Mathematics. — *On the theory of linear integral equations.* II. By A. C. ZAANEN. (Communicated by Prof. W. VAN DER WOUDE.)

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§ 1. *Introduction.*

We suppose the reader to be acquainted with the contents of the first paper bearing the same title, to which we shall refer with I. Let a_i, b_i $(i = 1, \ldots, m)$ be real and such that $a_i < b_i$ $(i = 1, \ldots, m)$. Then $\triangle = [a_1, b_1; a_2, b_2; \ldots; a_m, b_m]$ is an interval in the m-dimensional Euclidean space. The point $(x_1, x_2, \ldots, x_m)$ in this space will be denoted by x. Further we shall denote the function space of all functions $f(x)$, with complex values, such that $|f(x)|^2$ is summable (in the sense of LEBESGUE) over $\triangle$, by $L_2^{(m)}(\triangle)$ or $L_2(\triangle)$ or shortly by L_2. As well-known, L_2 is a HILBERT space, if addition and multiplication with complex numbers are defined in the usual way, and the inner product of f and g as

$$(f, g) = \int_{\triangle} f(x)\,\overline{g(x)}\,dx,$$

so that

$$\|f\| = \left(\int_{\triangle} |f|^2\,dx\right)^{1/2}.$$

Convergence of the sequence $s_n(x)$ to $f(x)$ in the space L_2 means that $\lim_{n\to\infty} \|f - s_n\| = 0$; convergence of the series $\sum_{i=1}^{\infty} f_i(x)$ to $f(x)$ means that $\lim_{n\to\infty} \|f - \sum_{i=1}^{n} f_i\| = 0$; to avoid misunderstandings, we shall say that $\sum_{i=1}^{\infty} f_i(x)$ converges *in mean* to $f(x)$, and write

$$f(x) \backsim \sum_{i=1}^{\infty} f_i(x),$$

reserving the term convergence for ordinary point-wise convergence. The interval $[a_1, b_1; \ldots; a_m, b_m; a_1, b_1; \ldots; a_m, b_m]$ in $2m$-dimensional Euclidean space will be denoted by $\triangle \times \triangle$, and the function space of all functions $f(x, y)$ $(x, y \in \triangle)$, with complex values, for which $|f(x, y)|^2$ is summable over $\triangle \times \triangle$, by $L_2^{(2m)}(\triangle)$ or $L_2^{(2m)}$. This function space is also a HILBERT space with inner product

$$(f, g)_{2m} = \int_{\triangle\times\triangle} f(x, y)\,\overline{g(x, y)}\,dx\,dy,$$

so that

$$\|f\|_{2m} = \left(\int\limits_{\triangle \times \triangle} |f(x, y)|^2 \, dx \, dy \right)^{1/2}.$$

Convergence in mean of $\sum_{i=1}^{\infty} f_i(x, y)$ to $f(x, y)$ means that

$$\lim_{n \to \infty} \| f(x, y) - \sum_{i=1}^{n} f_i(x, y) \|_{2m} = 0,$$

and we shall write

$$f(x, y) \backsim \sum_{i=1}^{\infty} f_i(x, y).$$

Let now the measurable function $K(x, y)$ be defined in $\triangle \times \triangle$, such that

$$\| K(x, y) \|_{2m} = \left(\int\limits_{\triangle \times \triangle} | K(x, y) |^2 \, dx \, dy \right)^{1/2}$$

is finite, in other words, $K(x, y) \in L_2^{(2m)}$. Then it is well-known that the linear transformation K in the space L_2, defined by

$$Kf = \int\limits_{\triangle} K(x, y) f(y) \, dy$$

is completely continuous. If, moreover, $\| K(x, y) \|_{2m} \neq 0$, the transformation K is not identical with the nulltransformation O. We shall consider now the *linear integral equation* $Kf - \lambda f = g$ or

$$\int\limits_{\triangle} K(x, y) f(y) \, dy - \lambda f(x) = g(x) \quad . \; . \; . \; . \; . \; . \quad (1)$$

where $f, g \in L_2$, the *kernel* $K(x, y) \in L_2^{(2m)}$, and λ is a complex number. If $K(x, y) = \overline{K(y, x)}$ for almost all points $(x, y) \in \triangle \times \triangle$, the kernel $K(x, y)$ is called *Hermitean*. It follows easily by FUBINI's Theorem on successive integrations that then $(Kf, g) = (f, Kg)$ for arbitrary $f, g \in L_2$, so that in this case the transformation K is completely continuous and self-adjoint.

§ 2. *Integral equation with Hermitean kernel, belonging to* $L_2^{(2m)}$.

Let $K(x, y) \in L_2^{(2m)}$ be Hermitean. Then the theorems, proved in I (for the special case, considered in I, § 4), yield the following results [1]):

1) A part of the theorems in this paragraph was proved in a different way, following the method of E. SCHMIDT, by F. SMITHIES, The eigenvalues and singular values of integral equations, Proc. of the London Math. Soc. (2) 43 (1937), 255—279.

Theorem 1. *Every characteristic value* $\lambda \neq 0$ *of* (1) *has finite multiplicity. The total number of different characteristic values* λ_i *is finite or enumerable, and in this last case* $\lim_{i=\infty} \lambda_i = 0$.

Proof. Follows from I, Theorem 2.

Theorem 2. *The characteristic values are real, and characteristic functions, belonging to different characteristic values, are orthogonal.*

Proof. Follows from I, Theorem 5.

Theorem 3. *If* $\lambda \neq 0$ *is a characteristic value of* (1), *this equation has, for a given function* $g(x) \in L_2$, *a solution* $f(x) \in L_2$ *for those and only those functions* $g(x)$ *that are orthogonal to all characteristic functions, belonging to the characteristic value* λ. *If* $\lambda \neq 0$ *is no characteristic value, so that it is a regular value, the equation* (1) *has a uniquely determined solution for every* $g(x) \in L_2$.

Proof. Follows from the preceding theorem and I, Theorem 3.

Theorem 4. *If* $\| K(x, y) \|_{2m} \neq 0$, *the equation* (1) *has a characteristic value* $\neq 0$.

Proof. Follows from I, Theorem 6.

Let now $\lambda_i (| \lambda_1 | \geqq | \lambda_2 | \geqq \ldots)$ be the sequence of all characteristic values $\neq 0$, each of them occurring as many times as denoted by its multiplicity, and $\varphi_i(x)$ a corresponding orthonormal sequence of characteristic functions. Then we have

Theorem 5. *If* $a_i = (f, \varphi_i) = \int_\Delta f(x) \overline{\varphi_i(x)}\, dx$, *then*

$$\int_\Delta K(x, y) f(y)\, dy \backsim \Sigma \lambda_i a_i \varphi_i(x) \quad \textit{(Expansion Theorem)},$$

$$\int_{\Delta \times \Delta} K(x, y) \overline{f(x)} f(y)\, dx\, dy = \Sigma \lambda_i |a_i|^2$$

for any $f(x) \in L_2$.

Proof. Follows from I, Theorem 9.

Theorem 6 (Courant's Theorem). *Let* λ_{n_i} $(i = 1, 2, \ldots)$ *be the subsequence of all positive characteristic values where* $\lambda_{n_1} \geqslant \lambda_{n_2} \geqslant \ldots$, *let the functions* $p_1(x), p_2(x), \ldots, p_{i-1}(x)$ *be arbitrarily given, and let*

$$\mu_i = \text{upper bound} \int_{\Delta \times \Delta} K(x, y) \overline{f(x)} f(y)\, dx\, dy \Big/ \int_\Delta |f|^2\, dx$$

for all $f(x) \in L_2$ *satisfying* $\int_\Delta |f|^2\, dx \neq 0$ *and*

$$\int_\Delta f \overline{p_1}\, dx = \ldots = \int_\Delta f \overline{p_{i-1}}\, dx = 0.$$

The number μ_i *depends on* $p_1(x), \ldots, p_{i-1}(x)$. *Letting now these functions run through the whole space* L_2, *we have* $\lambda_{n_i} = \min \mu_i$ [2]).

A similar statement holds for the negative characteristic values.

Proof. Follows from I, Theorem 11.

Theorem 7. *Let* $\lambda \neq 0$, *and let* $g(x) \in L_2$ *be orthogonal to all characteristic functions of* (1), *belonging to the characteristic value* λ. (*If* λ *is no characteristic value, the function* $g(x)$ *is therefore arbitrary.*) *Then the solution of* (1) *is given by*

$$f(x) \backsim -\frac{g(x)}{\lambda} - \Sigma' \frac{\lambda_i}{\lambda(\lambda - \lambda_i)} a_i \varphi_i(x),$$

where $a_i = \int\limits_{\triangle} g(x)\overline{\varphi_i(x)}dx$ *for* $\lambda_i \neq \lambda$, *and where* Σ' *denotes that for those values of i for which* $\lambda_i = \lambda$ *the coefficient of* $\varphi_i(x)$ *may have any arbitrary value.*

Proof. Follows from I, Theorem 13.

It is not possible to obtain an expansion theorem for the kernel itself directly from the theorems in I. A little consideration will show, however, that the following theorem holds:

Theorem 8. *We have*

$$\int\limits_{\triangle} |K(x, y)|^2 dy = \Sigma \lambda_i^2 |\varphi_i(x)|^2 \text{ for almost every } x \in \triangle, \quad \ldots \quad (2)$$

$$\int\limits_{\triangle \times \triangle} |K(x, y)|^2 dx\, dy = \Sigma \lambda_i^2, \quad \ldots \quad (3)$$

$$K(x, y) \backsim \Sigma \lambda_i \varphi_i(x) \overline{\varphi_i(y)} \text{ (Expansion Theorem)} \quad \ldots \quad (4)$$

Proof. From $\| K(x, y)\|_{2m} < \infty$ follows that the function $k(x) = K(x, y)$ belongs to the space $L_2^{(m)}$ for almost every $y \in \triangle$. We shall show now that the relations $(k, \varphi_i) = \lambda_i \overline{\varphi_i(y)}$ and $(k, g) = 0$ for every $g(x) \in L_2$ orthogonal to all $\varphi_i(x)$, hold for almost every $y \in \triangle$, so that it will be possible to write $k = \Sigma(k, \varphi_i)\varphi_i = \Sigma \lambda_i \overline{\varphi_i(y)} \cdot \varphi_i$ in the terminology of HILBERT space. Indeed,

$$(k, \varphi_i) = \int\limits_{\triangle} K(x, y) \overline{\varphi_i(x)} dx = \overline{\int\limits_{\triangle} K(y, x) \varphi_i(x) dx} = \lambda_i \overline{\varphi_i(y)}$$

[2]) R. COURANT, Zur Theorie der linearen Integralgleichungen, Math. Annalen 89 (1923), 161—178, proved his theorem for a continuous kernel, approximating this kernel by a sequence of kernels having only a finite number of characteristic values. He remarks that his result remains valid in the case that $\int\int |K(x, y)|^2 dx\, dy$ is finite, and $\int |K(x, y)|^2 dy$ is bounded. As we see here, the first condition alone is sufficient (and, as may be proved, not even necessary).

for almost every $y \in \triangle$, and, if $(g, \varphi_i) = 0$ for all values of i, so that by Theorem 5 $\int_\triangle K(x, y)\, g(y) dy = 0$ almost everywhere, we have

$$(k, g) = \int_\triangle K(x, y)\, \overline{g(x)}\, dx = \overline{\int_\triangle K(y, x)\, g(x)\, dx} = 0$$

for almost every $y \in \triangle$.

The relation $k = \Sigma(k, \varphi_i)\varphi_i$ implies

$$\| k - \sum_{i=1}^{n} (k, \varphi_i)\varphi_i \|^2 = \sum_{i=n+1} |(k, \varphi_i)|^2;$$

hence

$$\int_\triangle |K(x, y) - \sum_{i=1}^{n} \lambda_i \varphi_i(x) \overline{\varphi_i(y)}|^2 dx = \sum_{i=n+1} \lambda_i^2 |\varphi_i(y)|^2 \quad . \quad . \quad (5)$$

for almost every $y \in \triangle$. Taking $n = 0$, we obtain (2), and. integrating (5) over y, we see that

$$\int_{\triangle\times\triangle} |K(x, y) - \sum_{i=1}^{n} \lambda_i \varphi_i(x) \overline{\varphi_i(y)}|^2 dx\, dy = \sum_{i=n+1} \lambda_i^2 .$$

For $n = 0$ we have (3), and, making $n \to \infty$, we find (4).

By direct computation it may be verified readily that the *iterated* kernels $K_n(x, y) = \int_\triangle K(x, z) K_{n-1}(z, y) dz$ $(n = 2, 3, \dots)$ exist almost everywhere in $\triangle \times \triangle$, that they are also Hermitean and belong to the space $L_2^{(2m)}$. It is not difficult to prove that the sequence λ_i^n $(i = 1, 2, \dots)$ is the sequence of all characteristic values $\neq 0$ of $K_n(x, y)$, and that $\varphi_i(x)$ $(i = 1, 2, \dots)$ is a corresponding sequence of characteristic functions. The Theorems 1—8 hold therefore for the integral equation with kernel $K_n(x, y)$, replacing everywhere λ_i by λ_i^n.

The numbers $\int_\triangle K_n(x, x) dx$ are called the *traces* of $K(x, y)$. If n is even $(n = 2p)$, we have

$$\int_\triangle K_n(x, x)\, dx = \int_{\triangle\times\triangle} |K_p(x, z)|^2\, dx\, dz = \Sigma \lambda_i^{2p} = \Sigma \lambda_i^n .$$

We shall prove that this relation is also true for odd $n > 1$.

Theorem 9. $\int\limits_{\triangle} K_n(x,x)\,dx = \Sigma \lambda_i^n$ for $n \geq 2$.

Proof. Let $n \geq 2$. From

$$\left| K_n(x,x) - \sum_{i=1}^{p} \lambda_i^n \, |\varphi_i(x)|^2 \right| =$$

$$= \left| \int\limits_{\triangle} K(x,z) \left[K_{n-1}(z,x) - \sum_{i=1}^{p} \lambda_i^{n-1} \varphi_i(z) \overline{\varphi_i(x)} \right] dz \right| \leqslant \left(\int\limits_{\triangle} |K(x,z)|^2 dz \right)^{1/2} \cdot$$

$$\cdot \left(\int\limits_{\triangle} \left| K_{n-1}(z,x) - \sum_{i=1}^{p} \lambda_i^{n-1} \varphi_i(z) \overline{\varphi_i(x)} \right|^2 dz \right)^{1/2},$$

holding for almost every $x \in \triangle$, follows on account of SCHWARZ's inequality

$$\left| \int\limits_{\triangle} K_n(x,x)\,dx - \sum_{i=1}^{p} \lambda_i^n \right| = \left| \int\limits_{\triangle} \left\{ K_n(x,x) - \sum_{i=1}^{p} \lambda_i^n \, |\varphi_i(x)|^2 \right\} dx \right| \leqslant$$

$$\leqslant \int\limits_{\triangle} \left| K_n(x,x) - \sum_{i=1}^{p} \lambda_i^n \, |\varphi_i(x)|^2 \right| dx \leqslant$$

$$\left(\int\limits_{\triangle \times \triangle} |K(x,z)|^2 \,dx\,dz \right)^{1/2} \cdot \left(\int\limits_{\triangle \times \triangle} \left| K_{n-1}(z,x) - \sum_{i=1}^{p} \lambda_i^{n-1} \varphi_i(z) \overline{\varphi_i(x)} \right|^2 dx\,dz \right)^{1/2};$$

hence, since the last factor on the right tends to 0 as $p \to \infty$ by Theorem 8, (4),

$$\int\limits_{\triangle} K_n(x,x)\,dx = \Sigma \lambda_i^n .$$

Remark. It is well-known that if $K(x,y)$ satisfies the conditions that $\int\limits_{\triangle} |K(x,y)|^2 dy$ is finite for every $x \in \triangle$ and

$$\lim_{x_2 \to x_1} \int\limits_{\triangle} |K(x_2,y) - K(x_1,y)|^2 dy = 0,$$

several of the results in this paragraph can be improved. The convergence in mean in the Theorems 5 and 7, and also the convergence almost everywhere in Theorem 8, (2), may be replaced by uniform convergence. Moreover, for $n \geq 2$, the series $\Sigma \lambda_i^n \varphi_i(x) \overline{\varphi_i(y)}$ converges uniformly in $\triangle \times \triangle$ to $K_n(x,y)$.

§ 3. *Positive Hermitean kernels.*

It was proved by J. MERCER that, given the positive Hermitean kernel $H(x, y)$, continuous in $\triangle \times \triangle$, we have, uniformly in $\triangle \times \triangle$,

$$H(x, y) = \Sigma \lambda_i \varphi_i(x) \overline{\varphi_i(y)},$$

where $\lambda_i (i = 1, 2, \ldots)$ is the sequence of all characteristic values $\neq 0$ of $H(x, y)$ and $\varphi_i(x)$ a corresponding orthonormal sequence of characteristic functions. As well-known, all λ_i are non-negative (follows easily from Theorem 5).

Theorem 10. *If in the space $L_2(\triangle)$ the bounded, positive, self-adjoint transformation H is given by*

$$Hf = \int_{\triangle} H(x, y) f(y)\, dy,$$

where $H(x, y)$ is a positive Hermitean kernel, continuous in $\triangle \times \triangle$, then there exists a positive Hermitean kernel $H_{1/2}(x, y)$ for which

$$\int_{\triangle} |H_{1/2}(x, y)|^2\, dy$$

is bounded, such that the uniquely determined, bounded, positive, self-adjoint transformation $H^{1/2}$ is given by

$$H^{1/2} f = \int_{\triangle} H_{1/2}(x, y) f(y)\, dy.$$

Proof. From

$$H(x, y) = \Sigma \lambda_i \varphi_i(x) \overline{\varphi_i(y)},$$

holding uniformly in $\triangle \times \triangle$, follows

$$\int_{\triangle} H(x, x)\, dx = \Sigma \lambda_i = \Sigma (\lambda_i^{1/2})^2;$$

the series $\Sigma \lambda_i^{1/2} \varphi_i(x) \overline{\varphi_i(y)}$ converges therefore in mean (convergence in the HILBERT space $L_2^{(2m)}$) to a function $H_{1/2}(x, y) \in L_2^{(2m)}$, and it is not difficult to see that the transformation A, defined by

$$A f = \int_{\triangle} H_{1/2}(x, y) f(y)\, dy,$$

has $\lambda_i^{1/2} (i = 1, 2, \ldots)$ as the sequence of its characteristic values $\neq 0$, and

$\varphi_i(x)$ as a corresponding sequence of characteristic functions. This transformation is therefore identical with $H^{1/2}$, so that

$$H^{1/2} f = \int_{\Delta} H_{1/2}(x, y) f(y) \, dy.$$

By Theorem 8 we have

$$\int_{\Delta} |H_{1/2}(x, y)|^2 \, dy = \Sigma \lambda_i \, |\varphi_i(x)|^2 = H(x, x)$$

for almost every $x \in \Delta$, and, since it is allowed to change the values of $H_{1/2}(x, y)$ in a set of measure 0, we may even suppose this relation to hold for every $x \in \Delta$. Observing finally that $H(x, x)$, as a continuous function, is bounded, we obtain the desired result.