Mathematics. — *On the theory of linear integral equations.* I. By A. C. ZAANEN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of January 26, 1946.)

§ 1. *Introduction.*

Let R be a complete (not necessarily separable) HILBERT space. We shall use the following notations:

$f, g, \ldots\ldots$	, the elements of R.
$\lambda, \mu, \ldots\ldots$	, complex numbers.
$\bar{\lambda}, \bar{\mu}, \ldots\ldots$	, the conjugate complex numbers of $\lambda, \mu, \ldots\ldots$
(f, g)	, the inner product of f and g.
$\lVert f \rVert$	, the non-negative numbers $(f, f)^{1/2}$.
$T, K, \ldots\ldots$	, bounded, linear transformations in R, that is (for T), $\lVert Tf \rVert \leq M \lVert f \rVert$ for a certain $M \geq 0$ and $T(\lambda f + \mu g) = \lambda Tf + \mu Tg$ for arbitrary λ, μ, f, g.
$T^*, K^*, \ldots\ldots$	, the adjoint transformations of $T, K, \ldots\ldots$, we have therefore (for T) $(Tf, g) = (f, T^* g)$ for arbitrary f, g.
H	, a bounded, positive, self-adjoint transformation, that is, a bounded, linear transformation satisfying $(Hf, g) = (f, Hg)$ and $(Hf, f) \geq 0$ for arbitrary f, g.
$H^{1/2}$	, the uniquely determined, bounded, positive, self-adjoint transformation, satisfying $(H^{1/2})^2 = H$.
$N(f)$	, the non-negative number $(Hf, f)^{1/2} = \lVert H^{1/2} f \rVert$.
I	, the identical transformation, $If = f$ for every f.
O	, the nulltransformation, $Of = 0$ for every f.

We suppose that $H \neq O$. Then the set of all elements f, satisfying $Hf = 0$, is a subspace $[L]$, not identical with the whole space R. The orthogonal subspace will be denoted by $[M]$. As well-known, every element $f \in R$ can be written uniquely in the form $f = h + g$ with $h \in [L]$ and $g \in [M]$. By $g = Ef$ the *projection* E on $[M]$ is defined; the projection on $[L]$ is $I - E$, and we have $E \neq O$. From $H(I - E)f = 0$ for every $f \in R$ follows $Hf = HEf$, so that $H = HE$.

Two elements f and g will be called *H-orthogonal* when $(Hf, g) = 0$, and the system Q of elements is called *H-orthonormal* when, for $\varphi \in Q$, $\psi \in Q$, we have $(H\varphi, \psi) = 1$ for $\varphi = \psi$, and $= 0$ for $\varphi \neq \psi$. The elements $f_1, f_2, \ldots, f_n$ will be called *H-independent* when $H \sum_{i=1}^{n} \lambda_i f_i = 0$ implies $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$. Evidently, if $f_1, f_2, \ldots, f_n$ are H-independent, they are linearly independent. It is also not difficult to prove that if the elements $\varphi_1, \ldots, \varphi_n$ form an H-orthonormal system, they are H-independent.

If $Tf = \lambda f$ for an element $f \neq 0$, this element is called a *characteristic element* of the transformation T, belonging to the *characteristic value* λ. The set of all characteristic elements, belonging to the same characteristic value λ, is a subspace of R, and the dimension of this subspace is called the *multiplicity* of the characteristic value λ.

The bounded, linear transformation K is said to be *completely continuous* when every bounded, infinite set of elements contains a sequence f_n such that the sequence Kf_n converges. We shall assume the following theorems about transformations of this kind to be known:

Theorem 1. *If K is completely continuous, the same is true of K^*.*

Theorem 2. *If K is completely continuous, every characteristic value $\lambda \neq 0$ of K has finite multiplicity. The number of different characteristic values λ_n is finite or enumerable and in this last case $\lim_{n \to \infty} \lambda_n = 0$.*

Theorem 3. *If K is completely continuous, and $\lambda \neq 0$ is a characteristic value of K, having a certain multiplicity, then $\bar{\lambda}$ is a characteristic value of K^* with the same multiplicity. In this case the equation $Kf - \lambda f = g$ has, for a given element g, a solution f for those and only those elements g that are orthogonal to all characteristic elements of K^*, belonging to the characteristic value $\bar{\lambda}$. In the same way the equation $K^*f - \bar{\lambda} f = g$ has, for a given element g, a solution f for those and only those elements g that are orthogonal to all characteristic elements of K, belonging to the characteristic value λ.*

If $\lambda \neq 0$ is no characteristic value of K, both the equations $Kf - \lambda f = g$ and $K^*f - \bar{\lambda} f = g$ have uniquely determined solutions for every element g. In this case the complex number λ will be called a regular value of K.

§ 2. *Bounded, symmetrisable transformations.*

The bounded, linear transformation K is called *symmetrisable* (to the left, and relative to the transformation H), if the transformation HK is self-adjoint, that is, if $(HKf, g) = (f, HKg)$ for arbitrary f, g.

Theorem 4. *If K is symmetrisable, the same is true of $T = EK$. Further $Hf = 0$ implies $Tf = 0$.*

Proof. From $H = HE$ follows $HT = HEK = HK$; if therefore HK is self-adjoint, the same is true of HT.

Further $(HTg, f) = (g, HTf)$ or $(Tg, Hf) = (g, HTf)$ for arbitrary f, g; the relation $Hf = 0$ implies therefore $(g, HTf) = 0$ for every $g \in R$, hence $HTf = 0$. Then however $Tf \in [L]$, so that, since also $Tf = EKf \in [M]$, we have $Tf = 0$.

Theorem 5. *Let the symmetrisable transformation K be such that $Hf = 0$ implies $Kf = 0$. Then the characteristic values of K are real and* *characteristic elements, belonging to different characteristic values, are H-orthogonal.*

Proof. Let $f \neq 0$ and $Kf = \lambda f$. If $(Hf, f) = 0$ we see, since $(Hf, f) = \| H^{1/2} f \|^2$, that $H^{1/2} f = 0$, so that $Hf = 0$ or, by hypothesis, $\lambda f = Kf = 0$, from which follows, on account of $f \neq 0$, that $\lambda = 0$. If $(Hf, f) \neq 0$ we find $\lambda(Hf, f) = (H\lambda f, f) = (HKf, f) = (f, HKf) = (f, H\lambda f) = \bar{\lambda}(Hf, f)$ or $\lambda = \bar{\lambda}$, which shows that λ is real.

Let now $\lambda \neq \mu$, $f \neq 0$, $g \neq 0$, $Kf = \lambda f$ and $Kg = \mu g$. Then $\lambda(Hf, g) = (HKf, g) = (f, HKg) = \bar{\mu}\,(f, Hg) = \mu\,(Hf, g)$ or $(\lambda - \mu)(Hf, g) = 0$, from which follows, since $\lambda - \mu \neq 0$, that $(Hf, g) = 0$.

Remark. Since, even if $Hf = 0$ does not imply $Kf = 0$ for the symmetrisable transformation K, it does imply $Tf = EKf = 0$ by Theorem 4, Theorem 5 is in any case valid for the symmetrisable transformation $T = EK$.

In what follows now, we shall assume that K, and therefore also $T = EK$, is symmetrisable, while, moreover, $T = EK$ is completely continuous. Then the following theorems hold [1]):

Theorem 6. *If $HK \neq O$, the transformation $T = EK$ has a characteristic value $\lambda \neq 0$, that is, there exists an element $\varphi \neq 0$ such that $T\varphi = \lambda\varphi$. In the particular case that $Hf = 0$ implies $Kf = 0$, the transformation K itself has also the characteristic value λ with characteristic element $\psi = \varphi + \lambda^{-1}\,(I - E)K\varphi$, hence $K\psi = \lambda\psi$.*

Theorem 7. *In the case that $Hf = 0$ implies $Kf = 0$, the relations $\varphi = E\psi$, $\psi = \varphi + \lambda^{-1}\,(I - E)K\varphi$ define a one-to-one correspondence between all characteristic elements ψ of K, belonging to the characteristic values $\neq 0$, and all characteristic elements φ of $T = EK$, belonging to the characteristic values $\neq 0$. Corresponding elements have the same characteristic value.*

As a consequence of Theorem 2, it is possible to range the characteristic values $\neq 0$ of T into a sequence λ_n such that every characteristic value $\neq 0$ occurs in this sequence as many times as denoted by its multiplicity, while moreover $|\lambda_1| \geq |\lambda_2| \geq \ldots$. Choosing now in the unitary space (space of finite dimension) of all characteristic elements belonging to a certain characteristic value $\lambda \neq 0$ a maximal system of linearly independent elements, we see readily that these elements, say $\chi_1, \ldots, \chi_p$, are H-independent. Indeed, $H\Sigma a_i\chi_i = 0$ (a_i complex) implying $T\Sigma a_i\chi_i = 0$ or $\Sigma\lambda a_i\chi_i = 0$, we find in virtue of $\lambda \neq 0$ and the linear independence of $\chi_1, \ldots, \chi_p$ that $a_1 = \ldots = a_p = 0$. Applying now to the elements $\chi_1, \ldots, \chi_p$ a process, wholly similar to SCHMIDT's well-known orthogonalization process, we obtain an H-orthonormal system, consisting of p elements, such that the unitary space determined by this system is identical with the

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1) A. C. ZAANEN, Ueber vollstetige symmetrische und symmetrisierbare Operatoren. Nieuw Arch. v. Wisk. (2) 22, 57—80 (1943).

unitary space of all characteristic elements belonging to the characteristic value λ. Doing this for all characteristic values $\neq 0$, we may range the elements of all these H-orthonormal systems into a sequence φ_n, such that for every value of n the element φ_n belongs to the characteristic value λ_n. Evidently the whole sequence φ_n is also H-orthonormal, since for $\lambda_m = \lambda_n$ the relation $(H\varphi_m, \varphi_n) = 0$ follows from our definition of the sequence φ_n and for $\lambda_m \neq \lambda_n$ this relation follows from Theorem 5. In the special case that $Hf = 0$ implies $Kf = 0$, the transformations K and $T = EK$ have, by Theorem 7, the same characteristic values $\neq 0$, and it may be verified readily that every characteristic value $\neq 0$ has, for K and T, the same multiplicity. Consequently, denoting by ψ_n the characteristic element of K, corresponding by Theorem 7 with the characteristic element φ_n of T, we obtain the H-orthonormal sequence ψ_n.

Then we have [2])

Theorem 8. *$|\lambda_n| = max\ N(Kf)/N(f)$ for all f satisfying the conditions $N(f) \neq 0$ and $(Hf, \varphi_1) = \ldots = (Hf, \varphi_{n-1}) = 0$. For $f = \varphi_n$ the maximum is attained. Further $HKf = 0$ or, which comes to the same thing, $N(Kf) = 0$ if and only if $(Hf, \varphi_n) = 0$ for every value of n.*

In the particular case that $Hf = 0$ implies $Kf = 0$, the elements φ may be replaced by the corresponding elements ψ in both parts of the theorem.

Theorem 9. *If $a_n = (Hf, \varphi_n)$, then*

$$\lim_{k \to \infty} N(Kf - \sum_{n=1}^{k} \lambda_n a_n \varphi_n) = 0 \quad (\textit{Expansion Theorem}),$$

$$(HKf, f) = \sum \lambda_n |a_n|^2$$

for any element f.

In the particular case that $Hf = 0$ implies $Kf = 0$, the elements φ_n may be replaced by the corresponding elements ψ_n.

Theorem 10. *Let λ_{n_i} $(i = 1, 2, \ldots)$ be the subsequence of all positive characteristic values where $\lambda_{n_1} \geq \lambda_{n_2} \geq \ldots$. Then $\lambda_{n_i} = max\ (HKf, f)/N^2(f)$ for all f satisfying the conditions $N(f) \neq 0$ and $(Hf, \varphi_{n_1}) = \ldots = (Hf, \varphi_{n_{i-1}}) = 0$. For $f = \varphi_{n_i}$ the maximum is attained.*

A similar statement holds for the subsequence of all negative characteristic values.

In the particular case that $Hf = 0$ implies $Kf = 0$, we may replace the elements φ by the corresponding elements ψ.

In the last theorem the characteristic value λ_{n_i} was characterized by a certain maximum property. It is a disadvantage however that for this characterization the elements $\varphi_{n_1}, \ldots, \varphi_{n_{i-1}}$ must be known. The question arises therefore whether this may be avoided. That this is indeed the case, is shown by the following theorem:

Theorem 11. *Let the elements $p_1, \ldots, p_{i-1}$ be arbitrarily given and let $\mu_i =$ upper bound $(HKf, f)/N^2(f)$ for all f satisfying the conditions*

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[2]) See [1]).

$N(f) \neq 0$ and $(Hf, p_1) = \dots = (Hf, p_{i-1}) = 0$. The number μ_i depends evidently on the elements $p_1, \dots, p_{i-1}$. Letting now these elements run through the whole space R, we have $\lambda_{n_i} = \min \mu_i$.

A similar statement holds for the negative characteristic values.

Proof. We shall prove first that it is possible to find an element $f = \sum_{k=1}^{i} c_k \varphi_{n_k}$ such that the conditions $N(f) = 1$ and $(Hf, p_1) = \dots = (Hf, p_{i-1}) = 0$ are satisfied. These conditions are equivalent with

$$\sum_{k=1}^{i} |c_k|^2 = 1 \text{ and } \sum_{k=1}^{i} c_k (H\varphi_{n_k}, p_h) = 0 \quad (h = 1, \dots, i-1),$$

and it follows immediately from a well-known theorem that the $i-1$ homogeneous linear equations

$$\sum_{k=1}^{i} c_k (H\varphi_{n_k}, p_h) = 0 \quad (h = 1, \dots, i-1)$$

have indeed a solution $c_1, \dots, c_i$ for which $\sum_{k=1}^{i} |c_k|^2 = 1$. Observing that

$$(HK\varphi_{n_k}, \varphi_{n_k}) = (HT\varphi_{n_k}, \varphi_{n_k}) = \lambda_{n_k}, \text{ and}$$
$$(HK\varphi_{n_k}, \varphi_{n_l}) = (HT\varphi_{n_k}, \varphi_{n_l}) = 0$$

for $k \neq l$, we find then for $f = \sum_{k=1}^{i} c_k \varphi_{n_k}$ the inequality

$$(HKf, f)/N^2(f) = (HKf, f) = \sum_{k,l=1}^{i} c_k \bar{c}_l (HK\varphi_{n_k}, \varphi_{n_l}) =$$
$$= \sum_{k=1}^{i} \lambda_{n_k} |c_k|^2 \geq \lambda_{n_i} \sum_{k=1}^{i} |c_k|^2 = \lambda_{n_i};$$

it is clear therefore that $\mu_i \geq \lambda_{n_i}$. On the other hand we have, by Theorem 10, for

$$p_1 = \varphi_{n_1}, \dots, p_{i-1} = \varphi_{n_{i-1}}, (Hf, p_1) = \dots = (Hf, p_{i-1}) = 0,$$

the relation $\max (HKf, f)/N^2(f) = \lambda_{n_i}$. Hence $\lambda_{n_i} = \min \mu_i$ [3]).

§ 3. *Expression of the solutions of $Tf - \lambda f = g$ and $Kf - \lambda f = g$ in terms of the characteristic elements.*

We suppose again the transformation K to be symmetrisable and the transformation $T = EK$ to be completely continuous. Then, if $\lambda \neq 0$ is not one of the characteristic values $\lambda_n \neq 0$ of T, it is a regular value (see Theorem 3), on account of Theorem 3 the equation $Tf - \lambda f = g$ has therefore a uniquely determined solution for every element g. If on the other hand λ is identical with one of the characteristic values λ_n, the equation $Tf - \lambda f = g$ has a solution f for those and only those elements

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[3]) Compare R. COURANT und D. HILBERT, Methoden der Math. Physik I, Ch. III, § 4, 3.

g that are orthogonal to all characteristic elements of T^*, belonging to the characteristic value $\bar{\lambda}$. Since however all numbers λ_n are real, we see that g must be orthogonal to all characteristic elements of T^*, belonging to the characteristic value λ.

Theorem 12. *For $\lambda \neq 0$, the equation $Tf - \lambda f = g$ has a solution f for those and only those elements g that are H-orthogonal to all characteristic elements of T, belonging to the characteristic value λ. (If λ is no characteristic value of T, this means that g may be any element.)*

In the case that $Hf = 0$ implies $Kf = 0$, the same statement holds for the transformation K.

Proof. We shall prove first that, if the p-dimensional unitary space of all characteristic elements of T, belonging to the characteristic value λ, is determined by the linearly independent elements $\chi_1, \ldots, \chi_p$, the p-dimensional unitary space of all characteristic elements of T^*, belonging to the characteristic value λ, is determined by $H\chi_1, \ldots, H\chi_p$. Indeed, the relation $T\chi = \lambda\chi$ implies, for every f,

$$(T^*H\chi, f) = (H\chi, Tf) = (\chi, HTf) = (HT\chi, f) = (\lambda H\chi, f);$$

hence $T^*H\chi = \lambda H\chi$. The elements $H\chi_1, \ldots H\chi_p$ are therefore characteristic elements of T^*. We have still to show that they are linearly independent. This follows from the fact that

$$\sum_{i=1}^{p} a_i H\chi_i = 0 \text{ or } H \sum_{i=1}^{p} a_i \chi_i = 0$$

implies

$$T \sum_{i=1}^{p} a_i \chi_i = 0 \text{ or } \sum_{i=1}^{p} \lambda a_i \chi_i = 0,$$

so that, since $\lambda \neq 0$, we have $a_1 = \ldots = a_p = 0$. Finally we observe that, if g is orthogonal to an element $H\chi$, this means that g and χ are H-orthogonal. The result is therefore that $Tf - \lambda f = g$ has a solution f for those and only those elements that are H-orthogonal to all characteristic elements of T, belonging to the characteristic value λ. Evidently this solution is only determined to within an arbitrary linear combination of these characteristic elements. This completes the proof of the first part.

Let now $Hf = 0$ imply $Kf = 0$. Then, since $H = HE$ or $H(I - E) = O$, we have also $K(I - E) = O$ or $K = KE$. We shall prove now that if one of the equations $Kf - \lambda f = g$ and $Tf - \lambda f = g$ has a solution, so has the other. Indeed, from $Kf - \lambda f = g$ follows, since $K = KE$,

$$TEf - \lambda Ef = EKEf - \lambda Ef = E(Kf - \lambda f) = Eg = g - (I - E)g,$$

hence $TEf - \lambda(Ef - \lambda^{-1}(I - E)g) = g$ or, on account of $T(I - E)g = 0$,

$$T(Ef - \lambda^{-1}(I - E)g) - \lambda(Ef - \lambda^{-1}(I - E)g) = g.$$

The element $f_1 = Ef - \lambda^{-1}(I - E)g$ satisfies therefore the relation $Tf_1 - \lambda f_1 = g$.

Conversely, from $Tf - \lambda f = g$ we infer

$$Kf - \lambda f = EKf + (I - E)Kf - \lambda f = g + (I - E)Kf,$$

hence $Kf - \lambda(f + \lambda^{-1}(I - E)Kf) = g$ or, on account of $K(I - E)Kf = 0$,

$$K(f + \lambda^{-1}(I - E)Kf) - \lambda(f + \lambda^{-1}(I - E)Kf) = 0.$$

The element $f_2 = f + \lambda^{-1}(I - E)Kf$ satisfies therefore the relation $Kf_2 - \lambda f_2 = g$.

Thus we find that the equation $Kf - \lambda f = g$ has a solution f for those and only those elements g that are H-orthogonal to all characteristic elements φ of T, belonging to the characteristic value λ. Observing finally that $(Hg, \varphi) = 0$ is equivalent with $(Hg, \psi) = 0$, where ψ is the characteristic element of K corresponding with φ, we obtain the desired result. Evidently the solution of $Kf - \lambda f = g$ is only determined to within an arbitrary linear combination of the characteristic elements of K, belonging to the characteristic value λ.

Theorem 13. *Let $\lambda \neq 0$, and let the element g be H-orthogonal to all characteristic elements of T, belonging to the characteristic value λ. (If λ is no characteristic value, the element g is therefore arbitrary.) Then every solution of $Tf - \lambda f = g$ satisfies the relation*

$$\lim_{k \to \infty} N\left(f + \frac{g}{\lambda} + \sum_{n=1}^{k}{}' \frac{\lambda_n}{\lambda(\lambda - \lambda_n)} a_n \varphi_n\right) = 0,$$

where $a_n = (Hg, \varphi_n)$ for $\lambda_n \neq \lambda$, and where Σ' denotes that for those values of n for which $\lambda_n = \lambda$ the coefficient of φ_n has the value $-(Hf, \varphi_n)$. For every set of arbitrarily prescribed values of the latter coefficients there exists a solution of $Tf - \lambda f = g$.

In the case that $Hf = 0$ implies $Kf = 0$, every solution of $Kf - \lambda f = g$ satisfies the relation

$$\lim_{k \to \infty} N\left(f + \frac{g}{\lambda} + \sum_{n=1}^{k}{}' \frac{\lambda_n}{\lambda(\lambda - \lambda_n)} a_n \psi_n\right) = 0,$$

where $a_n = (Hg, \psi_n)$ for $\lambda_n \neq \lambda$, and where Σ' denotes that for those values of n for which $\lambda_n = \lambda$ the coefficient of ψ_n has the value $-(Hf, \psi_n)$. For every set of arbitrarily prescribed values of the latter coefficients there exists a solution of $Kf - \lambda f = g$.

Proof. Let $Tf - \lambda f = g$. Writing $b_n = (Hf, \varphi_n)$, we have by Theorem 9

$$\lim_{k \to \infty} N\left(Tf - \sum_{n=1}^{k} \lambda_n b_n \varphi_n\right) = \lim_{k \to \infty} N\left(Kf - \sum_{n=1}^{k} \lambda_n b_n \varphi_n\right) = 0,$$

hence

$$\lim_{k \to \infty} N\left(\lambda f + g - \sum_{n=1}^{k} \lambda_n b_n \varphi_n\right) = 0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

From

$$(HTf, \varphi_n) = (f, HT\varphi_n) = (Hf, \lambda_n \varphi_n) = \lambda_n b_n$$

we derive

$$\lambda_n b_n = (HTf, \varphi_n) = (H(g + \lambda f), \varphi_n) = a_n + \lambda b_n,$$

so that for $\lambda_n \neq \lambda$ we find $b_n = -a_n/(\lambda - \lambda_n)$. It follows therefore from (1) that

$$\lim_{k \to \infty} N\left(f + \frac{g}{\lambda} - \sum_{n=1}^{k} \frac{\lambda_n}{\lambda} b_n \varphi_n\right) =$$

$$= \lim_{k \to \infty} N\left(f + \frac{g}{\lambda} + \sum_{n=1}^{k}{}' \frac{\lambda_n}{\lambda(\lambda - \lambda_n)} a_n \varphi_n\right) = 0.$$

Since, if g is given, f is determined to within a linear combination of those elements φ_n for which $\lambda_n = \lambda$, there exists a solution f for every set of arbitrarily prescribed values of the coefficients $b_n = (Hf, \varphi_n)$ for these values of n. This completes the proof of the first part.

The proof of the second part runs in a similar way, substituting everywhere K for T and ψ_n for φ_n.

§ 4. *Self-adjoint transformations as a special case of symmetrisable transformations.*

Identifying the bounded, positive, self-adjoint transformation $H \neq O$ with the identical transformation I, we see that the subspace $[L]$ of all elements satisfying $Hf = If = 0$ contains only the nullelement, so that the orthogonal subspace $[M]$ coincides with the whole space R. The projection E on $[M]$ is therefore the transformation I. The notions of H-orthogonality and H-independence are now identical with the usual notions of orthogonality and linear independence, while $N(f) = \| f \|$. That the bounded linear transformation K is symmetrisable means now that $IK = K$ is self-adjoint, further we have $T = EK = K$.

A bounded symmetrisable transformation K such that $T = EK$ is completely continuous is therefore in this case simply a completely continuous self-adjoint transformation K. The condition that $Hf = If = 0$ implies $Kf = 0$, is always satisfied. The Theorems 4 and 7 lose their meaning; we leave it to the reader to pronounce the other theorems for this case.

§ 5. *Transformations of the form $K = AH$, where A is self-adjoint.*

Theorem 14. *If A is a bounded, self-adjoint transformation, the transformation $K = AH$ is symmetrisable. Further $Hf = 0$ implies $Kf = 0$.*

Proof. $(HKf, g) = (HAHf, g) = (f, HAHg) = (f, HKg)$; the transformation HK is therefore self-adjoint, in other words, K is symmetrisable. The proof of the second part is trivial.

Making now the assumption that one at least of the transformations A and H is completely continuous, the transformation $K = AH$ is symmetrisable and completely continuous. The same holds then for $T = EK$. All theorems in §§ 2—3 are therefore valid for the transformation K. It is possible, however, to prove somewhat more.

Theorem 15. (*Expansion Theorem.*) *If ψ_i is the H-orthonormal sequence of characteristic elements of $K = AH$, belonging to the sequence* *of characteristic values $\lambda_i \neq 0$, and if $a_i = (Hf, \psi_i)$, then*

$$Kf = \Sigma \lambda_i a_i \psi_i + p,$$

where $Hp = 0$. For $n \geq 2$ we have

$$K^n f = \Sigma \lambda_i^n a_i \psi_i.$$

Proof [4]. We observe first that, the system ψ_i being H-orthonormal, the system $H^{1/2} \psi_i$ is orthonormal, since $(H\psi_i, \psi_k) = (H^{1/2} \psi_i, H^{1/2} \psi_k)$. Writing $a_i = (g, H^{1/2} \psi_i)$ for an arbitrary g, the sums $s_k = \sum_{i=1}^{k} a_i H^{1/2} \psi_i$ converge therefore to an element h. Taking $g = H^{1/2} f$, we find then $\Sigma a_i H^{1/2} \psi_i = h$, where $a_i = (H^{1/2} f, H^{1/2} \psi_i) = (Hf, \psi_i)$. From this we derive

$$A H^{1/2} h = A H^{1/2} \Sigma a_i H^{1/2} \psi_i = \Sigma a_i A H \psi_i = \Sigma \lambda_i a_i \psi_i.$$

The convergence of the series $\Sigma \lambda_i a_i \psi_i$ enables us now to make $k \to \infty$ in the relation

$$\lim_{k \to \infty} N\left(Kf - \sum_{i=1}^{k} \lambda_i a_i \psi_i\right) = 0,$$

proved in Theorem 9. Writing $Kf - \Sigma \lambda_i a_i \psi_i = p$, we obtain then $N(p) = 0$; hence, $N(p) = 0$ being equivalent with $Hp = 0$,

$$Kf = \Sigma \lambda_i a_i \psi_i + p,$$

where $Hp = 0$.

From this we deduce

$$K^2 f = \Sigma \lambda_i a_i K \psi_i + K p = \Sigma \lambda_i^2 a_i \psi_i + Kp,$$

but, since $Hp = 0$, we have $Kp = AHp = 0$; hence

$$K^2 f = \Sigma \lambda_i^2 a_i \psi_i.$$

The relation

$$K^n f = \Sigma \lambda_i^n a_i \psi_i$$

for $n > 2$ follows easily by induction.

It may be asked whether the element p, occurring in Theorem 15, is not always identical with the nullelement. The answer to this question is given by

Theorem 16. *The element p in Theorem 15 is not necessarily identical with the nullelement.*

Proof. Let R be a complete, separable HILBERT space in which φ_n is a complete orthonormal system, and let $\mu_3, \mu_4, \ldots$ and $\nu_3, \nu_4, \ldots$ be sequences of positive numbers for which $\lim \mu_n = \lim \nu_n = 0$ and $\mu_3 > \mu_4 > \ldots$, $\nu_3 > \nu_4 > \ldots$. Defining the positive, self-adjoint transformation H and the self-adjoint transformation A by

$$H\varphi_1 = \varphi_1, \quad H\varphi_2 = 0, \quad H\varphi_i = \mu_i \varphi_i \qquad (i = 3, 4, \ldots),$$
$$A\varphi_1 = \varphi_2, \quad A\varphi_2 = \varphi_1, \quad A\varphi_1 = \nu_i \varphi_i \qquad (i = 3, 4, \ldots),$$

[4]) This proof is simpler than that in [1]).

it is not difficult to see that both H and A are completely continuous. We have

$$AH\varphi_1 = \varphi_2,\ AH\varphi_2 = 0,\ AH\varphi_i = \nu_i \mu_i \varphi_i \qquad (i = 3, 4, \ldots).$$

To find the characteristic elements of AH belonging to characteristic values $\neq 0$, we write $AHf = \lambda f$ for $f = \sum_{i=1}^{\infty} a_i \varphi_i$ and $\lambda \neq 0$. From this we derive

$$a_1 \varphi_2 + \sum_3 \nu_i \mu_i a_i \varphi_i = \sum_1 \lambda a_i \varphi_i;$$

hence $a_1 = a_2 = 0$ and $\nu_i \mu_i a_i = \lambda a_i (i = 3, 4, \ldots)$. Since $\nu_i \mu_i \neq \nu_k \mu_k$ for $i \neq k$ we have therefore $\lambda = \nu_k \mu_k$ for a certain value of $k (\geq 3)$ and $a_i = 0$ for $i \neq k$, which shows that the elements $a_k \varphi_k$ $(k \geq 3)$ are the only characteristic elements with characteristic values $\neq 0$. Making them H-normal, we obtain $a_k = \mu_k^{-1/2}$, so that, by Theorem 15,

$$AHf = \sum_3 \nu_i (Hf, \varphi_i) \varphi_i + p$$

for every f. Taking $f = \varphi_1$, we have $(Hf, \varphi_i) = (H\varphi_1, \varphi_i) = (\varphi_1, \varphi_i) = 0$ $(i \geq 3)$ and $AHf = AH\varphi_1 = \varphi_2$; hence $p = \varphi_2 \neq 0$.

Theorem 17. *Let $\lambda \neq 0$, and let the element g be H-orthogonal to all characteristic elements of $K = AH$ belonging to the characteristic value λ. (If λ is no characteristic value, the element g is therefore arbitrary.) Then every solution of $Kf - \lambda f = g$ satisfies a relation of the form*

$$f = -\frac{g}{\lambda} - \sum{}' \frac{\lambda_i}{\lambda(\lambda - \lambda_i)} a_i \psi_i + q,$$

where $a_i = (Hg, \psi_i)$ for $\lambda_i \neq \lambda$, $Hq = 0$, and where Σ' denotes that for those values of i for which $\lambda_i = \lambda$ the coefficient of ψ_i has the value (Hf, ψ_i). For every set of arbitrarily prescribed values of the latter coefficients there exists a solution of $Kf - \lambda f = g$.

Proof. Let $Kf - \lambda f = g$. By Theorem 15 we have

$$Kf = AHf = \sum \lambda_i b_i \psi_i + p,$$

where $b_i = (Hf, \psi_i)$ and $Hp = 0$. Since, as we have proved in Theorem 13, $b_i = -a_i/(\lambda - \lambda_i)$ for $\lambda_i \neq \lambda$, we obtain

$$\lambda f + g = -\sum{}' \frac{\lambda_i}{\lambda - \lambda_i} a_i \psi_i + p$$

or

$$f = -\frac{g}{\lambda} - \sum{}' \frac{\lambda_i}{\lambda(\lambda - \lambda_i)} a_i \psi_i + q,$$

where we have written $q = p/\lambda$. The last statement of the theorem has been proved already in Theorem 13.

Theorem 18. *The self-adjoint transformation $\widetilde{K} = H^{1/2} A H^{1/2}$ possesses the same sequence λ_i of characteristic values $\neq 0$ as the transformation $K = AH$. If ψ_i is an H-orthonormal sequence of characteristic elements* *of K, corresponding with the characteristic values λ_i, then $H'^{1/2}\psi_i$ is an orthonormal sequence of characteristic elements of $\widetilde{K}$, also corresponding with the characteristic values λ_i.*

Proof. Let $K\psi = AH\psi = \lambda\psi$ where $\lambda\psi \neq 0$. Then, writing $H'^{1/2}\psi = \chi$, we have

$$\widetilde{K}\chi = H'^{1/2} A H \psi = \lambda H'^{1/2} \psi = \lambda\chi,$$

where $\lambda\chi \neq 0$ since $\lambda \neq 0$ and $AH'^{1/2}\chi = AH\psi = \lambda\psi \neq 0$. Conversely, if $\widetilde{K}\chi = \lambda\chi$ where $\lambda\chi \neq 0$, we find, writing $\psi = \lambda^{-1}AH'^{1/2}\chi$, that

$$H'^{1/2}\psi = \lambda^{-1}\widetilde{K}\chi = \chi;$$

hence

$$K\psi = A H'^{1/2} H'^{1/2} \psi = A H'^{1/2} \chi = \lambda\psi,$$

where $\lambda\psi \neq 0$ since $H'^{1/2}\lambda\psi = \lambda\chi \neq 0$. This shows that K and $\widetilde{K}$ have the same characteristic values $\neq 0$, and that with the H-orthonormal sequence ψ_i of characteristic elements of K corresponds the orthonormal sequence $H'^{1/2}\psi_i$ of characteristic elements of $\widetilde{K}$.