Mathematics. — *A theorem concerning analytic continuation.* By J. DE GROOT. (Communicated by Prof. J. G. VAN DER CORPUT.)

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1. Two one-valued analytic functions $f_1(z)$ and $f_2(z)$, originally defined in the respective regions G_1 and G_2, are identical in $G_1 + G_2$ if they take the same values in the points of an arbitrarily small region belonging both to G_1 and G_2; and, what is more, they are identical even if they take the same values in infinitely many points which have a limit-point belonging both to G_1 and G_2. In this case we say that the original $f_1(z)$, defined only in G_1, may be continued analytically in the region G_2.

Conversely we may ask ourselves the following question: in the complex Z-plane be given infinitely many points $z_1, z_2, z_3, \ldots$ converging to the point z'. Each point z_i $(i = 1, 2, \ldots)$ is given a certain complex "function-value" w_i. To which extent is it possible to find a one-valued analytic function $f(z)$, defined in a certain region in the Z-plane, which exactly in the points z_i takes the given values w_i; in other words, to which extent is it possible to continue the function defined in z_i to a certain region? In the following we shall — after giving the problem a more accurate form — give a necessary and sufficient condition for analytic continuation (theorem I). For part of the proof we need HANS FREUDENTHAL's mean value theorem of the theory of complex functions, which is generalized by us for higher difference-quotients (theorems II and III).

2. In the problem of continuity mentioned in 1. one must make a distinction between the two following principally different cases: 1°. the region where one wishes the function to be continued does not necessarily contain the limit-point z'; 2°. the region in question does contain z'. Case 1°. immediately leads to the well-known problem of composing an integral function which in the points z_i $(i = 1, 2, \ldots)$, that nowhere in the finite have a limit-point, takes the prescribed value w_i (comp. K. KNOPP, Funktionentheorie (Sammlung Göschen) II, 4th ed., p. 38 problem 3, p. 45 problem 4). The following function $f(z)$ (composed by J. VAN IJZEREN) satisfies this demand:

$$f(z) = \prod_{i=1}^{\infty} \left(1 - \frac{z}{z_i}\right) e^{\frac{z}{z_i} + \frac{1}{2}\left(\frac{z}{z_i}\right)^2 + \ldots \frac{1}{k_i}\left(\frac{z}{z_i}\right)^{k_i}} \cdot \sum_{i=1}^{\infty} \left(\frac{w_i}{p_i\left(1 - \frac{z}{z_i}\right)} - g_i(z) \right).$$

Here Π is a WEIERSTRASZ-product, being 0 in the points z_i; p_i is the value of this infinite product if it is divided by $\left(1 - \frac{z}{z_i}\right)$, while $g_i(z)$ are suitably chosen integral rational functions securing convergence (in case a $z_j = 0$ the shape of $f(z)$ must be slightly altered). $f(z)$ is an obviously integral function, which in the points z_i takes the values w_i. By submitting the complex plane and thereby the points z_i and the variable z to the following linear transformation

$$z_i = \frac{1}{z_i' - z'}, \qquad z = \frac{1}{\bar{z} - z'},$$

where the sequence of points z_i' converges to z', and by substituting these values in $f(z)$ one finds an analytic function $f(\bar{z})$ which is everywhere one-valuedly analytic except perhaps in the point z', and which in the given points z_i' takes the given values w_i. Thus $f(\bar{z})$ offers a solution to the problem put in case 1°.

We may further confine ourselves to case 2°., where analytic continuation must be possible in a region containing the limit-point z' of the point-sequence z_i. If such a function exists it is (according to 1.) uniquely defined (contrary to the problem put in 1°., where several continuation-functions $f(\bar{z})$ are possible). The required function must take the given values w_i in the points z_i. It is, however, asking too much when we demand that the region in which analytic continuation is possible contains besides z' also all points z_i, as in this case — because of the uniqueness of the function — the required continuation would be impossible by every change, however small, of but one of the values w_i. So we finally ask ourselves the following question: on which conditions is it possible to find an analytic function, defined in a region containing the limit-point z' of the sequence z_i (and so nearly all points z_i), which in almost all points z_i takes prescribed values w_i?

3. It must be possible to expand the required continued function $f(z)$ into a power-series in z':

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z')}{n!}(z - z')^n. \tag{1}$$

Thus necessarily all derivates of $f(z)$ in z' exist. The values w_i in z_i therefore must be chosen such that the limit of the nth difference-quotient exists for every natural number n if the $(n + 1)$ points $z_{i_0}, z_{i_1}, \ldots, z_{i_n}$ on which this difference-quotient is defined, tend to z'. Further we must demand that these limits do not tend too quickly to the infinite for $n \to \infty$ and that in such a way that the convergence-radius of (1) is > 0. At first sight one might expect these two conditions to be sufficient for the required analytic continuation. We shall, however, show by an example that this is not the case. We define $f(x)$ with real x by

$$f(x) = e^{-\frac{1}{x^2}} \quad \text{for } x \neq 0 \quad ; \quad f(0) = 0.$$

Apparently all derivates in 0 exist and have the value 0. Now, if on the real axis we take a sequence of points converging to 0 and give these points the corresponding function-values $f(x)$, the above-mentioned conditions are satisfied, i.e., when one runs through the sequence of points the nth difference-quotient tends to the fixed limit 0 for every n, and these limits, i.e., the nth derivates in 0, all are 0, so they all certainly are (uniformly) bounded. Yet analytic continuation to a neighbourhood of the origin is impossible as $e^{-\frac{1}{z^2}}$ is not analytic in the origin.

Now it is possible by sharpening the above-mentioned conditions to find a sufficient condition, namely by demanding as well that the nth difference-quotients "tend uniformly in n to their limits". We prefer, however, to derive a sufficient condition (which will prove to be necessary as well), which moreover, as we may see immediately, is a direct result of the first-mentioned sufficient condition.

4. The nth difference-quotient of $f(z)$ may, as is generally known, be defined like follows:

$$[z_1](f) = f(z_1) \qquad \text{and } [z_1\, z_2](f) = \frac{[z_1](f) - [z_2](f)}{z_1 - z_2}$$

are the 0th and first difference-quotients of $f(z)$ in z_1, and z_1 and z_2 respectively. By induction one defines the nth difference-quotient by

$$[z_1\, z_2 \ldots . z_n\, z_{n+1}](f) = n. \frac{[z_1\, z_3 \ldots . z_{n+1}](f) - [z_2\, z_3 \ldots . z_{n+1}](f)}{z_1 - z_2}.$$

For shortness we introduce the following notation:

$$[z_i\, z_{i+1} \ldots z_{i+n}](f) = \triangle_i^n \qquad (n = 1, 2, \ldots; i = 1, 2, \ldots).$$

Our function $f(z)$ now has already been defined in the points z_j; for $f(z_j) = w_j$. Thus we may compose the difference-quotients $\triangle_j^n$ defined on a number of points z_j.

We now demand *that there may be found a sufficiently great index i* (which we may fix from then onwards), *a positive number r and a natural number n_0 such that for every natural number $n \geqq n_0$*

$$|\triangle_i^n| \leqq n!\, r^n \qquad (n = n_0, n_0 + 1, \ldots). \tag{2}$$

We shall now prove

Theorem I. *If the values $w_j = f(z_j)$ satisfy condition (2) it is possible to find a neighbourhood of the limit-point z' where f may be continued analytically; i.e., it is possible to find a uniquely defined analytic function $f(z)$ (mentioned in (3)), which in almost all points z_j takes the given values w_j. Conversely, if $f(z)$ is a one-valued analytic function in a neighbourhood of point z' and if $z_1, z_2, \ldots$ is a sequence of points converging to z', then condition (2) holds true for every sequence of this kind. Therefore condition (2) is a necessary and sufficient condition for the required analytic continuation.*

5. *Condition (2) is sufficient for the required analytic continuation.*

We contend that

$$f(z) = w_i + \sum_{n=1}^{\infty} \left(\frac{\Delta_i^n}{n!} \cdot \prod_{j=0}^{n-1} (z - z_{i+j}) \right) \tag{3}$$

is the required analytic function in a sufficiently small neighbourhood of z'. In the first place it is evident that $f(z)$ has exactly the required values w_{i+k} in the points z_{i+k} $(k = 0, 1, 2, \ldots)$, as

$$w_{i+k} = w_i + \sum_{n=1}^{k} \left(\frac{\Delta_i^n}{n!} \cdot \prod_{j=0}^{n-1} (z_{i+k} - z_{i+j}) \right) \qquad (k = 1, 2, \ldots)$$

according to the interpolation-formula of NEWTON. Further the series (3) is uniformly convergent in a neighbourhood of z'. For it is possible to find an $m \geqq n_0$ and a neighbourhood of z' such that for all points z of that neighbourhood and for all points z_{i+m+l} $(l = 1, 2, \ldots)$

$$| z - z_{i+m+l} | < \delta \qquad (\delta \text{ arbitrarily small}, > 0). \tag{4}$$

The absolute value of (3) then is smaller than or equal to

$$\left| w_i + \sum_{n=1}^{m+1} \left(\frac{\Delta_i^n}{n!} \cdot \prod_{j=0}^{n=1} (z - z_{i+j}) \right) \right| + \prod_{s=0}^{m} | z - z_{i+s} | \cdot$$
$$\cdot \left| \sum_{n=m+2}^{\infty} \left(\frac{\Delta_i^n}{n!} \cdot \prod_{j=m+1}^{n+1} (z - z_{i+j}) \right) \right| .$$

Further, in connection with (2) and (4),

$$\left| \sum_{n=m+2}^{\infty} \left(\frac{\Delta_i^n}{n!} \cdot \prod_{j=m+1}^{n-1} (z - z_{i+j}) \right) \right| \leqq \sum_{n=m+2}^{\infty} r^n, \delta^{n-m-1} = \frac{1}{\delta^{m+1}} \cdot \sum_{n=m+2}^{\infty} (r.\, \delta)^n.$$

We now choose $\delta < 1/r$, from which follows that the last series is convergent and so the given series (3) is uniformly convergent in a sufficiently small neighbourhood of z'. As all terms of the series (3) are analytic functions, $f(z)$ is (according to a well-known theorem of WEIERSTRASZ) an analytic function in that neighbourhood of z'.

6. For the second part of the proof of theorem I the mean value theorem of the theory of complex functions is of importance. This theorem may be formulated as follows:

Theorem II (HANS FREUDENTHAL). *$f(z)$ be defined and analytic in a convex region G. Consider all values of the derivate $f'(z)$ and the convex closure G' of the corresponding points. We now contend that all difference-quotients $\dfrac{f(z_1) - f(z_2)}{z_1 - z_2}$ (where z_1 and z_2 belong to G) belong to G'*[1].

To our purpose we have to generalize this theorem for higher difference-quotients. We shall prove

[1]) And even: this difference-quotient belongs to the convex closure of the values $f'(z')$, where z' runs through the segment (z_1, z_2).

Theorem III. *$f(z)$ be defined and analytic in a convex region G. Consider all values of the n^{th} derivate $f^{(n)}(z)$ on G (n is an arbitrarily chosen but fixed natural number) and the convex closure $G^{(n)}$ of the corresponding points. We now contend that all n^{th} difference-quotients $[z_0 z_1 \ldots z_n](f)$, where $z_0, z_1, \ldots, z_n$ are arbitrary points belonging to G, belong to $G^{(n)}$* [2]) [3]).

Proof. We start with two remarks:

1°. Be $v_0, v_1, \ldots, v_n$ $(n+1)$ arbitrary complex numbers, and so as well points situated in the complex plane (for shortness we shall often identify the point of the complex plane with the corresponding complex number), then point v is situated within the convex closure of $v_0, v_1, \ldots, v_n$ only if

$$\left.\begin{array}{r} v = \lambda_0 v_0 + \lambda_1 v_1 + \ldots . + \lambda_n v_n \\ \lambda_i \text{ real with } 0 \leqq \lambda_i \leqq 1 \text{ for } i = 0, 1, \ldots n \text{ ; and} \\ \lambda_0 + \lambda_1 + \ldots . + \lambda_n = 1 \end{array}\right\} . \tag{5}$$

2°. We may, as known, write the n^{th} difference-quotient

$$[z_0 z_1 \ldots z_n](f),$$

which in this proof is denoted by $[z_0 z_1 \ldots z_n]$, in the following symmetrical form:

$$[z_0\, z_1 \ldots . z_n] = [z_0\, z_1 \ldots . z_n]\,(f) = n!\, \sum_{i=0}^{n} \frac{f(z_i)}{\prod\limits_{\substack{j=0 \\ j \neq i}}^{n} (z_i - z_j)} . \tag{6}$$

Let us first consider the case where $n = 2$. We take three arbitrary points z_0, z_1 and z_2 of G (fig. 1) and consider the middles p_0, p_1 and p_2

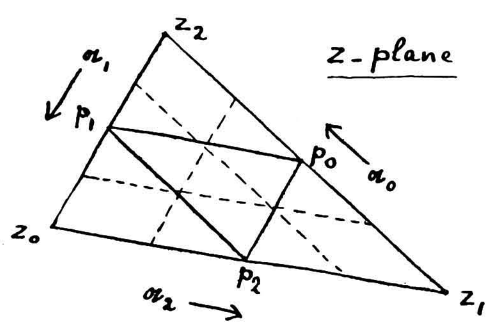

Fig. 1.

of the sides of the corresponding triangle. We shall prove that the following

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2) And even: this difference-quotient belongs to the convex closure of the values $f^{(n)}(z')$, where z' runs through all values of the convex closure of the points $z_0, z_1, \ldots, z_n$.

3) By specializing the FREUDENTHAL proof and ours for the real axis one reaches an (especially for higher derivates) fairly short proof of the (extended) mean value theorem of the common calculus.

equation holds true for $[z_0 z_1 z_2]$:

$$[z_0\, z_1\, z_2] = \tfrac{1}{4}\,[p_0\, p_1\, p_2] + \tfrac{1}{4}\,[z_0\, p_1\, p_2] + \tfrac{1}{4}\,[p_0\, z_1\, p_2] + \tfrac{1}{4}\,[p_0\, p_1\, z_2]. \quad (7)$$

For the equation (7), which we must prove, we may write (according to (6)):

$$\frac{f(z_0)}{(z_0 - z_1)(z_0 - z_2)} + \frac{f(z_1)}{(z_1 - z_0)(z_1 - z_2)} + \frac{f(z_2)}{(z_2 - z_0)(z_2 - z_1)} =$$

$$= \frac{1}{4}\left\{ \frac{f(p_0)}{(p_0 - p_1)(p_0 - p_2)} + \dots + \frac{f(z_2)}{(z_2 - p_0)(z_2 - p_1)} \right\}.$$

If we put $z_2 - z_1 = \mathfrak{a}_0$, $z_0 - z_2 = \mathfrak{a}_1$ and $z_1 - z_0 = \mathfrak{a}_2$, then the coefficient of $f(z_0)$ on the left is $-\dfrac{1}{\mathfrak{a}_2 \cdot \mathfrak{a}_1}$, and on the right $-\frac{1}{4}\dfrac{1}{\frac{1}{2}\mathfrak{a}_2 \cdot \frac{1}{2}\mathfrak{a}_1}$; so these coefficients are equal. The coefficient of $f(p_0)$ on the right is

$$-\frac{1}{\mathfrak{a}_2 . \mathfrak{a}_1} - \frac{1}{\mathfrak{a}_1 . \mathfrak{a}_0} - \frac{1}{\mathfrak{a}_2 . \mathfrak{a}_0} = -\frac{\mathfrak{a}_0 + \mathfrak{a}_2}{\mathfrak{a}_0\, \mathfrak{a}_1\, \mathfrak{a}_2} - \frac{1}{\mathfrak{a}_2 . \mathfrak{a}_0} = \frac{\mathfrak{a}_1}{\mathfrak{a}_0\, \mathfrak{a}_1\, \mathfrak{a}_2} - \frac{1}{\mathfrak{a}_0\, \mathfrak{a}_2} = 0.$$

By changing the letters for the rest of the coefficients (7) is generally proved.

Geometrically (7) may be interpreted thus, where we suppose the second difference-quotients to be points in the plane of $f''(z)$: the difference-quotient $[z_0 z_1 z_2]$ is situated in the centre of gravity of the four points $[p_0 p_1 p_2]$, $[z_0 p_1 p_2]$, $[p_0 z_1 p_2]$ and $[p_0 p_1 z_2]$ (fig. 2). Therefore $[z_0 z_1 z_2]$ certainly lies within the convex closure of these four points. After this we

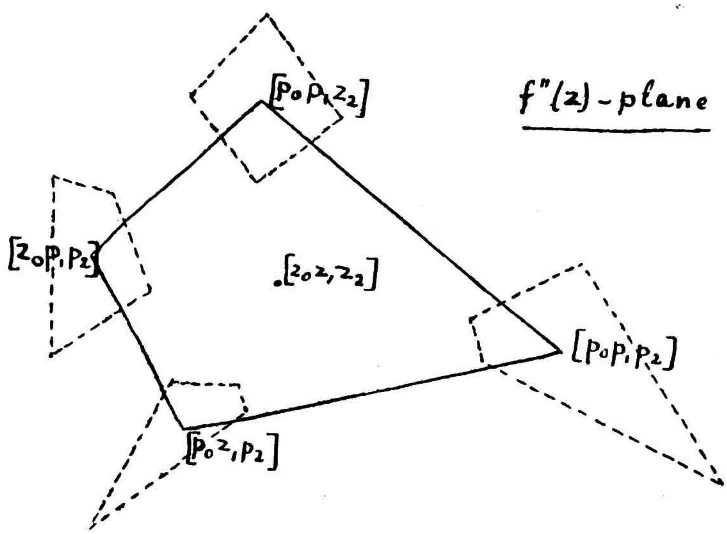

Fig. 2.

subdivide each of the four triangles of fig. 1 in the same way into four triangles, namely by taking the middles of the sides. In each subdivided triangle there is again a relation analogical to (7), in other words the four difference-quotients, each of which is defined by the vertices of a triangle, again group themselves "around" the corresponding difference-quotient (fig. 2). Therefore $[z_0 z_1 z_2]$ certainly lies within the convex closure of the sixteen points which were formed by the second division. This process may be continued infinitely.

We shall now try to find a relation like (7) and an infinitely continued division for the nth difference-quotients with $n > 2$. Let us first consider the case $n = 3$. Though the four points z_0, z_1, z_2 and z_3, on which a third difference-quotient is defined, lie in the complex plane, it is better to imagine these points to be spacial, as in that way the generalisation may be

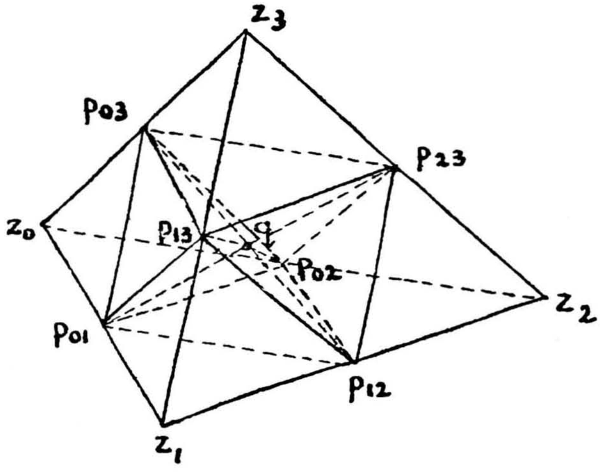

Fig. 3.

more easily realized. One divides every triangle of the tetrahedron $z_0 z_1 z_2 z_3$ (fig. 3) in the same way as in fig. 1 and joins all thus formed points p_{ij}. Three of these joining lines meet in the centre of gravity q. One easily proves — by comparing the coefficients on both sides — that the following relation holds true:

$$\left.\begin{aligned}[z_0 z_1 z_2 z_3] = \frac{1}{2^3}[z_0 p_{01} p_{02} p_{03}] + \frac{1}{2^3}[z_1 p_{01} p_{12} p_{13}] + \\ + \frac{1}{2^3}[z_2 p_{02} p_{12} p_{23}] + \frac{1}{2^3}[z_3 p_{03} p_{13} p_{23}] + \\ + \frac{1}{2^4}[q p_{01} p_{12} p_{13}] + \frac{1}{2^4}[q p_{01} p_{02} p_{12}] + \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots + \frac{1}{2^4}[q p_{03} p_{13} p_{23}]\end{aligned}\right\} \quad (7').$$

In the right side of this relation we find twelve terms corresponding with the twelve tetrahedra into which $z_0\, z_1\, z_2\, z_3$ is divided. In connection with

(5) the point $[z_0\, z_1\, z_2\, z_3]$ lies in the $f''(z)$-plane within the convex closure defined by the twelve difference-quotients of the right side. Each of the twelve tetrahedra is now likewise divided into twelve tetrahedra, and this process is infinitely continued. Always after the mth division $[z_0\, z_1\, z_2\, z_3]$ lies within the convex closure of the 12^m corresponding difference-quotients. By induction it is now possible, though complicated, to generally determine the division of the n-dimensional simplex $z_0\, z_1 \ldots z_n$, corresponding with the nth difference-quotient $[z_0\, z_1 \ldots z_n]$ and the corresponding relation $(7^{(n)})$. Because the general formulas are complicated we shall not enter further into these, but shall close this subject by the following remarks. It appears that one must distinguish between odd and even dimension. For the n-simplices with odd n the centre of gravity always acts as point of division, while for the n-simplices with even n this is not the case (in case $n = 2$ one had a "centre-simplex" $p_0\, p_1\, p_2$, for $n = 4$ one finds a "centre-simplex" $q_0\, q_1\, q_2\, q_3\, q_4$, where the points q_i are the centres of gravity of the five tetrahedra by which the 4-simplex $z_0\, z_1\, z_2\, z_3\, z_4$ is bounded, etc.). In case $n = 3$ one may also attain one's purpose by a simpler division, namely a division into eight tetrahedra, where instead of the inner eight tetrahedra of fig. 3 we have the four tetrahedra $p_{01}\, p_{12}\, p_{02}\, p_{13}$, $p_{13}\, p_{02}\, p_{12}\, p_{23}$, $p_{01}\, p_{02}\, p_{23}\, p_{13}$ and $p_{03}\, p_{02}\, p_{13}\, p_{23}$. The point q now does not appear in this division. This division is, however, no longer symmetrical and I do not think it will be easy to generalize this division, although it is simpler, for a general n.

We denote $[z_0\, z_1 \ldots z_n]$ by $\triangle_1^n$, and have stated that the corresponding n-simplex is then subdivided into k_n n-simplices, where we denote the nth difference-quotient of such an n-simplex by $\triangle^n_{1a_1}$ $(a_1 = 1, 2, \ldots, k_n)$. An equation of the character $(7')$ holds true:

$$\triangle_1^n = \sum_{a_1=1}^{k_n} \lambda_{a_1} \triangle^n_{1a_1} \qquad \left(0 < \lambda_{a_1} < 1 \; ; \; \sum_{a_1=1}^{k_n} \lambda_{a_1} = 1\right). \tag{$7^{(n)}$}$$

From now on, if no mistake is possible, we shall denote by $\triangle^n$ not only the nth difference-quotients but also the corresponding n-simplices.

Each of the simplices $\triangle^n_{1a_1}$ is again subdivided in the same way into k_n n-simplices $\triangle^n_{1a_1a_2}$ $(a_2 = 1, 2, \ldots, k_n)$, and this division is infinitely continued. Thus in general:

$$\triangle^n_{1a_1a_2\ldots a_j} = \sum_{a_{j+1}=1}^{k_n} \lambda_{a_{j+1}} \cdot \triangle^n_{1a_1a_2\ldots a_j a_{j+1}} \qquad (j = 1, 2, \ldots) \tag{$7^{(n)}$}$$

with the conditions

$$0 < \lambda_{a_{j+1}} < 1 \quad ; \quad \sum_{a_{j+1}=1}^{k_n} \lambda_{a_{j+1}} = 1.$$

The diameter of a sequence of monotonicly decreasing sets of n-simplices

$$\triangle_1^n \supset \triangle^n_{1a_1} \supset \triangle^n_{1a_1a_2} \supset \ldots. \tag{8}$$

apparently tends to 0, in other words the sets of (8) have one point $d_{1a_1a_2a_3\dots}$ as intersection; so the sets of (8) converge to that point. The corresponding nth difference-quotients then tend to the nth derivate $f^{(n)}(d_{1a_1a_2\dots})$:

$$\lim_{j\to\infty} \Delta^n_{1a_1a_2\dots a_j} = f^{(n)}(d_{1a_1a_2a_3\dots}).$$

We now consider the $f^{(n)}(z)$-plane, where the values $\Delta^n_{1a_1a_2\dots a_j}$ are introduced as points (comp. fig. 2). From ($7^{(n)}$) it follows that point Δ^n_1 lies within the convex closure of the points $\Delta^n_{1a_1a_2\dots a_j}$ ($j = 1, 2, \dots, i$; $a_j = 1, 2, \dots, k_n$). We still have to prove that point Δ^n_1 lies within the convex closure V of the set of points $f^{(n)}(z)$, where z runs through the convex closure of $z_0, z_1, \dots, z_n$, i.e., the n-simplex Δ^n_1. Suppose on the contrary that *point* Δ^n_1 lay outside V. V is, as the convex closure of the continuous image of a bounded closed set, bounded and closed. Between point Δ^n_1 and V there is a certain distance 2ε. Consider an ε-neighbourhood V_ε of the set V. This apparently is again a convex set while point Δ^n_1 is at a distance ε of V_ε.

The *points* $\Delta^n_{1a_1}$ ($a_1 = 1, 2, \dots, k_n$) of the first division have a convex closure to which point Δ^n_1 belongs according to the afore-said. Therefore there exists at least one point $\Delta^n_{1b_1}$ lying outside V_ε. With the simplex $\Delta^n_{1b_1}$ corresponds a division into k_n n-simplices $\Delta^n_{1b_1a_2}$ ($a_2 = 1, 2, \dots, k_n$). For at least one of these points again the corresponding point $\Delta^n_{1b_1b_2}$ lies outside V_ε. By infinitely continuing this process we find a sequence of points $\Delta^n_1, \Delta^n_{1b_1}, \Delta^n_{1b_1b_2}, \dots\dots$ such that for the corresponding n-simplices an equation of the character (8) holds true, in other words, this sequence of points converges to a point determining a value $f^{(n)}(d_{1b_1b_2\dots})$. Here $d_{1b_1b_2\dots}$ apparently belongs to the convex closure of $z_0\, z_1\, z_2 \dots z_n$. But t hen $f^{(n)}(d_{1b_1b_2\dots})$ bust belong to V, which is impossible as the points Δ^n_1 $\Delta^n_{1b_1}$, ..., converge to a point of the boundary of V_ε or outside V_ε. Thus the required contradiction is reached, by which theorem III (and then by specialization also theorem II) has been proved.

7. Following directly from the mean value theorem derived in 6. is

Theorem IV. *Be $f(z)$ defined and analytic in a convex region G. We consider the nth difference-quotient Δ^n ($n = 1, 2, \dots$) defined on $(n+1)$ arbitrary points of G. Now*

$$|\Delta^n| \leqq |f^{(n)}(z')|, \tag{9}$$

where z' is a suitably chosen point of G [4]. *Or in other words: the upper bound of $|\Delta^n|$ in G is smaller than or equal to the upper bound of $|f^{(n)}(z)|$ in G.*

Proof. Point Δ^n in the complex plane of the points $f^{(n)}(z)$ lies, accor-

[4]) z' even is a point of the convex closure of the $(n+1)$ points. — REMARK. As G is a region the $=$-mark in (9) may even be left out.

ding to theorem III, on the joining line of two points $f^{(n)}(p)$ and $f^{(n)}(q)$, where p and q are suitably chosen points of G. In the triangle determined by the points 0, $f^{(n)}(p)$, $f^{(n)}(q)$ the distance $|\triangle^n|$ between 0 and $\triangle^n$ is smaller than or equal to [5]) $|f^{(n)}(p)|$ or $|f^{(n)}(q)|$ by which the theorem has already been proved.

8. *Condition* (2) *mentioned in theorem I is also necessary.* $f(z)$ is a given function, which certainly is one-valued and analytic in a circle-region around point z'. According to an extension of a well-known inequality of CAUCHY

$$|f^{(n)}(z)| \leqq \frac{2\, n!\, M}{\varrho^n}.$$

Here z is an arbitrary point of a (sufficiently small) circle-region C (within the given circle-region) with z' for centre, and ϱ for radius; M is the maximum of $|f(z)|$ on the circumference of a circle with z' as centre and radius $2\,\varrho$. According to (9) now for all points of C

$$|\triangle^n| \leqq |f^{(n)}(z')| \leqq \frac{2n!\, M}{\varrho^n}. \tag{10}$$

If $2M \leqq 1$, then (10) is $\leqq n!\, r^n$ where $1/\varrho = r$. If $2M > 1$, then (10) is also $\leqq n!\, r^n$, where $2M/\varrho = r$.

Thus in both cases the required condition (2) holds true as one may take i so great that all points $z_i, z_{i+1}, \ldots$, on which $\triangle_i^n$ is defined, belong to the circle-region C.

By this we have also proved the following

Theorem V. *If $f(z)$ be analytic in a point z', we may write $f(z)$ — besides the power-series-expansion in z' — by an expansion of the shape*

$$f(z) = f(z_1) + \sum_{n=1}^{\infty} \left(\frac{\triangle^n}{n!} \cdot \prod_{j=1}^{n} (z - z_j) \right)$$

where $z_1, z_2, z_3, \ldots$ is an arbitrary sequence of points converging to z'; these points have to lie within a sufficiently small neighbourhood of z' [6]). *$\triangle^n$ is the difference-quotient* $[z_1 z_2 \ldots z_{n+1}](f)$.

As this expansion is a generalization of the interpolation-formula of NEWTON (for infinitely many terms) one might call this the NEWTON-*expansion of* $f(z)$ to the sequence $z_1, z_2, \ldots$.

[5]) When p and q coincide.

[6]) For that neighbourhood we may always take the above-mentioned circle-region C. In this region the series in question is certainly converging.