Mathematics. — *On the G-function.* I. By C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT.)

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REFERENCES.

W. N. BAILEY.

1. Generalized hypergeometric series (Cambridge Tracts in Mathematics and Mathematical Physics, **32** (1935)).

E. W. BARNES.

2. The asymptotic expansion of integral functions defined by TAYLOR's series. Phil. Trans. Royal Soc. London, (A) **206**, 249—297 (1906).
3. The asymptotic expansion of integral functions defined by generalized hypergeometric series. Proc. London Math. Soc., (2) **5**, 59—116 (1907).
4. On functions defined by simple types of hypergeometric series. Trans. Cambridge Phil. Soc., **20**, 253—279 (1908).
5. A new development of the theory of the hypergeometric functions. Proc. London Math. Soc., (2) **6**, 141—177 (1908).

J. G. VAN DER CORPUT.

6. On KUMMER's solutions of the hypergeometric differential equation. Proc. Kon. Akad. v. Wetensch., Amsterdam, **39**, 1056—1059 (1936).

P. DEBYE.

7. Semikonvergente Entwickelungen für die Zylinderfunktionen und ihre Ausdehnung ins Komplexe. Sitzungsberichte der math.-phys. Klasse der K. B. Akad. der Wissensch. zu München, **40**, no. 5 (1910).

A. R. FORSYTH.

8. A treatise on differential equations (fifth edition, 1921).

C. FOX.

9. The asymptotic expansion of generalized hypergeometric functions. Proc. London Math. Soc., (2) **27**, 389—400 (1928).

C. F. GAUSS.

10. Determinatio seriei nostrae per aequationem differentialem secundi ordinis. Werke, **3**, 207—229 (1866).

É. GOURSAT.

11. Sur l'équation différentielle linéaire qui admet pour intégrale la série hypergéométrique. Ann. de l'école normale, (2) **10**, (1881), Supplément, 3—142.

F. KLEIN.

12. Vorlesungen über die hypergeometrische Funktion (1933).

E. E. KUMMER.

13. Über die hypergeometrische Reihe .... Journ. für die reine und angewandte Math., **15**, 39—83 and 127—172 (1836).

T. M. MACROBERT.

14. Induction proofs of the relations between certain asymptotic expansions and corresponding generalized hypergeometric series. Proc. Royal Soc. Edinburgh, **58**, 1—13 (1938).
15. A proof by induction of the analytic continuations of certain generalized hypergeometric functions. Philosophical Magazine, (7) **25**, 848—851 (1938).
16. Solution in multiple series of a type of generalized hypergeometric equation. Proc. Royal Soc. Edinburgh, **59**, 49—54 (1939).
17. Expressions for generalized hypergeometric functions in multiple series. Ibidem, **59**, 141—144 (1939).

C. S. MEIJER.

18. Über WHITTAKERsche bezw. BESSELsche Funktionen und deren Produkte. Nieuw Archief voor Wiskunde, (2) **18**, (4tes Heft) 10—39 (1936).

19. Einige Integraldarstellungen aus der Theorie der BESSELschen und WHITTAKERschen Funktionen. Proc. Kon. Akad. v. Wetensch., Amsterdam, **39**, 394—403 and 519—527 (1936).

20. Über Produkte von WHITTAKERschen Funktionen. Ibidem, **40**, 133—141 and 259—263 (1937).

21. Über BESSELsche, STRUVEsche und LOMMELsche Funktionen. Ibidem, **43**, 198—210 and 366—378 (1940).

22. Neue Integraldarstellungen für WHITTAKERsche Funktionen. Ibidem, **44**, 81—92, 186—194, 298—307, 442—451 and 590—598 (1941).

23. Integraldarstellungen für WHITTAKERsche Funktionen und ihre Produkte. Ibidem, **44**, 435—441 and 599—605 (1941).

24. Multiplikationstheoreme für die Funktion $G_{p,q}^{m,n}(z)$. Ibidem, **44**, 1062—1070 (1941).

25. Integraldarstellungen für STRUVEsche und BESSELsche Funktionen. Compositio Mathematica, **6**, 348—367 (1939).

L. POCHHAMMER.

26. Über die Differentialgleichung der allgemeineren hypergeometrischen Reihe mit zwei endlichen singulären Punkten. Journ. für die reine und angewandte Math., **102**, 76—159 (1888).

F. C. SMITH.

27. Relations among the fundamental solutions of the generalized hypergeometric equation when $p = q + 1$. I. Non-logarithmic cases. Bulletin American Math. Soc., **44**, 429—433 (1938).

G. G. STOKES.

28. Note on the determination of arbitrary constants which appear as multipliers of semi-convergent series. Proc. Cambridge Phil. Soc., **6**, 362—366 (1889). [Math. and Phys. Papers, V, 221—225 (1905).]

J. THOMAE.

29. Über die höheren hypergeometrischen Reihen, insbesondere über die Reihe: .... Math. Ann., **2**, 427—444 (1870).

G. N. WATSON.

30. A class of integral functions defined by TAYLOR's series. Trans. Cambridge Phil. Soc., **22**, 15—37 (1923).

31. A treatise on the theory of BESSEL functions (1922).

E. T. WHITTAKER and G. N. WATSON.

32. A course of modern analysis (fourth edition, 1927).

E. WINKLER.

33. Über die hypergeometrische Differentialgleichung n^{ter} Ordnung mit zwei endlichen Singulären Punkten. Dissertation München (1931).

34. Die Unbestimmtheitstelle der ausgearteten hypergeometrischen Differentialgleichung. Sitzungsberichte der math.-naturwissensch. Abteilung der Bayerischen Akad. der Wissensch. zu München, 1933, 237—256.

E. M. WRIGHT.

35. The asymptotic expansion of the generalized hypergeometric function. Journ. London Math. Soc., **10**, 286—293 (1935).

36. The asymptotic expansion of integral functions defined by TAYLOR series. Phil. Trans. Royal Soc. London, (A) **238**, 423—451 (1940).

D. WRINCH.

37. An asymptotic formula for the hypergeometric function ${}_0\triangle_4(z)$. Philosophical Magazine, (6) **41**, 161—173 (1921).

38. A generalized hypergeometric function with n parameters. Ibidem, (6) **41**, 174—186 (1921).

39. Some approximations to hypergeometric functions. Ibidem, (6) **45**, 818—827 (1923).

§ 1. Definition of the function $G_{p,q}^{m,n}(z)$.

Suppose that m, n, p and q are integers with

$$q \geqq 1,\ 0 \leqq n \leqq p \leqq q \text{ and } 0 \leqq m \leqq q;$$

suppose further that the number z satisfies the inequalities

$$z \neq 0 \text{ and } |z| < 1 \text{ if } q = p,$$
$$z \neq 0 \text{ if } q > p;$$

moreover that the numbers $a_1, \dots, a_n$ and $b_1, \dots, b_m$ fulfil the condition

$$a_j - b_h \neq 1, 2, 3, \dots \qquad (j = 1, \dots, n;\ h = 1, \dots, m). \quad . \quad . \quad (1)$$

Then the function $G_{p,q}^{m,n}(z)$ is defined as follows [1])

$$G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) = \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} z^s \, ds. \quad (2)$$

The contour C runs from $\infty - i\tau$ to $\infty + i\tau$ (τ is a positive number) and encloses all the poles

$$b_j, b_j + 1, b_j + 2, \dots \qquad (j = 1, \dots, m), \quad . \quad . \quad . \quad . \quad (3)$$

but none of the poles

$$a_j - 1, a_j - 2, a_j - 3, \dots \qquad (j = 1, \dots, n) \quad . \quad . \quad . \quad . \quad (4)$$

of the integrand [2]).

It follows from the asymptotic expansion of the Gamma function that the integral (2) is convergent and independent of τ.

If $m = 0$, the integrand in (2) is analytic inside C. Hence we find

$$G_{p,q}^{0,n}\left(z \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

More generally: Suppose $m \geqq 0$ and

$$b_j - b_h \neq 0, \pm 1, \pm 2, \dots \qquad (j = 1, \dots, m;\ h = 1, \dots, m;\ j \neq h); \quad (6)$$

then the integrand has simple poles at the points (3) and the value of the integral (2) is equal to minus the sum of the residues of the integrand at these poles. By evaluating these residues we obtain [3])

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1) If $p = 0$, the function G is denoted by $G_{0,q}^{m,0}(z \mid b_1, \dots, b_q)$.

2) Such a contour C can always be drawn, since, because of (1), none of the poles (3) coincides with one of the poles (4).

3) We employ the usual notation for the generalized hypergeometric function:

$${}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) = \sum_{h=0}^{\infty} \frac{z^h \prod_{j=1}^{p} \{\alpha_j(\alpha_j + 1) \dots (\alpha_j + h - 1)\}}{h! \prod_{j=1}^{q} \{\beta_j(\beta_j + 1) \dots (\beta_j + h - 1)\}}.$$

$$G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)=\sum_{h=1}^{m}\frac{\prod\limits_{\substack{j=1\\ j\neq h}}^{m}\Gamma(b_j-b_h)\prod\limits_{j=1}^{n}\Gamma(1+b_h-a_j)}{\prod\limits_{j=m+1}^{q}\Gamma(1+b_h-b_j)\prod\limits_{j=n+1}^{p}\Gamma(a_j-b_h)}z^{b_h}\times$$
$$\times\,{}_pF_{q-1}(1+b_h-a_1,\ldots,1+b_h-a_p;1+b_h-b_1,\ldots^{*}\ldots,1+b_h-b_q;(-1)^{p-m-n}z); \qquad (7)$$

the asterisk denotes that the number $1+b_h-b_h$ is to be omitted in the sequence $1+b_h-b_1$, ..., $1+b_h-b_q$. This expansion holds, provided that a_j and b_j satisfy not only (1) but also the additional condition (6) [4].

I may remark here that in my former papers [5]) the function $G_{p,q}^{m,n}(z)$ has been defined by means of (7).

From (7) it is apparent that $G_{p,q}^{m,n}(z)$ is a many-valued function of z with a branch-point at $z=0$.

The function $G_{p,q}^{m,n}(z)$ is a symmetric function of a_1, ..., a_n, of a_{n+1}, ..., a_p, of b_1, ..., b_m and of b_{m+1}, ..., b_q; this follows immediately from the definition (2).

When there is no risk of ambiguity, we shall in our notation omit the parameters a_1, ..., a_p and b_1, ..., b_q. The function G defined by (2) will then be denoted by $G_{p,q}^{m,n}(z)$. When the parameters are not explicitly mentioned, it will always be supposed that they are a_1, ..., a_p and b_1, ..., b_q. For instance by the symbols $G_{p,q}^{q,0}(z)$ and $G_{p,q}^{k,l}(z)$ we shall denote the functions

$$G_{p,q}^{q,0}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right),\ \text{respect.}\ G_{p,q}^{k,l}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right).$$

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4) The generalized hypergeometric function ${}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z)$ is meaningless if $\beta_j=0,-1,-2,\ldots$ $(1\leqq j\leqq q)$. The function

$$\frac{1}{\prod\limits_{j=s+1}^{q}\Gamma(\beta_j)}\,{}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z)\qquad(0\leqq s\leqq q)$$

is only meaningless when $\beta_j=0,-1,-2,\ldots$ $(1\leqq j\leqq s)$. If one or more of the numbers β_j $(s+1\leqq j\leqq q)$ is equal to zero or a negative integer, the last function is in the usual way defined as follows:

$$\frac{1}{\prod\limits_{j=s+1}^{q}\Gamma(\beta_j)}\,{}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z)$$
$$=\sum_{h=0}^{\infty}\frac{z^h\prod\limits_{j=1}^{p}\{\alpha_j(\alpha_j+1)\ldots(\alpha_j+h-1)\}}{h!\prod\limits_{j=1}^{s}\{\beta_j(\beta_j+1)\ldots(\beta_j+h-1)\}\prod\limits_{j=s+1}^{q}\Gamma(\beta_j+h)};$$

when $b_j=0,-1,-2,\ldots$ $(s+1\leqq j\leqq q)$, then some initial terms of the series on the right-hand side of this relation vanish.

5) MEIJER, [18] — [25].

Now let

$$q \geqq 1,\ 0 \leqq l-1 \leqq n \leqq p \leqq q \text{ and } 0 \leqq k \leqq q;$$

we consider

$$G_{p,q}^{k,l-1}(z) = G_{p,q}^{k,l-1}\left(z \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right).$$

In this function we change the order of the parameters $a_1, \dots, a_p$ in a special way; we replace the sequence

$$a_1, \dots, a_p \text{ by } a_{n-l+2}, \dots, a_p, a_1, \dots, a_{n-l+1}.$$

Then we obtain

$$G_{p,q}^{k,l-1}\left(z \,\middle|\, \begin{matrix} a_{n-l+2}, \dots, a_p, a_1, \dots, a_{n-l+1} \\ b_1, \dots, b_q \end{matrix}\right). \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This function we will, briefly, denote by $G_{p,q}^{k,l-1,n}(z)$.

We now suppose

$$1 \leqq l \leqq n \leqq p \leqq q,\ 1 \leqq t \leqq n-l+1,\ 0 \leqq k \leqq q$$

and consider

$$G_{p,q}^{k,l}(z) = G_{p,q}^{k,l}\left(z \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right).$$

In this function we change the order of the parameters $a_1, \dots, a_p$ in the following way: we replace the sequence

$$a_1, \dots, a_p \text{ by } a_t, a_{n-l+2}, \dots, a_p, a_1, \dots, a_{t-1}, a_{t+1}, \dots, a_{n-l+1}.$$

Then we get

$$G_{p,q}^{k,l}\left(z \,\middle|\, \begin{matrix} a_t, a_{n-l+2}, \dots, a_p, a_1, \dots, a_{t-1}, a_{t+1}, \dots, a_{n-l+1} \\ b_1, \dots, b_q \end{matrix}\right). \quad . \quad . \quad (9)$$

This function shall be denoted by $G_{p,q}^{k,l,n}(z \| a_t)$.

Since $G_{p,q}^{k,\lambda}(z)$ is a symmetric function of $a_{\lambda+1}, \dots, a_p$, the functions (8) and (9) are independent of n if $l = 1$. Hence we find

$$G_{p,q}^{k,0,n}(z) = G_{p,q}^{k,0}\left(z \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) = G_{p,q}^{k,0}(z) \quad . \quad . \quad . \quad (10)$$

and

$$G_{p,q}^{k,1,n}(z \| a_t) = G_{p,q}^{k,1}\left(z \,\middle|\, \begin{matrix} a_t, a_1, \dots, a_{t-1}, a_{t+1}, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right).$$

The function

$$G_{p,q}^{k,1}\left(z \,\middle|\, \begin{matrix} a_t, a_1, \dots, a_{t-1}, a_{t+1}, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) \qquad (1 \leqq t \leqq p)$$

shall henceforth be denoted by $G_{p,q}^{k,1}(z \| a_t)$. So we obtain

$$G_{p,q}^{k,1,n}(z \| a_t) = G_{p,q}^{k,1}(z \| a_t). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

§ 2. Asymptotic expansions of BARNES' type for $G_{p,q}^{m,n}(z)$ $(q > p)$.

In some cases an asymptotic expansion of $G_{p,q}^{m,n}(z)$ $(q > p)$ for large values of $|z|$ can be deduced by means of the methods of BARNES [6]). I will communicate here the results. First I give two definitions.

Definition 1. *Suppose that p, q and t are integers with* $1 \leqq t \leqq p < q$; *suppose further that the numbers* a_t *and* $b_1, \ldots, b_q$ *satisfy the condition*

$$a_t - b_j \neq 1, 2, 3, \ldots \qquad (j = 1, \ldots q) \; . \; . \; . \; . \; . \quad (12)$$

Then, for brevity, we formally [7]) *write*

$$E_{p,q}(z \| a_t) = z^{-1+a_t} \sum_{h=0}^{\infty} \frac{(-z)^{-h} \prod_{j=1}^{q} \Gamma(1 + b_j - a_t + h)}{h! \prod_{\substack{j=1 \\ j \neq t}}^{p} \Gamma(1 + a_j - a_t + h)} \; . \; . \quad (13)$$

If the divergent series on the right-hand side of (13) *is written by means of the hypergeometric notation, the formal expansion* (13) *takes the form*

$$E_{p,q}(z \| a_t) = \frac{z^{-1+a_t} \prod_{j=1}^{q} \Gamma(1 + b_j - a_t)}{\prod_{\substack{j=1 \\ j \neq t}}^{p} \Gamma(1 + a_j - a_t)} \, {}_qF_{p-1}\left(\begin{matrix} 1 + b_1 - a_t, \ldots, 1 + b_q - a_t \, ; \\ 1 + a_1 - a_t, \ldots^{*} \ldots, 1 + a_p - a_t \, ; -z^{-1} \end{matrix} \right), \; . \quad (14)$$

the asterisk denoting that the number $1 + a_t - a_t$ *is to be omitted in the sequence* $1 + a_1 - a_t, \ldots, 1 + a_p - a_t$.

From this definition it follows

$$E_{p,q}(z e^{2\gamma\pi i} \| a_t) = e^{2\gamma\pi i a_t} E_{p,q}(z \| a_t) \qquad (\gamma = 0, \pm 1, \pm 2, \ldots). \quad (15)$$

Definition 2. *Suppose that m, n, q and t are integers with*

$$1 \leqq t \leqq n \text{ and } 0 \leqq m \leqq q;$$

suppose further that the numbers $a_1, \ldots, a_n$ *satisfy the condition*

$$a_j - a_t \neq 0, \pm 1, \pm 2, \ldots \qquad (j = 1, \ldots, n; j \neq t).$$

Then the coefficient $\Delta_q^{m,n}(t)$ *is defined as follows:*

$$\Delta_q^{m,n}(t) = \pi^{m+n-q-1} \frac{\prod_{j=m+1}^{q} \sin(b_j - a_t)\pi}{\prod_{\substack{j=1 \\ j \neq t}}^{n} \sin(a_j - a_t)\pi} \; . \; . \; . \; . \; . \quad (16)$$

This coefficient may also be written in the form [8])

$$\Delta_q^{m,n}(t) = (-1)^{m+n-q-1} \frac{\prod_{\substack{j=1 \\ j \neq t}}^{n} \{\Gamma(a_t - a_j)\Gamma(1 + a_j - a_t)\}}{\prod_{j=m+1}^{q} \{\Gamma(a_t - b_j)\Gamma(1 + b_j - a_t)\}} \; . \; . \quad (17)$$

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[6]) BARNES, [3]. Some of BARNES' formulae are also proved by WINKLER, [34] and MACROBERT [14].

[7]) The series on the right-hand side of (13) is divergent because of $q > p$.

[8]) (17) follows from (16) by means of

$$\frac{\pi}{\sin z\pi} = -\Gamma(-z)\Gamma(1 + z).$$

The results obtained by BARNES can now be stated as follows [9]):

Theorem A. *The function* $G^{q,1}_{p,q}(z\|a_t)$ *with* $1 \leqq t \leqq p < q$ *admits for large values of* $|z|$ *with* $|\arg z| < (\frac{1}{2}q - \frac{1}{2}p + 1)\pi$ *the asymptotic expansion* [10])

$$G^{q,1}_{p,q}(z\|a_t) \backsim E_{p,q}(z\|a_t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

More generally [11]):

Theorem B. *Suppose that m, n, p and q are integers with*

$$1 \leqq n \leqq p < q, \quad 1 \leqq m \leqq q \text{ and } m + n > \tfrac{1}{2}p + \tfrac{1}{2}q;$$

suppose further that the numbers $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_m$ *satisfy the conditions*

$$a_t - b_j \neq 1, 2, 3, \ldots \quad (t = 1, \ldots, n; j = 1, \ldots, m), \quad . \quad . \quad . \quad (19)$$

$$a_j - a_t \neq 0, \pm 1, \pm 2, \ldots \quad (j = 1, \ldots, n; t = 1, \ldots, n; j \neq t). \quad (20)$$

Then for large values of $|z|$ *with* $|\arg z| < (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi$ *the following asymptotic expansion holds* [12])

$$G^{m,n}_{p,q}(z) \backsim \sum_{t=1}^{n} e^{(m+n-q-1)\pi i a_t} \triangle^{m,n}_{q}(t)\, E_{p,q}(z\, e^{(q-m-n+1)\pi i} \| a_t) \quad . \quad (21)$$

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[9]) BARNES, [2], 295—296 and [3], 65. The formulae of BARNES are written in another notation. The symbols $G^{m,n}_{p,q}$, $E_{p,q}$ and $\triangle^{m,n}_{q}$ do not occur in his papers.

[10]) In (18) it is of course supposed that the numbers a_t and $b_1, \ldots, b_q$ satisfy the condition (12).

Formula (18) is not true uniformly for $|\arg z| < (\frac{1}{2}q - \frac{1}{2}p + 1)\pi$; it is true uniformly only for $|\arg z| \leqq (\frac{1}{2}q - \frac{1}{2}p + 1)\pi - \delta$ for every $\delta > 0$. A similar remark holds with regard to the asymptotic expansions (21) and (22).

[11]) BARNES, [3], 70. The formulae (21) and (18) are proved by evaluating the residues of the integrand in (2) at the poles outside the contour C.

The result of BARNES with regard to formula (21) is not entirely complete; he seems to consider only the case with $m \geqq q - p$.

[12]) The expansion of the term

$$e^{(m+n-q-1)\pi i a_t} \triangle^{m,n}_{q}(t)\, E_{p,q}(z\, e^{(q-m-n+1)\pi i} \| a_t)$$

can according to (17) and (13) be written in the form

$$\frac{z^{-1+a_t} \prod\limits_{\substack{j=1\\ j\neq t}}^{n} \Gamma(a_t - a_j)}{\prod\limits_{j=m+1}^{q} \Gamma(a_t - b_j)} \sum_{h=0}^{\infty} \frac{\{(-1)^{q-m-n} z\}^{-h} \prod\limits_{j=1}^{m} \Gamma(1 + b_j - a_t + h) \prod\limits_{j=m+1}^{q} \{(1 + b_j - a_t)(2 + b_j - a_t) \ldots (h + b_j - a_t)\}}{h! \prod\limits_{\substack{j=1\\ j\neq t}}^{n} \{(1 + a_j - a_t)(2 + a_j - a_t) \ldots (h + a_j - a_t)\} \prod\limits_{j=n+1}^{p} \Gamma(1 + a_j - a_t + h)}.$$

The coefficients in this divergent series are significant, if the numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ satisfy the conditions (19) and (20).

BARNES has also investigated the behaviour of $G_{p,q}^{q,0}(z)$ for $|z|$ tending to infinity. His result runs as follows [13]):

The function $G_{p,q}^{q,0}(z)$ possesses for large values of $|z|$ an asymptotic expansion of the form

$$G_{p,q}^{q,0}(z) \backsim \left[\exp\left((p-q)\, z^{\frac{1}{q-p}}\right)\right] z^{\vartheta} \left\{ \frac{(2\pi)^{\frac{q-p-1}{2}}}{\sqrt{q-p}} + \frac{M_1}{z^{\frac{1}{q-p}}} + \frac{M_2}{z^{\frac{2}{q-p}}} + \dots \right\} \quad (22)$$

where

$$\vartheta = \frac{1}{q-p} \left\{ \tfrac{1}{2}(p-q+1) + \sum_{h=1}^{q} b_h - \sum_{h=1}^{p} a_h \right\} \quad . \quad . \quad . \quad . \quad (23)$$

This expansion holds for $|\arg z| < (q-p+\varepsilon)\pi$; herein is

$$\varepsilon = \tfrac{1}{2} \text{ if } q = p+1, \quad \varepsilon = 1 \text{ if } q \geqq p+2. \quad . \quad . \quad . \quad . \quad (24)$$

The coefficients M_1, M_2, ... do not depend on z, but are complicated functions of the parameters $a_1, \dots, a_p$ and $b_1, \dots, b_q$ which I will not record here [14]).

In what follows we shall, for brevity, formally write

$$H_{p,q}(z) = \left[\exp\left((p-q)\, z^{\frac{1}{q-p}}\right)\right] z^{\vartheta} \left\{ \frac{(2\pi)^{\frac{q-p-1}{2}}}{\sqrt{q-p}} + \frac{M_1}{z^{\frac{1}{q-p}}} + \frac{M_2}{z^{\frac{2}{q-p}}} + \dots \right\}, \quad (25)$$

where ϑ is defined by (23).

The results contained in (22) and (23) can then be formulated as follows:

Theorem C. *The function $G_{p,q}^{q,0}(z)$ possesses for large values of $|z|$ with $|\arg z| < (q-p+\varepsilon)\pi$ the asymptotic expansion*

$$G_{p,q}^{q,0}(z) \backsim H_{p,q}(z). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

It appears from (25) and (23) that the formal function $H_{p,q}(z)$ satisfies the relation

$$(-1)^{p-q+1} \exp\left\{ 2\pi i \left(\sum_{h=1}^{q} b_h - \sum_{h=1}^{p} a_h \right) \right\} H_{p,q}(z\, e^{(2p-2q)\pi i}) = H_{p,q}(z). \quad (27)$$

§ 3. The analytic continuation of the function $G_{p,p}^{m,n}(z)$ (simple cases).

The function $G_{p,p}^{m,n}(z)$ has been defined in § 1 only for $|z|<1$. The analytic continuation of this function when $|z| \geqq 1$ will be denoted too by the symbol $G_{p,p}^{m,n}(z)$.

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13) BARNES, [2], 296—297; [3], 80 and 108—110.
The proof of (22) is much more difficult than that of (18) and (21).

14) The reader is referred to BARNES' paper [3].

Now it is well-known [15]) that the generalized hypergeometric function ${}_pF_{p-1}\ (\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_{p-1}; z)$ has a branch-point at $z = 1$, and that, if a cross-cut is made along the real axis of the z-plane from 1 to ∞, then the function is analytic and one-valued throughout the cut plane. Considering the function $G^{m,n}_{p,p}(z)$, which can by (7) be written as a linear combination of functions of the type $z^{\lambda}\,{}_pF_{p-1}((-1)^{p-m-n}z)$, we shall make a cross-cut in the z-plane along the real axis from $(-1)^{m+n-p}$ to $\infty \cdot (-1)^{m+n-p}$; we may expect that $G^{m,n}_{p,p}(z)$ is analytic (not one-valued in general) in the cut plane.

I will now shew that a somewhat more general result holds when $m + n \geqq p + 2$.

For this purpose I consider the integral

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{p}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\,z^s\,ds, \quad . \quad . \quad . \quad (28)$$

where $1 \leqq m \leqq p$, $1 \leqq n \leqq p$, $m + n \geqq p + 1$ and $a_j - b_h \neq 1, 2, 3, \ldots$ $(j = 1, \ldots, n; h = 1, \ldots, m)$.

The contour L runs from $-\infty i + \sigma$ to $\infty i + \sigma$ (σ is an arbitrary real number) and is curved, if necessary, so that the points

$$b_j, b_j + 1, b_j + 2, \ldots \qquad (j = 1, \ldots, m)$$

lie on the right and the points

$$a_j-1,\ a_j-2,\ a_j-3, \ldots \qquad (j = 1, \ldots, n)$$

lie on the left of the contour.

The integral (28) is a function of z which is analytic for each value of z whether $|z|$ is greater than, equal to, or less than unity, provided that $z \neq 0$ and $|\arg z| < (m + n - p)\pi$ [16]).

If $|z| < 1$, we may [17]), without altering the value of the integral, bend round the contour L so as to coincide with the contour C defined in § 1. We therefore get in view of (2).

Theorem D. *The function* $G^{m,n}_{p,p}(z)$ *with* $m + n \geqq p + 1$ *is an analytic function of* z *in the sector* $|\arg z| < (m + n - p)\pi$; *this function may be represented by the integral* (28).

Now, when $m + n \geqq p + 2$, the point $(-1)^{m+n-p}$ lies in the interior of

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[15]) POCHHAMMER, [26]; WINKLER, [33]; MACROBERT, [16]. Comp. also MACROBERT, [17].

[16]) The reader may compare BAILEY, [1], 5—6 or WHITTAKER and WATSON, [32], 286—288, where a proof for a special case is elaborated. The proof of the general case runs along the same lines.

[17]) Comp. BAILEY, loc. cit. or WHITTAKER and WATSON, loc. cit..

the sector $|\arg z| < (m+n-p)\pi$. So, if we restrict ourselves to values of z for which $|\arg z| < (m+n-p)\pi$, the function $G_{p,p}^{m,n}(z)$ with $m+n \geqq p+2$ has no singularity at $z = (-1)^{m+n-p}$.

Hence, if $m+n \geqq p+2$ *and at the same time* $|\arg z| < (m+n-p)\pi$, *the cross-cut from* $(-1)^{m+n-p}$ *to* $\infty \,.\, (-1)^{m+n-p}$ *is superfluous.*

In all other cases, in continuing the function $G_{p,p}^{m,n}(z)$ *outside the circle* $|z| = 1$, *we will make a cross-cut along the real axis from* $(-1)^{m+n-p}$ *to* $\infty \,.\, (-1)^{m+n-p}$.

Replacing s by $-s$ in (28) we get

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{n}\Gamma(1-a_j-s)\prod_{j=1}^{m}\Gamma(b_j+s)}{\prod_{j=n+1}^{p}\Gamma(a_j+s)\prod_{j=m+1}^{p}\Gamma(1-b_j-s)}\left(\frac{1}{z}\right)^s ds.$$

On comparing this integral with (28) we find in virtue of theorem D that it is equal to

$$G_{p,p}^{n,m}\left(\frac{1}{z}\,\middle|\,\begin{matrix}1-b_1,\ldots,1-b_p\\1-a_1,\ldots,1-a_p\end{matrix}\right).$$

So we obtain

Theorem E. *The function* $G_{p,p}^{m,n}(z)$ *with* $m+n \geqq p+1$ *possesses in the sector* $|\arg z| < (m+n-p)\pi$ *the following analytic continuation outside the circle* $|z| = 1$

$$G_{p,p}^{m,n}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_p\end{matrix}\right) = G_{p,p}^{n,m}\left(\frac{1}{z}\,\middle|\,\begin{matrix}1-b_1,\ldots,1-b_p\\1-a_1,\ldots,1-a_p\end{matrix}\right). \quad (29)$$

In order to obtain a convenient notation for the continuation of the function $G_{p,p}^{p,1}(z\|a_t)$ I give still another definition.

Definition 3. (Comp. Definition 1.) *If* p *and* t *are integers with* $1 \leqq t \leqq p$ *and if the numbers* a_t *and* $b_1, \ldots, b_p$ *satisfy the condition*

$$a_t - b_j \neq 1, 2, 3, \ldots \qquad (j = 1, \ldots, p),$$

then the function $E_{p,p}(z\|a_t)$ *is defined as follows:*

$$E_{p,p}(z\|a_t) = G_{p,p}^{1,p}\left(\frac{1}{z}\,\middle|\,\begin{matrix}1-b_1,\ldots,1-b_p\\1-a_t, 1-a_1,\ldots,1-a_{t-1}, 1-a_{t+1},\ldots,1-a_p\end{matrix}\right). \quad (30)$$

This function may because of (7) *also be written* (comp. (14))

$$E_{p,p}(z\|a_t) = \frac{z^{-1+a_t}\prod_{j=1}^{p}\Gamma(1+b_j-a_t)}{\prod_{\substack{j=1\\j\neq t}}^{p}\Gamma(1+a_j-a_t)}\,{}_pF_{p-1}\left(\begin{matrix}1+b_1-a_t,\ldots,1+b_p-a_t;\\1+a_1-a_t,\ldots{}^{*}\ldots,1+a_p-a_t;-z^{-1}\end{matrix}\right); \quad (31)$$

the asterisk denotes, as usual, that the number $1 + a_t - a_t$ *is to be omitted in the sequence* $1+a_1-a_t, \ldots, 1+a_p-a_t$ [18]).

Now it follows from (7), (17) and (31), if the numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ satisfy the conditions (19) and (20),

$$G_{p,p}^{n,m}\left(\frac{1}{z}\,\middle|\,\begin{matrix}1-b_1, \ldots, 1-b_p\\ 1-a_1, \ldots, 1-a_p\end{matrix}\right)$$

$$= \sum_{t=1}^{n} e^{(m+n-p-1)\pi i a_t} \triangle_p^{m,n}(t)\, E_{p,p}(z\, e^{(p-m-n+1)\pi i} \,\|\, a_t).$$

Theorem E may therefore be stated in another way, namely

Theorem E*. (Comp. Theorem B.) *If* $m + n \geqq p + 1$ *and the numbers* $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_m$ *satisfy the conditions* (19) *and* (20), *then the function* $G_{p,p}^{m,n}(z)$ *possesses in the sector* $|\arg z| < (m+n-p)\pi$ *an analytic continuation outside the unit-circle which can be expressed in the form*

$$G_{p,p}^{m,n}(z) = \sum_{t=1}^{n} e^{(m+n-p-1)\pi i a_t} \triangle_p^{m,n}(t)\, E_{p,p}(z\, e^{(p-m-n+1)\pi i} \,\|\, a_t). \quad (32)$$

The particular case with $m = p$ and $n = 1$ of formula (29) (which is, of course, also the corresponding particular case of (32)) is worth while to mention it separately; because of theorem E and (30) we obtain:

Theorem F. (Comp. Theorem A.) *The function* $G_{p,p}^{p,1}(z \,\|\, a_t)$ *possesses in the sector* $|\arg z| < \pi$ *the following analytic continuation outside the unit-circle*

$$G_{p,p}^{p,1}(z \,\|\, a_t) = E_{p,p}(z \,\|\, a_t). \quad \ldots\ldots \quad (33)$$

This formula and the particular case with $m = 1$ and $n = p$ of (32) are well-known. For $p = 2$ the formulae are proved by GAUSS, KUMMER, GOURSAT, BARNES and VAN DER CORPUT [19]). The general case has been treated by THOMAE, WINKLER, MACROBERT and F. C. SMITH [20]).

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18) Comp. footnote [4]).

19) GAUSS, [**10**], 220; KUMMER, [**13**], 60; GOURSAT, [**11**], 29; BARNES, [**5**], 146—152; VAN DER CORPUT, [**6**]. Comp. also: FORSYTH, [**8**], Chapter VI; KLEIN, [**12**], § 10; WHITTAKER and WATSON, [**32**], § 14. 51. VAN DER CORPUT and WHITTAKER and WATSON give the formula in question with some misprints.

20) THOMAE, [**29**]; WINKLER, [**33**]; MACROBERT, [**15**]; SMITH, [**27**]. Comp. also MEIJER, [**19**], formula (40).