Mathematics. — *The foundation of the invariant theory of linear systems of curves on an algebraic surface.* By B. L. VAN DER WAERDEN. (Communicated by Prof. J. G. VAN DER CORPUT.)

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The birational transformation of a linear system of curves on a surface can easily be so defined that the transform of a system without fixed components is again a system without fixed components. But according to this definition the notion of a "complete system" (VOLLSCHAR), that is of a linear system which is not contained in a larger one, is not invariant. For instance, a Cremona transformation of the well-known type with 3 singular points:

$$x_1' : x_2' : x_3' = x_2 x_3 : x_3 x_1 : x_1 x_2$$

transforms the complete system of all straight lines of the plane into a system of conics with three basic points, which is not complete, because it is contained in the larger system of all conics of the plane.

In order to establish the invariance of the notion of a complete system, the Italian geometers have introduced the notion of virtual multiplicity of an assigned base-point of a linear system. The system of conics with 3 base-points, obtained by the above transformation, is complete, if these base-points are *assigned* with *virtual multiplicity* 1, that means, if only those larger systems are let into consideration, whose curves have in these points at least these multiplicities.

But it is not enough to consider ordinary base points with assigned multiplicities; one has to consider also "infinitely near" base points. For instance, the transformation

$$x_1' : x_2' : x_3' = x_2 x_3 : x_1 x_3 : x_2^2$$

transforms the system of all straight lines of the plane into a system of conics with two ordinary base points P and Q and a "neighbour point" P_1, infinitely near to P. The conics of the system have at least multiplicity 1 at P and Q, and the sum of their multiplicities at P and P_1 is at least 2, some of them having a double point at P and others passing through P and P_1. With these assigned virtual multiplicities the system is complete.

The notions of neighbour points and multiplicities in them can be strictly defined, according to NOETHER, ENRIQUES, and ZARISKI, but the analysis is very complicated and the extension to more than 2 dimensions seems hardly possible. Therefore it is desirable to find a simpler notion, by which the analysis of neighbour points is avoidable and which can easily be generalised. This simpler notion turns out to be the notion of valuation.

Let ξ be a general point of an algebraic surface F. Rational functions of ξ are quotients of forms $f(\xi)/g(\xi)$ of equal degrees. They form a field, invariant on birational transformation. Now consider valuations w of this field, whose values $w(\eta)$ are real numbers, with the well-known properties

$$\begin{aligned} &w(\eta\zeta) = w(\eta) + w(\zeta) \\ &w(\eta + \zeta) \geqq \text{Min}\,\{ w(\eta), w(\zeta)\} \\ &w(a) = 0 \text{ for constant } a \end{aligned}$$

Any such valuation defines a *valuation ideal* $\mathfrak{p}$, consisting of those elements η whose values are positive, situated in the *valuation ring* $\mathfrak{o}$, which consists of the elements η of non-negative value, and a *rest class field* $\mathfrak{o}/\mathfrak{p}$, which is a field of algebraic functions of 0 or 1 variable.

This number 0 or 1 is called the *dimension* of the valuation. In the more general case, when F is replaced by a variety of r dimensions, the dimension of the valuation can be any one of the numbers 0, 1, ..., $(r-1)$. We restrict ourselves, however, to valuations of dimension $(r-1)$. As ZARISKI has shown, these valuations are discrete, that is, they can be normed in such a way that the values $w(\eta)$ are whole rational numbers.

Any valuation w can be extended to forms f of arbitrary degree g, by the definition

$$w(f) = \underset{j}{\text{Max.}}\, w(\xi_j^{-g} f(\xi)).$$

The values $w(f)$, so defined, are non-negative whole numbers. Secondly, if the surface F is free from singular curves, the valuation w can be extended to curves C on F, the value $w(C)$ being defined as the minimum of all values $w(f)$ for all forms f, whose intersections with the surface contain the curve C. These values $w(C)$ are again non-negative, and the value $w(C+D)$ for a reducible curve $C+D$ is the sum of the values $w(C)+w(D)$. Moreover, the value $w(C)$ of any curve C of a linear system of curves is at least equal to the minimum of the values of the basic curves of the system.

An example of such a valuation is the multiplicity $w(C)$ of the curve C in a point P of F. Another example is the sum of the multiplicities of C in P and in a neighbour point P_1 of P. This sum can also be defined as the minimum of the multiplicities of intersection of C with an arbitrary curve branch passing by P and P_1. In the same way a valuation can be defined for any neighbour point of P, also to those of higher order, by means of branches passing through them.

On comparing these examples with those given before for assigned virtual multiplicities of base points of linear systems of conics, one sees that the notion of virtual multiplicity can be defined just as well with the aid of valuations without making use of the concept of infinitely near points. For instead of assigning virtual multiplicities to base points and their neighbour points, one can assign "virtual (minimal) values" w_0 for the values $w(C)$ of the curves C of a linear system in certain valuations w.

In order to work out this idea, we must introduce a distinction between "higher" and "lower" valuations. To any valuation w corresponds an irreducible manifold I on F, such that all forms f containing I have a positive value $w(f)$, and all forms f not containing I have the value 0. The dimension of I can be any one of the numbers 0, 1, ..., d, where d is the dimension of the valuation (hence in our case $d = r - 1$). If the dimension of I is $(r - 1)$, the valuation is called a *higher valuation*, if it is less, a *lower* one. The higher valuations are also called *prime divisors* ("Kurvenprimteiler" in the terminology of H. W. E. JUNG). In the case of a surface F the higher valuations corresponds to irreducible curves I on F: if C contains the curve I with multiplicity m, the value $w(C)$ is m. The lower valuations correspond to points P on F and their neighbour points $P_1, P_2 \ldots$: the value $w(C)$ is the minimum multiplicity of intersection of the curve C with an arbitrary curve branch passing through the neighbour point in question.

It is a generally adopted convention, not to assign virtual values w_0 in higher valuations, but only in lower valuations w. This means geometrically, that no fixed components are assigned for linear systems of curves, but only basic points and their neighbour points. But even these lower virtual values w_0 cannot be assigned arbitrarily, for there is a very useful general theorem in the Italian theory, to the effect that any complete linear system with assigned virtual values can be represented as the difference of two linear systems without fixed components, whose virtual values are equal to their effective ones, the *effective value* of a linear system in any valuation w being defined as the minimum of the values $w(C)$ for all curves of the system. In order to verify this theorem it is necessary to restrict the virtual values to those which can be represented as differences of the effective values of two linear systems $|C_1|$ and $|C_2|$ without fixed components, so that the equation

$$w_0 = \mathrm{Min}\ w(C_1) - \mathrm{Min}\ w(C_2)$$

holds for all valuations w. Adopting this restriction, the theorem just mentioned can be proved and generalised to r dimensions.

As I said before, the aim of the theory is birational invariance. Now the notion of valuation is invariant by itself, and so is the restriction to valuations of dimension $(r - 1)$. The distinction between higher and lower valuations, however, is not invariant: any lower valuation can be transformed birationally into a higher one. The transformation of a linear system is governed in the Italian theory by the following rules:

1°. If a basic point with virtual multiplicity w_0 and effective multiplicity w_1 is transformed into a curve, this curve is to be taken into the transformed system as a fixed component with multiplicity $m = w_1 - w_0$.

2°. If, conversely, a fixed component of the linear system with multiplicity m is transformed into a point, the virtual multiplicity of this point is defined by the equation $m = w_1 - w_0$.

3°. The multiplicities of all other fixed components remain unchanged.

4°. For all basic points and neighbour points which are not transformed into curves the differences $w_1 - w_0$ remain unchanged.

These rules can be combined into one general principle of invariance, valid for all higher and lower valuations:

The difference $w_1 - w_0$ between effective and virtual value of a linear system in any valuation shall remain unchanged upon birational transformation.

This principle can be extended without any change to r dimensions. The virtual value in a higher valuation being zero, the effective multiplicities of all fixed components of the transformed system are determined by the principle of invariance. The variable components are simply the transforms of the variable components of the original system. The fixed and variable components of all curves of the transformed system being thus determined, the effective values w_1 are fixed, and as the differences $w_1 - w_0$ are again determined by the principle of invariance, the virtual values w_0 are fixed also.

If a linear system $|C|$ is contained in a larger system $|D|$ with the same virtual multiplicities, the transformed system $|C'|$ is contained in $|D'|$. From this theorem follows at once the invariance of the notion of a complete system.

The notions sum and difference of complete systems are also invariant on birational transformation, if their virtual values are defined as sums and differences of the virtual values of the component systems. The complete systems with assigned virtual values form upon addition an abelian semi-group, which can be extended to a group by the adjunction of "virtual" differences $|C| - |D|$, to which may or may not correspond effective linear systems.

The proofs of the theorems announced in this note are contained in a manuscript, which I hope is not lost and shall be published some time.