frac{1}{k^2-1} d_k\right] \sin k\varphi
\end{aligned}\right\} \quad (18b)$$

In the following sections attention will only be paid to A-load systems.

3. *The differential equation of the buckling ring.* The differential equation of a bent circular ring, the central line of which does not change its length is represented by

$$\frac{1}{\varrho}-\frac{1}{r}=\frac{Mr^2}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

($1/r$ = curvature of the unbent ring, $1/\varrho$ = curvature of the bent ring; M = bending moment, positive if it increases the curvature of the ring; EI = flexural rigidity of the ring). From geometrical deductions it follows that

$$\frac{1}{\varrho}-\frac{1}{r}=-\frac{u''+u}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

u representing the increase of the radius vector r. From eqs. (1) it follows, by eliminating N and D:

$$M''+M'=-(q'+t)\,r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and from (21) and (19) we find

$$(u''+u)'''+(u''+u)'=\frac{(q'+t)\,r^4}{EI} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

This equation can be applied to a buckling ring subjected to a critical A-loadsystem (q^*, t^*) if only q and t be replaced by those supplementary loadcomponents to which — by the deformation of the ringelement — the prevailing external and internal loads $q^*rd\varphi$, $t^*rd\varphi$, N and $N+dN$ give rise. Two different cases must be distinguished, according as the external loads q^* and t^* keep their directions with respect to the ring-element or with respect to a fixed system of coordinates. In this paper we confine ourselves to the first one, so that q has to be replaced by $-N\left(\frac{1}{\varrho}-\frac{1}{r}\right)=N(u''+u)/r^2$ (comp. 20) and t by zero.

Consequently eq. (22) changes into:

$$(u''+u)'''+(u''+u')=\frac{[N(u''+u)]'r^2}{EI} \quad . \quad . \quad . \quad . \quad (23)$$

If we denote $u''+u$ by U, eq. (23) can be replaced by

$$U''+U=\frac{Nr^2}{EI}U+C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where C represents a constant of integration to be found in the course of our investigation.

Both quantities N and U are periodic functions of φ with the period 2π, so that they may be expanded into Fourier-series:

$$\left.\begin{aligned} N = \lambda N_0 = \lambda \left[A_0 + \sum_{k=1}^{\infty} A_k \cos k\varphi + \sum_{k=1}^{\infty} B_k \sin k\varphi \right] \\ U = a_0 + \sum_{l=1}^{\infty} a_l \cos l\varphi + \sum_{l=1}^{\infty} b \sin l\varphi \end{aligned}\right\} \quad . \quad . \quad (25)$$

These expressions, however, are subject to some distinct restrictions, which must be discussed beforehand.

Firstly it must be remembered that we have to deal with a closed ring, so that — if the ring was cut at say $\varphi = 0$, and if the disturbed internal forces were restored — neither relative displacements nor relative rotation of the opposite ends of the ring would occur. These conditions are analytically expressed by the equations

$$\int_0^{2\pi} \frac{M}{EI} r d\varphi = 0, \quad \int_0^{2\pi} \frac{Mr \sin\varphi}{EI} r\, d\varphi = 0, \quad \int_0^{2\pi} \frac{Mr(1-\cos\varphi)}{EI} r d\varphi = 0 \quad . \quad . \quad (26)$$

relating to well-known properties of the so-called "reduced" bending moment M/EI. As, according to (19) M/EI is proportional to $u'' + u = U$, we find

$$\int_0^{2\pi} U\, d\varphi = 0, \qquad \int_0^{2\pi} U \sin\varphi\, d\varphi = 0, \qquad \int U \cos\varphi\, d\varphi = 0 \quad . \quad . \quad (27)$$

from which it follows that

$$a_0 = a_1 = b_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

All further restrictions (relating to N) follow from the fact, that (24) must *identically* be fulfilled if N and U are replaced by the expressions (25). To control this required identity the product NU, occurring at the right hand side of (24), evidently must be written in terms of cos and sin of multiples of φ. In doing so it is seen at once, that no terms A_1, $\cos\varphi$ and $B_1 \sin\varphi$ in N are admitted if U contains the terms $a_2 \cos 2\varphi$ and $b_2 \sin 2\varphi$; otherwise the product NU would give rise to terms $\cos\varphi$ and $\sin\varphi$ in the right hand side of (24), which in virtue of (28) are missing in the lefthand side. The presence of $A_1 \cos\varphi$ and $B_1 \sin\varphi$ would require the disappearance of $a_2 \cos 2\varphi$ and $b_2 \sin 2\varphi$ in U, so that the Fourier-series for U should have to start with $a_3 \cos 3\varphi$ and $b_3 \sin 3\varphi$. However, the product of these terms with $A_1 \cos\varphi$ and $B_1 \sin\varphi$ would in the righthand side of (24) now produce terms with $\cos 2\varphi$ and $\sin 2\varphi$ which, in consequence of the preceding remark, are absent in the lefthand side. Proceeding in this way *all* terms of U must disappear, and therefore we conclude, that no equilibrium position of the ring, different from the circular one, exists if A_1 and B_1 are different from zero; any departure of the circular shape sets the ring into motion.

Moreover all multiples of φ_1 occurring in the remaining terms of N must have a factor, different from one, in common. The statement is proved by showing first, that the presence of two terms cos $k_1\varphi$ and cos $k_2\varphi$, k_1 and k_2 having no factor in common, leads to an absurdity. It is seen at once, that in such a case U is deprived from its terms cos $(k_1 \pm 1)\varphi$, sin $(k_1 \pm 1)\varphi$, cos $(k_2 \pm 1)\varphi$, sin $(k_2 \pm 1)\varphi$, because otherwise the righthand side of (24) would contain the terms cos φ and sin φ, which do not occur in the left hand side. But then terms of the type cos $(2k_1 \pm 1)\varphi$, sin $(2k_1 \pm 1)\varphi$, cos $(2k_2 \pm 1)\varphi$, sin $(2k_2 \pm 1)\varphi$ are as well forbidden terms for U; if such terms would exist, the righthand side of (24) would contain terms cos $(k_1 \pm 1)$ a.s.o. which — after the preceding statement — do not occur in the lefthand side. Proceeding in this way, we find that all terms cos $(\alpha k_1 \pm 1)\varphi$, sin $(\alpha k_1 \pm 1)\varphi$, cos $(\beta k_2 \pm 1)\varphi$, sin $(\beta k_2 \pm 1)\varphi$, (α and β representing arbitrary pos. or neg. integers) are forbidden terms for U. A similar reasoning leads to the conclusion firstly, that terms of the type cos $(\alpha k_1 \pm k_2 \pm 1)\varphi$, sin $(\alpha k_1 \pm k_2 \pm 1)\varphi$ are inadmissable, then that all terms cos $(\alpha k_1 \pm \beta k_2 \pm 1)\varphi$, sin $(\alpha k_1 \pm \beta k_2 \pm 1)\varphi$ must be excluded in U (α and β representing arbitrary pos. or neg. integers). But this means that *all* terms in U have to be excluded because of the fact that any integer may be written as $\alpha k_1 + \beta k_2$, provided only that k_1 and k_2 have no factor in common. Again, if N should contain the terms cos $k_1\varphi$, cos $k_2\varphi$, cos $k_3\varphi_1$ — k_1, k_2, k_3 having no factor in common — all terms cos resp. sin $(\alpha k_1 + \beta k_2 + \gamma k_3 \pm 1)\varphi$ (α, β, γ pos. or neg. integers) would be excluded from U, but this again would mean that all terms of U should vanish; a.s.o. Our final conclusion with respect to the possibility of buckling of the ring therefore is:

1°. All multiples of φ in the Fourier series of N must have a factor in common.

2°. If the greatest factor in common of all these multiples is called p, the terms to be excluded from U are a_0, $a_1 \cos\varphi$, $b_1 \sin\varphi$, $a_{\alpha p \pm 1}$ cos $(\alpha p \pm 1)\varphi$, $b_{\alpha p \pm 1}$ sin $(\alpha p \pm 1)\varphi$, α representing any integer pos. number $\neq 0$.

One way in which we could now pursue the solution of eq. (24) would consist in substituting the expressions (25) — liable to restrictions laid upon them — into (24), and by identifying the corresponding terms of both sides of this equation; this would lead to an infinite system of recurrent relations between the coefficients a_l and b_l. Though in fact we do not intend to follow this way, one useful remark, to which this method would lead us, must be made, viz. that the system of relations just mentioned breaks up into a number of minor systems, each of which relates to a distinct class of coefficients a_l, b_l, l being defined by

$$l \equiv \pm q, \bmod p \qquad \left.\begin{array}{ll} q = 0, 2, 3 \ldots \frac{p-1}{2}, & p \text{ odd} \\ q = 0, 2, 3 \ldots \frac{p}{2}, & p \text{ even} \end{array}\right\} \quad . \quad . \quad . \quad (29)$$

(If $q = 0$, the value $l = 0$ obviously must be excluded). With any prescribed p our buckling problem therefore is split up into $\frac{p-1}{2}$ or $\frac{p}{2}$ cases in every one of which U is composed exclusively of terms relating to one of the congruences (29).

We conclude this section by a remark with respect to the constant of integration C, occurring in eq. (24). If the product NU in the righthand-side of this equation happens to miss a constant term after being written as an ordinary Fourier-series, then C must be suppressed; if not so C serves to annul this term.

4. *The integral equation of the problem.* In the next sections up from sect. **5**, eq. (24) will be solved in an iterative way. The justification of this method, however, makes it desirable to express our problem in terms of an integral equation, which therefore will be deduced first. To this end we apply to the ring an equilibrium loadsystem consisting of a concentrated force P of unit magnitude, acting at the fixed point (ψ) and a complementary continuous radial load q_φ, which at any point (φ) of the ring amounts to $q_\varphi = -P \cos(\varphi - \psi)/\pi r$. The bending moment at the point φ of the ring, due to this loadsystem is called $K(\varphi, \psi)$. It has been stated in sect. **3** that the deformation of the buckling ring must be maintained by the continuous radial load $\lambda N_0(\psi)\, U(\psi)/r^2$, where $\lambda N_0(\psi)$ stands for the normal force in the cross-section (ψ) and $-U(\psi)/r^2$ for the local change in curvature. If every infinitesimal part $\lambda N_0(\psi)\, U(\psi)\, r d\psi/r^2$ of this load (acting on the ring element $r d\psi$) is supplemented by a continuous load all over the ring of the specific amount $-\lambda N_0(\psi)\, r d\psi/r^2 - \cos(\varphi - \psi)/\pi r$, then the total bending moment M_φ at the angle φ amounts — in accordance to the just given definition of $K(\varphi, \psi)$ — to

$$M_\varphi = \int_0^{2\pi} \frac{\lambda\, N_0(\psi)\, U(\psi)\, K(\varphi, \psi)}{r}\, d\psi \quad . \quad . \quad . \quad . \quad . \quad (30)$$

whereas at the other hand (comp. (20))

$$M_\varphi = -\frac{EI\, U(\psi)}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Consequently U_φ satisfies the integral equation

$$U_\varphi = -\int_0^{2\pi} \frac{\lambda\, r\, N_0(\psi)\, U(\psi)\, K(\varphi, \psi)}{EI}\, d\psi \quad . \quad . \quad . \quad . \quad . \quad (32)$$

which also may be written as follows:

$$\left.\begin{aligned} \left[\sqrt{\frac{N_0(\varphi)\, r}{EI}}\, U(\varphi)\right] = -\lambda \int_0^{2\pi} \left[\sqrt{\frac{N_0(\psi) \cdot r}{EI}}\, U(\psi)\right] \left[\sqrt{\frac{N_0(\psi)\, r}{EI}}\, \cdot \right. \\ \left. \cdot \sqrt{\frac{N_0(\psi) \cdot r}{EI}} \cdot K(\varphi, \psi)\right] d\psi \end{aligned}\right\} . \quad (33)$$

The kernel $\sqrt{\dfrac{N_0(\varphi)\,.\,r}{EI}}\sqrt{\dfrac{N_0(\psi)\,.\,r}{EI}}\;K(\varphi,\psi)$ of this equation is symmetrical with respect to φ and ψ in consequence of the fact that $K(\varphi,\psi)$ — by its mechanical meaning — is symmetrical with respect to its arguments. Therefore all general theorems, relating to homogeneous integral equations with a symmetrical kernel can be applied to this special equation. In particular it may be remembered: 1°. that there exists an infinity of characteristic numbers λ for which eq. (32) is satisfied in general by one and exceptionally by more corresponding characteristic functions U; 2°. that any two characteristic functions U_1 and U_2 corresponding to different characteristic number λ_1 and λ_2 are orthogonal in that $\int N_0(\varphi)\,U_1(\varphi)\,U_2(\varphi)d\varphi = 0$; 3°. that any arbitrary function $V(\varphi)$ can be expanded into a series of the characteristic functions U.

It must be emphasized that our deductions only have a bearing on our proper problem if the infinite collection of introduced supplementary continuous loads $-\lambda N_0(\psi)U(\psi)\cos(\varphi-\psi)d\psi/r^2$ does influence the ring nowhere. The effect of these loads at a fixed angle φ is represented by

$$-\int_0^{2\pi}\frac{\lambda N_0(\psi)\,U(\psi)\cos(\varphi-\psi)}{r}\,d\psi = -\lambda\cos\varphi\int_0^{2\pi}\frac{N_0(\psi)\,U(\psi)\cos\psi}{r}\,d\psi -$$

$$-\lambda\sin\varphi\int_0^{2\pi}\frac{N_0(\psi)\,U(\psi)\sin\psi}{r}\,d\psi$$

which really is zero in virtue of the equilibrium-equations

$$\int_0^{2\pi}\frac{N_0(\psi)\,U(\psi)\cos\psi}{r}\,d\psi = 0,\quad \int_0^{2\pi}\frac{N_0(\psi)\,U(\psi)\sin\psi}{r}\,d\psi = 0$$

of the buckling ring.

Yet the problems represented by the eqs. (24) and (32) are not identical, for eq. (24) restricts itself to those loads NU/r^2 which are in equilibrium whereas eq. (32) equally refers to loads NU/r^2, which — not in equilibrium themselves — are balanced by a suitable load $a\cos\varphi + b\sin\varphi$. The reason of this discrepancy obviously is due to the fact, that in section 3 in the expression for N (25) the terms $\cos\varphi$ and $\sin\varphi$ and in the expression U (25) the terms $\cos l\varphi$ and $\sin l\varphi$ ($l \equiv \pm 1 \bmod p$) explicitely have been suppressed. Not before these exclusions should have been raised and the congruences (29) should have been completed with $l \equiv \pm 1 \bmod p$, (mechanically spoken, not until the buckling problem treated in sect. 3 should have been extended to such cases, in which the supplementary buckling loadsystem — eventually not in equilibrium itself — is balanced by a suitable radial loadsystem of the type $(a\cos\varphi + b\sin\varphi)$), complete conformity between eqs. (24) can exist. Obviously this conformity is formally indispensable if the theorems connected with eq. (32) shall be transferred to eq. (24); especially if it is stated that any arbitrary function $V(\varphi)$ can be expanded into a series of the characteristic functions of (24). On the other hand it is seen at once that two characteristic functions U^{q_1} and U^{q_2} relating to two different congruences $l = q_1$ mod p and $l = q_2$ mod p, never will have any cos. or sin. term in common. Consequently the statement can be made — which is essential for the method to be developed hereafter — that any arbitrary Fourier-series, containing only terms cos $l\varphi$ and sin $l\varphi$ for which $l \equiv q_1$ mod p, can be expanded in the characteristic functions $U_q{}^1$ relating to the same congruence $l = q_1$ mod p. For if we put

$$V^{q_1}(\varphi) = \Sigma a_k U_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

where in the righthand side the summation provisionally must be extended over *all* characteristic functions of (32), the coefficient a_k is given by

$$a_k = N_0(\varphi) \int_0^{2\pi} V^{q_1}(\varphi)\, U_k(\varphi)\, d\varphi \quad . \quad . \quad . \quad . \quad . \quad (35)$$

provided that the characteristic functions have been normalized such that

$$\int_0^{2\pi} N_0(\varphi)\, U_k{}^2(\varphi)\, d\varphi = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Obviously the righthand side of (35) is zero for every function $U_k^{q_2}$, which does not belong to the class of characteristic functions, defined by the congruence $l \equiv q_1$ mod p. For (possibly apart from a constant term), the product $N_0(\varphi) U_k^{q_2}$ consists of the same terms as $U_k^{q_2}$ itself (comp. (24)) and therefore has no terms in common with $V^{q_1}(\varphi)$, which was supposed to consist of the same cos and sin-functions as $U_k^{q_1}$. The expansion (34) therefore reduces to

$$V^{q_1}(\varphi) = \Sigma a_k U_l^{q_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

5. *The iterative method.* It has already been stated that every congruence (29) defines a distinct infinite class of characteristic functions U_k and a set of corresponding characteristic numbers λ_k ($|\lambda_1| < |\lambda_2| < |\lambda_3| \ldots$). To find the smallest of these numbers we start with an arbitrary function $V_1(\varphi)$, containing only such cos and sin terms as appear in the functions U_k, so that it can be represented by

$$V_1 = \sum_1^{\infty} a_k U_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

From this function we derive another one, defined by the differential equation

$$V_2'' + V_2 = \frac{N_0 r^2}{EI} V_1 + C = \sum_1^{\infty} \frac{a_k r^2 N_0 U_k}{EI} + C \quad . \quad . \quad . \quad (39)$$

and the condition that it does not contain terms of the type $\alpha + \beta \cos \varphi + \gamma \sin \varphi$ (C being a constant to be determined a posteriori). Obviously V_2 is represented by

$$V_2 = \Sigma \frac{a_k\, U_k}{\lambda_k} \quad \ldots \ldots \ldots \quad (40)$$

for on account of the relations

$$U_k'' + U_k = \frac{\lambda_k N_0 r^2}{EI} U_k + l_k \quad (k = 1, 2, \ldots) \quad \ldots \quad (41)$$

(comp. (24)) eq. (39) can be written as

$$V_2'' + V_2 = \left[\sum_1^{\infty} \frac{a_k\, U_k}{\gamma_k}\right]'' + \left[\sum_1^{\infty} \frac{a_k\, U_k}{\lambda_k}\right] + C - \sum_1^{\infty} \frac{a_k\, l_k}{\lambda_k} \quad \ldots \quad (42)$$

By the appointment, that C must be chosen such that $C - \sum_1^{\infty} a_k l_k / \lambda_k$ is zero, (40) follows from (42). If now analogously a third function V_3 is derived from V_2, a.s.o. we find successively

$$V_3 = \sum_1^{\infty} \frac{a_k\, U_k}{\lambda_k^2},\ V_3 = \sum_1^{\infty} \frac{a_k\, U_k}{\lambda_k^3}, \ldots V_n = \sum^{\infty} \frac{a_k\, U_k}{\lambda_k^{n-1}},\ V_{n+1} = \sum_1^{\infty} \frac{a_k\, U_k}{\lambda_k^{n+1}}$$

and therefore

$$\lim_{n \to \infty} \frac{V_n}{V_{n+1}} = \lim_{n \to \infty} \frac{a_k\, U_k}{\lambda_k^{n-1}} : \sum_1^{\infty} \frac{a_k\, U_k}{\lambda_k^n} = \lim_{n \to \infty} \frac{U_1 + \sum_2^{\infty} \left(\frac{\lambda_1}{\lambda_k}\right)^{n-1} \frac{a_k}{a_1} U_k}{U_1 + \sum_2^{\infty} \left(\frac{\lambda_1}{\lambda_k}\right)^{n} \frac{a_k}{a_1} U_k} \lambda_1 = \lambda_1 \quad (43)$$

In practice the iteration process can be stopped as soon as two consecutive functions V_m and V_{m+1} are practically similar, λ_1 then being approximated by the slightly varying factor of similarity $V_m(\varphi) : V_{m+1}(\varphi)$. The approximation can be refined and the numerical work efficiently reduced by putting [3])

$$\text{either } \lambda_1 \backsim \frac{\int_0^{2\pi} N_0\, V_m\, V_{m+1}\, d\varphi}{\int_0^{2\pi} N_0\, V_{m+1}^2\, d\varphi} \text{ or } \lambda_1 \backsim \frac{\int_0^{2\pi} N_0\, V_m^2\, d\varphi}{\int_0^{2\pi} N_0\, V_m\, V_{m+1}\, d\varphi} \quad . \quad (44)$$

By using these formulae the iteration can be stopped at very low values of the index m ($m = 1$ or 2).

The iterative method is not restricted to the calculation of the smallest characteristic number λ, but can as well be used if λ_2 (or any higher characteristic) number should be required. We only have to start with an

[3]) Comp. footnote 4.

initial function V_1, which does not contain the first characteristic function U_1. The way, in which an arbitrary function V_1 can (approximately) be "cleaned" from U_1, and the way in which the iterations of such a nearly cleaned function V_1 can themselves be freed from rests of U_1, which they might contain, will not be treated here [4]).

6. *The iteration scheme.* The only thing still to be described is the scheme along which the iteration has to be performed in fact. Let, to fix our mind, N be given in the simple form

$$N = \lambda N_0 = \lambda (1 + \varepsilon_k \cos k\varphi) \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

and let

$$V_1 = \cos l\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

be our starting function, l representing any number, which is compatible with our problem. Then V_2 must satisfy the equation

$$\left. \begin{aligned} V_2'' + V_2 = -\frac{r^2}{EI} \cos l\varphi (1 + \varepsilon_k \cos k\varphi) + C = -\frac{r^2}{EI} \Big[\cos l\varphi + \\ + \frac{\varepsilon_k}{2} \cos (l+k)\varphi + \frac{\varepsilon_k}{2} \cos (l-k)\varphi \Big] + C \end{aligned} \right\} \quad (47)$$

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[4]) Comp. f. i. J. J. KOCH: Eenige toepassingen van de leer der eigenfuncties op vraagstukken uit de Toegepaste Mechanica, Doctor Thesis 1929 Delft, or C. B. BIEZENO und R. GRAMMEL, Technische Dynamik, III, **14**, **15**, Springer 1939, Berlin. In these treatises only the first mentioned approximation is discussed but the second one can analogously be verified. It may be stated here without proof, that from the four approximations

$$\lambda_1 = \frac{\int_0^{2\pi} N_0 V_{m-1}^2 \, d\varphi}{\int_0^{2\pi} N_0 V_{m-1} V_m \, d\varphi}; \quad \lambda_1 = \frac{\int_0^{2\pi} N_0 V_{m-1} V_m \, d\varphi}{\int_0^{2\pi} N_0 V_m^2}; \quad \lambda_1 = \frac{\int_0^{2\pi} N_0 V_m^2 \, d\varphi}{\int_0^{2\pi} N_0 V_m V_{m+1} \, d\varphi};$$

$$\lambda_1 = \frac{\int_0^{2\pi} N_0 V_m V_{m+1} \, d\varphi}{\int_0^{2\pi} N_0 V_{m+1}^2 \, d\varphi}$$

each one excels the foregoing in accuracy provided that all characteristic numbers are positive. If negative characteristic numbers occur, the statement becomes doubtful with respect to the third and second approximation, though *in general* it may be expected that in this case too the statement holds true.

from which we deduce

$$\frac{r^2}{(l^2-1)\,EI}\left[\cos l\varphi+\frac{\varepsilon_k}{2}\frac{(l^2-1)}{(l+k)^2-1}\cos(l+k)\varphi+\right.$$
$$\left.+\frac{\varepsilon_k}{2}\frac{(l^2-1)}{(l-k)^2-1}\cos(l-k)\varphi+C\right] \quad \ldots \quad (48)$$

The condition that no constant term shall be present in V_2 requires either $C=0$ (if $k\neq l$) or $C=-\frac{\varepsilon_k}{2}(l^2-1):[(l-k)^2-1]$ (if $k=l$).

Furthermore it may be emphasized that the factor $r^2:(l^2-1)EI$ represents the constant normal force N in the crosssections of a ring subjected to the critical normal pressure $q=(l^2-1)EI/r^3$. By placing this factor before the [ ] brackets a convenient comparison is made possible with this case of buckling. For if we write

$$V_3=\left(\frac{r^2}{(l^2-1)\,EI}\right)^2 V_3^*(\varphi);\ldots.\; V_n=\left(\frac{r^2}{(l^2-1)\,EI}\right)^{n-1} V_n^*(\varphi) \quad (49)$$

the characteristic number λ_1 will be represented by

$$\lambda_1=\lim_{n\to\infty}\frac{V_n}{V_{n+1}}=\frac{(l^2-1)\,EI}{r^2}\lim_{n\to\infty}\frac{V_n^*}{V_{n+1}^*} \quad \ldots \quad (50)$$

and the corresponding critical load by

$$q=\frac{\lambda_1}{r}(1+\varepsilon_k\cos k\varphi)=\frac{(l^2-1)\,EI}{r^3}(1+\varepsilon_k\cos k\varphi)\lim_{n\to\infty}\frac{V_n^*}{V_{n+1}^*} \quad . \quad (51)$$

If we compare the functions V_1^* ($\equiv V_1$) and V_2^* we see that V_2^* (apart from the constant C) is composed of the unchanged function V_1^* and of well defined multiples of the two functions cos $(l+k)\varphi$ and cos $(l-k)\varphi$. The function V_3^* consequently can be deduced from V_2^* by applying the same law of development to each of the individual terms of V_2^*; and obviously the same holds for any following iteration $V_2^*\to V_3^*$, $V_3^*\to V_4^*$ a.s.o.

Table I shows for $k=4$, $\varepsilon_k=\frac{1}{2}$, $l=2$ now the successive iterations best can be performed.

The columns of Table I are provided with the superscriptions cos 2φ, cos 6φ, ... cos $(2+(n-1)4)\varphi$, and the corresponding factors $(2^2-1):(2^2-1)$, $(2^2-1):(6^2-1)$ a.s.o. As starting function has been chosen cos 2φ; it is represented by the number 1 in the first row and first column. As stated before this term gives in the first iteration rise to certain multiples of cos -2φ and cos 6φ (comp. 48). If, provisionally, only attention is given to the multiplicatior $\varepsilon_k/2=0{,}25$, the number $0{,}25\times 1$ should be shifted one place to the left and one place to the right. As cos $-2\varphi=$ cos 2φ, the numbers destined for column cos -2φ (not present in table I), can be placed in the column cos 2φ, and this indeed has been done as can be seen from the first number in the second row in table I. Thereupon the numbers placed above the first short horizontal line in Table I have been summed up to find the coefficients of the product $W_1^* = N_0 V_1^*$ (comp. eq. (47)). Multiplication of these results with the multiplicators mentioned in the heads of the columns gives the coefficients of the required function V_2^*

$$V_2^* = 1{,}25000 \cos 2\varphi + 0{,}02143 \cos 4\varphi. \quad . \quad . \quad . \quad . \quad (52)$$

TABLE I.

$l=2$ $k=4$ $\varepsilon_k=\frac{1}{2}$	$\frac{2^2-1}{2^2-1}=1$ cos 2φ	$\frac{2^2-1}{6^2-1}=\frac{3}{35}$ cos 6φ	$\frac{2^2-1}{10^2-1}=\frac{1}{33}$ cos 10φ	$\frac{2^2-1}{14^2-1}=\frac{1}{65}$ sos 14φ	
$V_1 = V_1^*$	1.00000 0.25000	 0.25000			
$W_1^* = N_0 V_1^*$	1.25000	0.25000			
V_2^*	1.25000 0.31250 0.00536	0.02143 0.31250 	 0.00536		
$W_2^* = N_0 V_2^*$	1.56786	0.33393	0.00536		
V_3^*	1.56786 0.39197 0.00716	0.02862 0.39197 0.00004	0.00017 0.00716	 0.00004	
$W_3^* = N_0 V_3^*$	1.96699	0.42063	0.00733	0.00004	
V_4^*	1.96699 0.49175 0.00901	0.03605 0.49175 0.00006	0.00022 0.00901	0.00000 0.00006	
$W_4^* = N_0 V_4^*$	2.46775	0.52786	0.00923	0.00006	
V_5^*	2.46775	0.04525	0.00028	0.00000	
V_3^* / V_4^* V_4^* / V_5^*	0.7971 0.7910	0.7939 0.7967	0.7727 0.7856	— —	
$W_1^* V_1^*$ $W_1^* V_2^*$	1.25000 1.56250	0.00000 0.00536	0.00000 0.00000	0.00000 0.00000	$\frac{\Sigma_1}{\Sigma_2}=0.79727$
$W_2^* V_2^*$ $W_2^* V_3^*$	1.95983 2.45818	0.00710 0.00956	0.00000 0.00000	0.00000 0.00000	$\frac{\Sigma_1}{\Sigma_2}=0.79708$
$W_3^* V_3^*$ $W_3^* V_4^*$	3.08396 3.86905	0.01204 0.01516	0o00000 0.00000	0.00000 0.00000	$\frac{\Sigma_1}{\Sigma_2}=0.79708$
$W_4^* V_4^*$ $W_4^* V_5^*$	4.85404 6.08979	0.01903 0.02389	0.00000 0.00000	0.00000 0.00000	$\frac{\Sigma_1}{\Sigma_2}=0.79708$

(In controling this step, the reader has to pay due attention to the fact, that the factor $(l^2 - 1) : [(l-k)^2 - 1]$ of the third term between brackets [ ] of eq. (48) has the value one in virtue of $k = -2l$). The coefficients of V_2^* again are shifted one place to the left and one place to the right after having been multiplied with the factor $\varepsilon_k/2 = 0{,}25$, under the understanding that every number destined for column cos -2φ be placed in the column cos 2φ. The numbers thus obtained (and placed beneath the first long horizontal line of Table I) are added, — giving as result the coefficients of $W_2^* = N_0 V_2^*$ — and these coefficients in their turn have been multiplied by the factors 1, 3/35, 1/33 .... In this way we find

$$V_2^* = 1{,}56786 \cos 2\varphi + 0{,}02862 \cos 6\varphi + 0{,}00017 \cos 10\varphi \quad \ldots \quad (53)$$

As will be seen from Table I the iteration has been pursued up to V_5^*. Then the quotients of the corresponding coefficients of V_3^* and V_4^* (resp. of V_4^* and V_5^*) have been calculated and it may be stated that a high degree of similarity between these coefficients exists. Clearly the factor of proportionality is best approximated by the figures in the first column and therefore we put:

$$\lambda_1 = 0{,}7971 \,.\, \frac{3\,EI}{r^2} \quad \ldots \ldots \ldots \quad (54)$$

If we should have used the second of the formulae (44)

$$\left.\begin{aligned} \lambda_1 = \frac{\int_0^{2\pi} N_0\, V_m^2\, d\varphi}{\int_0^{2\pi} N_0\, V_m\, V_{m+1}\, d\varphi} &= \frac{(l^2-1)\,EI}{r^2}\, \frac{\int_0^{2\pi} N_0\, V_m^{2*}\, d\varphi}{\int_0^{2\pi} N_0\, V_m^*\, V_{m+1}^*\, d\varphi} = \\ &= \frac{3\,EI}{r^2}\, \frac{\int_0^{2\pi} W_m^*\, V_m^*\, d\varphi}{\int_0^{2\pi} W_m^*\, V_{m+1}^*\, d\varphi} \end{aligned}\right\} \quad (55)$$

we should have found (with $m = 1, 2, 3,$ and 4): $\lambda_1 = 0{,}7973$; $0{,}7971$; $0{,}7971$; $0{,}7971 \dfrac{3EI}{r^2}$ respectively [5]). It appears that, even if we had stopped the iteration with V_2^*, the relative error in λ_1 would have been less than 1/4000.

Some additional remarks with respect to the iterative process have to be made. Firstly it must be stated, that in the case just treated the iteration could as well have been started with $V_1 \equiv V_1^* \equiv \sin l\varphi$, l again represen-

[5]) The work to be done in computing these successive approximations for λ_1 is represented in the last 8 rows of Table I. Obviously an integral such as $\int_0^{2\pi} W_1^*\, V_2^*\, d\varphi$ can be evaluated by multiplying the corresponding coefficients of W_1^* and V_2^* and by summing up the so-acquired products.

ting a number compatible with our problem. Then we should have found [6])

$$V_2 = \frac{r^2}{(l^2-1)EI}\left[\sin l\varphi + \frac{\varepsilon_k}{2}\frac{l^2-1}{(l+k)^2-1}\sin(l+k)\varphi + \frac{\varepsilon_k}{2}\frac{l^2-1}{(l-k)^2-1}\sin(l-k)\varphi\right] \tag{56}$$

The same scheme of calculation as given in Table I can be used here, provided that the superscriptions cos 2φ, cos 6φ ... be replaced by sin 2φ, sin 6φ ...; it must, however, be put in mind, that the transition of a term sin $(l - k)\varphi$ from the column sin $(l - k)\varphi$ to the column sin $|l - k|\varphi$ involves the introduction of a factor (-1). It would be found that the characteristic value λ_1, calculated in this way is greater than the first one. If $V_1 = \cos l\varphi + \sin l\varphi$ had been chosen as the starting function one is invariably led to the first characteristic value (54) and to the corresponding characteristic function. A slight modification would occur if the special case $N = \lambda\,(1 + \varepsilon_2 \cos 2\varphi)$, $(k = 2)$ would be examined. It may be left to the reader to iterate once with $V_1^* = \cos 2\varphi$ and once with $V_1^* = \sin 2\varphi$ as starting functions. He will be led to the same characteristic number λ_1 and to two linearly independent characteristic functions!

A second remark refers to the fact, that a slight complication appears if $N = \lambda\,(1 + \varepsilon_k \cos k\varphi)$ is replaced by $N = \lambda\,(1 + \varepsilon_k \sin k\varphi)$. Obviously we then have to start with a function composed of cos and sin terms, for instance $V_1 = \alpha \cos l\varphi + \beta \sin l\varphi$. The first iteration, defined by

$$V_2'' + V_2 = -\frac{r^2}{EI}[\alpha\cos l\varphi + \beta\sin l\varphi]\,[1 + \varepsilon_k \sin k\varphi] + C \quad . \tag{57}$$

then proves to be

$$V = \frac{\alpha r^2}{(l^2-1)EI}\left[\cos l\varphi + \frac{\varepsilon_k}{2}\frac{l^2-1}{(l+k)^2-1}\sin(l+k)\varphi - \frac{\varepsilon_k}{2}\frac{l^2-1}{(l-k)^2-1}\sin(l-k)\varphi\right] + \frac{\beta r^2}{(l^2-1)EI}\left[\sin l\varphi - \frac{\varepsilon_k}{2}\frac{l^2-1}{(l+k)^2-1}\cos(l+k)\varphi + \frac{\varepsilon_k}{2}\frac{l^2-1}{(l-k)^2-1}\cos(l-k)\varphi\right] + C. \tag{58}$$

Again the constant C is determined by the condition, that ultimately no constant term in V_2 is allowed.

Table II represents the scheme of iteration, adapted to this case, with $\varepsilon_k = 1$, $k = 2$, and $\cos 2\varphi + \sin 2\varphi$ as starting function. It has, for the first three iterations been indicated by asterisks and cross-signs how the

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[6]) The constant of integration C of eq. (48) vanishes here under all circumstances as the sine-functions in (56) for no single value of k give rise to a constant term.

TABLE II.

$l=2$ $k=2$ $\varepsilon_k=1$	$\frac{2^2-1}{2^2-1}=1$		$\frac{2^2-1}{4^2-1}=\frac{3}{35}$		$\frac{2^2-1}{6^2-1}=\frac{3}{35}$		$\frac{2^2-1}{8^2-1}=\frac{1}{33}$		$\frac{2^2-1}{10^2-1}=\frac{1}{33}$	
	$\cos 2\varphi$	$\sin 2\varphi$	$\cos 4\varphi$	$\sin 4\varphi$	$\cos 6\varphi$	$\sin 6\varphi$	$\cos 8\varphi$	$\sin 8\varphi$	$\cos 10\varphi$	$\sin 10\varphi$
V_1^*	1.00000*	1.00000×	−0.50000×	0.50000*						
W_1^*	1.00000	1.00000	−0.50000	0.50000××						
V_2^*	1.00000* 0.05000××	1.00000× 0.05000**	−0.10000** −0.50000×	1.00000×× 0.50000*	−0.05000××	−0.05000**				
W_2^*	1.05000	1.05000	−0.60000	+0.60000	−0.05000	−0.05000				
V_3^*	1.05000* 0.06000××	1.05000× 0.06000××	−0.12000** −0.52500× −0.00214×××	0.12000×× 0.52500 0.00214***	−0.00429*** −0.06000××	−0.00429××× −0.06000**	0.00214×××	−0.00214***		
W_3^*	1.11000	1.11000	−0.64714	0.64714	−0.06429	−0.06429	0.00214	−0.00214		
V_4^*	1.11000 0.06471	1.11000 0.06471	−0.12943 −0.55500 −0.00276	0012943 0.55500 0.00276	−0.00551 −0.06471 −0.00005	−0.00551 −0.06471 −0.00005	0.00010 0.00276	−0.00010 −0.00276	0.00005	0.00005
W_4^*	1.17471	1.17471	−0.68719	0.63719	−0.07027	−0.07027	0.00286	−0.00286	0.00005	0.00005
V_5^*	1.17471	1.17471	−0.13744	0.13744	−0.00600	−0.00600	0.00014	−0.00014		
V_4^*/V_5^*	0.946	0.946	0.942	0.942	0.918	0.918	—	—	—	—
$W_1^*\ V_1^*$	1.00000	1.00000					$\Sigma_1 \equiv 2.00000$		$\frac{\Sigma_1}{\Sigma_2}=0.95238$	
$W_1^*\ V_2^*$	1.00000	1.00000	0.05000	0.05000			$\Sigma_2 = 2.10000$			
$W_2^*\ V_2^*$	1.05000	1.05000	0.06000	0.06000	0.00000	0.00000	$\Sigma_1 = 2.22000$		$\frac{\Sigma_1}{\Sigma_2}=0.94491$	
$W_2^*\ V_3^*$	1.10250	1.10250	0.07200	0.07200	0.00021	0.00021	$\Sigma_2 = 2.34942$			
$W_3^*\ V_3^*$	1.16550	1.16550	0.07766	0.07766	0.00028	0.00028	$\Sigma_1 = 2.48688$		$\frac{\Sigma_1}{\Sigma_2}=0.94473$	
$W_3^*\ V_4^*$	1.23210	1.23210	0.08373	0.08373	0.00035	0.00035	$\Sigma_2 = 2.63236$			
$W_4^*\ V_4^*$	1.30393	1.30393	0.08894	0.08894	0.00039	0.00039	$\Sigma_1 = 2.78552$		$\frac{\Sigma_1}{\Sigma_2}=0.94470$	
$W_4^*\ V_3^*$	1.37994	1.37994	0.09445	0.09445	0.00042	0.00042	$\Sigma_2 = 2.94962$			

different figures have to be shifted. As to the iteration it will be seen, that the first figure of V_2^* has been placed (multiplied by $\varepsilon_2/2 = 1/2$) on the fourth place of the next row; the second figure multiplied by $-\varepsilon_2/2$ on the third place; the third one, multiplied by $\varepsilon_2/2$ and $-\varepsilon_2/2$ respectively on the sixth and second places; the fourth one, multiplied by $-\varepsilon_2/2$ and $\varepsilon_2/2$ respectively on the fifth and first place. Then the corresponding figures of the two rows under consideration have been added, and the results have been multiplied by the factors inserted in the headings of the columns. The iteration has been stopped with V_5^* and it is seen by comparison of V_4^* and V_5^*, that the required smallest characteristic number λ_1 (with considerable approximation) can be represented by $\lambda = 3\mathrm{EI}/r^2 \,.\, 0{,}946$. The second formula (44) leads for $m = 1, 2, 3, 4$ to the following values

$$\lambda_1 = 0{,}95238, \qquad 0{,}94491, \qquad 0{,}94473, \qquad 0{,}94470$$

and it is confirmed again that much labour can be saved by the use of this formula.

7. *The compressive force* $N = \lambda N_0 = \lambda(1 + 2 \cos k\varphi)$. In this section the numerical results are collected, with regard to the special normal force distribution $N = \lambda\,(1 + 2 \cos k\varphi)$, up to $k = 12$. The example has chiefly been chosen to get an insight in the influence of a strong fluctuation of N with respect to its mean value on the buckling force of the ring. As has already been stated in sects. **3** and **4** that for every value of k the buckling problem is split up into $(k-1)/2$ or $k/2$ separate problems connected with the congruences $l \equiv \pm q \bmod k$ $\left(q = 0, 2, 3 \ldots \frac{k-1}{2} \text{ or } \frac{k}{2}\right)$; the characteristic

TABLE III. Giving the values of $\dfrac{\lambda r^2}{(l^2-1)\,EI}$ if $N = \lambda\,(1 + 2 \cos k\varphi)$.

k \ l	2	3	4	5	6	7	8	9	10	11	12
2	0.832	—	—	—	—	—	—	—	—	—	—
3	—	0.812	—	—	—	—	—	—	—	—	—
4	0.489	—	0.808	—	—	—	—	—	—	—	—
5	0.708	—	—	0.802	—	—	—	—	—	—	—
6	0.810	0.488	—	—	0.801	—	—	—	—	—	—
7	0.866	0.634	—	—	—	0.799	—	—	—	—	—
8	0.899	0.728	0.487	—	—	—	0.799	—	—	—	—
9	0.922	0.789	0.598	—	—	—	—	0.798	—	—	—
10	0.937	0.831	0.679	0.486	—	—	—	—	0.798	—	—
11	0.948	0.862	0.738	0.577	—	—	—	—	—	0.797	—
12	0.957	0.885	0.782	0.647	0.486	—	—	—	—	—	0.797

functions of each of these problems is built up of terms cos $l\varphi$ and sin $l\varphi$ in which l is determined by one of these congruences.

Table III contains all values of $\lambda^2 : (l^2 - 1)EI$ for $2 \leqq \frac{k}{l} \leqq 12$. It may be emphasized that for any prescribed number l the smallest value of λ corresponds to $k = 2l$. This fact becomes explicable by the consideration that for $k = 2l$ the angles *both* of maximum and minimum deflection coincide with those of maximum compressive force N.

8. *The compressive force* $N = \lambda N_0 = \lambda\ (1 + \varepsilon_2 \cos 2\varphi + \varepsilon_4 \cos 4\varphi)$, $\varepsilon_2 = 2$, $\varepsilon_4 = 1$; *the second iteration.* As an illustration how to handle the iterative method if the compressive normal force has a more complicated form like $N = \lambda(1 + 2 \cos 2\varphi + 4\varphi)$ and how to act if apart from the first characteristic function the second one should be required, we insert in this section Tables IV*a*, and IV*b*, from which all necessary data can be borrowed. The starting function V_1^* is represented by cos 2φ. This term has to be shifted one place to the right and one place to the left after multiplication with the factor $\varepsilon_2/2 = 1$ on account of the term $\varepsilon_2 \cos 2\varphi$ in N. The shifting to the left gives a term in the column cos 0φ which must be suppressed (comp. sect. 6). Furthermore the starting term cos 2φ has to be shifted two places to the right and to the left after multiplication with the factor $\varepsilon_4/2 = 1/2$ on account of the term $\varepsilon_2 \cos 4\varphi$ in N. The shifting to the left provides us with a term in the column cos -2φ, which therefore must be placed in the column cos 2φ. Summation of all terms occurring in the different columns gives the coefficients of the function W_1^*. Finally these coefficients have to be multiplied by the factors 1, 1/5, 3/35 ... mentioned in the heads of the columns, to obtain the coefficients of the first iteration V_2^*. Analogously the terms of V_2^* have to be shifted *one* place to the right and left after multiplication with the factor $\varepsilon_2/2$ and *two* places to the right and left after multiplication with the factor $\varepsilon_4/2$, with due regard to the fact, that every number in the column cos 0. φ has to be suppressed and every number in the column cos -2φ has to be transported to the column cos 2φ. Summing up all terms in the different columns provides us with the coefficients of W_2^*, whereas multiplication of these coefficients with their respective multiplicators 1, 1/15 ... furnishes the coefficients of V_3^*; a.s.o. The iterative process has been carried on to the sixth iteration V_7^*, though the computation of the first characteristic number λ_1 in itself would require no more than the iterations V_2^* and V_3^*. By proceeding as far as V_3^*, however, great accuracy is obtained in the first characteristic function U_1 which is approximated by

$$\left. \begin{aligned} U_1 \backsim V_7^* = 18{,}493 \cos 2\varphi + 2{,}627 \cos 4\varphi + 0{,}650 \cos 6\varphi + \\ + 0{,}058 \cos 8\varphi + 0{,}007 \cos 10\varphi \end{aligned} \right\} \quad (59)$$

TABLE IVa.

$N = \lambda\ [1 + 2\cos 2\varphi + \cos 4\varphi]$; $\varepsilon_2 = 2$, $\varepsilon_4 = 1$.

	1	1/5	3/35	1/21	1/33	3/143
	cos 2 φ	cos 4 φ	cos 6 φ	cos 8 φ	cos 10 φ	cos 12 φ
V_1^*	1.000^o					
		1.000^{or}				
			0.500^{oor}			
	5.000^{ool}					
W_1^*	1.500	1.000	0.500			
V_2^*	1.500^o	0.200^o	0.043^x			
		1.500^{or}	0.200^{or}	0.043^{xr}		
	0.200^{ol}	0.043^{xl}				
			0.750^{oor}	0.100^{oor}	0.022^{xxr}	
	0.022^{xxl}					
	0.750^{ool}					
W_2^*	2.472	1.743	0.993	0.143	0.043	
V_3^*	2.472	0.349	0.085	0.007	0.001	
		2.472	0.349	0.085	0.007	0.001
	0.349	0.085	0.007	0.001		
			1.236	0.175	0.043	0.004
	0.043	0.004	0.001			
	1.236					
W_3^*	4.100	2.910	1.678	0.268	0.051	0.005
V_4^*	4.100	0.582	0.144	0.012	0.002	0.000
		4.100	6.582	0.144	0.012	0.002
	0.582	0.144	0.012	0.002		
			2.052	0.291	0.072	0.006
	0.072	0.006	0.001			
	2.050					
W_4^*	6.704	4.832	2.789	0.449	0.086	0.008
V_5^*	6.704	0.966	0.239	0.021	0.003	0.000
		6.704	0.966	0.239	0.021	0.003
	0.966	0.239	0.021	0.003		
		3.352	0.483	0.120	0.011	
	0.120	0.011	0.002			
	3.352					
W_5^*	11.142	7.920	4.580	0.746	0.144	0.014
V_6^*	11.142	1.584	0.392	0.036	0.004	0.000
		11.142	1.584	0.392	0.036	0.004
	1.584	0.392	0.036	0.004		
			5.571	0.792	0.196	0.018
	0.196	0.018	0.002			
	5.571					
W_6^*	18.493	13.136	7.585	1.224	0.236	0.022
V_7^*	18.493	2.627	0.650	0.058	0.007	0.000
		18.493	2.627	0.650	0.058	0.007
	2.627	0.650	0.058	0.007		
			9.247	1.314	0.325	0.029
	0.325	0.029	0.004			
	9.247					
W_7^*	30.692	21.799	12.586	2.029	0.390	0.036
$W_7^*\ V_6^*$	341.96	34.53	4.93	6.07	$\Sigma_1 = 381.50$	$\frac{\Sigma_1}{\Sigma_2} = \lambda_1 = 0.60253$
$W_7^*\ V_7^*$	467.59	57.27	8.18	0.12	$\Sigma_2 = 633.16$	

TABLE IV*b*. $N = \lambda\ [1 + 2\cos 2\varphi + \cos 4\varphi]$. $\varepsilon_2 = 2$, $\varepsilon_4 = 1$.

	1	1/5	3/35	1/21	1/33	3/143
	$\cos 2\varphi$	$\cos 4\varphi$	$\cos 6\varphi$	$\cos 8\varphi$	$\cos 10\varphi$	$\cos 12\varphi$
V_8^*	100.000					
$V_8^* W_7^*$	3069.200					
$-4.8474\ V_7^*$	−89.643	−12.724	−3.151	−0.281	−0.034	—
V_8^*	10.357	−12.734	−3.151	−0.281	−0.034	
		10.357	−12.734	−3.151	−0.281	−0.034
	−12.734	−3.151	−0.281	−0.034		
			5.179	−6.367	−1.576	−0.141
	−1.576	−0.141	−0.017			
	5.179					
W_8^*	1.226	−5.669	−11.004	−9.833	−1.891	−0.175
V_9^*	1.226	−1.134	−0.943	−0.468	−0.057	−0.004
$V_9^*\ V_7^*$	37.628	−24.720	−11.869	−0.950	−0.022	
$-0.000106 \cdot V_7^*$	−0.002	—	—	—	—	
V_9^*	1.224	−1.134	−0.943	−0.468	−0.057	−0.004
		1.224	−1.134	−0.943	−0.468	−0.057
	−1.134	−0.943	−0.468	−0.057	−0.004	
			0.612	−0.567	−0.472	−0.234
	−0.472	−0.234	−0.029	−0.002		
	+0.612					
W_9^*	0.230	−1.087	−1.962	−2.037	−1.001	−0.295
V_{10}^*	0.2300	−0.2174	−0.1682	−0.0970	−0.0303	−0.0062
$V_{10}^* W_7^*$	7.059	−4.739	−2.117	−0.197	−0.012	—
$+0.0000095\ V_7^*$	0.0002					
V_{10}^*	0.2302	−0.2174	−0.1682	−0.0970	−0.0303	−0.0062
$\overline{V}_{11}^*$	0.04379	−0.04078	−0.03280	−0.01940	−0.00659	−0.00178
$\overline{V}_{12}^*$	0.008495	−0.007900	−0.006375	−0.003813	−0.001338	−0.000379
$\overline{V}_{11}^* : \overline{V}_{12}^*$	5.155	5.162	5.145	5.064	4.889	4.670

The characteristic number λ_1 has been computed with the aid of the first formula (44)

$$\left.\begin{aligned} \lambda_1 &= \frac{\int_0^{2\pi} N_0\, V_6\, V_7\, d\varphi}{\int_0^{2\pi} N_0\, V_7^2\, d\varphi} = \frac{3\,EI}{r^2}\,\frac{\int_0^{2\pi} N_0\, V_6^*\, V_7^*\, d\varphi}{\int_0^{2\pi} N_0\, V_7^{*2}\, d\varphi} = \\ &= \frac{3\,EI}{r^2}\,\frac{\int_0^{2\pi} W_7^*\, V_6^*\, d\varphi}{\int_0^{2\pi} W_7^*\, V_7^*\, d\varphi} = 0{,}60253\ \frac{3\,EI}{r^2} \end{aligned}\right\} \quad . \quad . \quad (60)$$

Table IVb is devoted to the calculation of λ_2. The starting function is represented by $V_8^* = 100 \cos 2\varphi$. The coefficient a_1 in the development of V_8^* into a series of the characteristic functions U

$$V_8^* = a_1 U_1 + a_2 U_2 + \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

is given by

$$\left. \begin{aligned} a_1 &= \frac{\int\limits_0^{2\pi} N_0 V_8^* U_1 \, d\varphi}{\int\limits_0^{2\pi} N_0 U_1^2 \, d\varphi} \text{ and approximately by} \\ a_1^* &= \frac{\int\limits_0^{2\pi} N_0 V_8^* V_7^* \, d\varphi}{\int\limits_0^{2\pi} N_0 V_7^{*2} \, d\varphi} = 4{,}847 \end{aligned} \right\} \quad . \quad . \quad (62)$$

If a_1 and U_1 would have been obtained *exactly*, the iteration process applied to $V_8^* - a_1 U_1$ obviously would lead to the enumeration of the *second* characteristic number λ_2. In reality both U_1 and a_1 are only approximately known and therefore, if the process is applied to the function $\overline{V}_8^* = V_8^* - a_1 V_7^*$ which to a certain (but small) amount contains the first characteristic function U_1 — serious difficulties are to be expected. For indeed, however small the contribution of U_1 in any starting function may be, the iteration process always and invariably leads to λ_1 if U_1 initially is present. These difficulties are surmounted by iterating V_9^* from $\overline{V}_8^*$, by cleaning V_9^* from $U_1 \infty V_7^*$ in the same way as V_8^* has been cleaned from U_1, and by repeating this combined iteration and cleaning process till two consecutive "cleaned" functions $\overline{V}_m^*$ and $\overline{V}_{m+1}^*$ are sufficiently proportional. Here too considerable abbreviation of the cipherwork can be attained by the use of the formulae (44) if the characteristic number λ_2 alone should be required, as will be seen from the results

$$\lambda_2 = 5{,}2602, \quad 5{,}1331, \quad 5{,}1234, \quad 5{,}1228 \quad . \quad . \quad . \quad . \quad (63)$$

which have been found by the first of the formulae (44) with $m = 8$, 9, 10 and 11 respectively. The second of these values already approximates λ_2 quite satisfactory.

The determination of the second characteristic function, however, requires the continuation up to V_{12}^*

$$\left. \begin{aligned} V_{12}^* = 0{,}008495 \cos 2\varphi - 0{,}007900 \cos 4\varphi - 0{,}006375 \cos 6\varphi - \\ - 0{,}003813 \cos 8\varphi - 0{,}001338 \cos 10\varphi - 0{,}000379 \cos 12\varphi. \end{aligned} \right\} (64)$$

9. *Negative characteristic numbers.* It has already been stated that dealing with buckling problems one has to expect negative characteristic values in all such cases, where the normal force N changes its sign. A negative value of λ interchanges the compressed and the stretched parts of the construction, and a sufficiently great negative λ therefore causes the buckling of the initially stretched parts. As an example it may be stated that with a compressive force $N = \lambda\,[1 + 4\cos 2\varphi]$ the *second* characteristic number proves to be negative, $\lambda_2 = -2{,}1133\,\frac{3EI}{r^2}$, whereas the first characteristic numbers λ_1 equals to $\lambda_1 = 0{,}6275\,.\,\frac{3EI}{r^2}$.