Mathematics. — *Continuous classification of all microcompact 0-dimensional spaces.* By J. DE GROOT. (Communicated by Prof. L. E. J. BROUWER.)

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1. All continuous invariants of the family of all countable sets having been classified in a previous paper ([3], chapter II; comp. also [4]), we set ourselves as chief task the classification of all continuous invariants of the family of *all microcompact 0-dimensional sets* (theorem III). To accomplish this purpose we begin by proving a theorem concerning retractions of 0-dimensional sets. The subset A of B is, as known (BORSUK [1]), by definition a *retract* of B if there exists a continuous mapping f of B on A which in A is the identity ($A = f(B)$, $f(x) = x$ for $x \subset A$). f is called a retraction of B in A. We shall now prove (theorem I) that, given an arbitrary 0-dimensional set B, a not-vacuous subset $A \subset B$ is a retract of B only if A is *closed in* B.

The notations and results of [3] are supposed to be known. *All sets* (or spaces) *considered will be separable.*

2. **Theorem I.** *A not-vacuous subset A of an 0-dimensional space N is a retract of N only if A is closed in N.*

P r o o f. According to a well-known theorem of SIERPINSKI [6] N is homoeomorphic with a subset of the *Cantor discontinuum* D (i.e., the set of all real numbers expressible in the form $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ where $a_n = 0$ or 2). We denote this homoeomorphic subset of D again by N. Be A an (in N) closed subset of N. We shall construct a retraction of N in A.

We consider an infinite system of coverings $\{I\}$ of D, consisting of lump-sets (i.e., both open and closed subsets of D). Each lump-set $I_{a_1 a_2 \ldots a_n}$ ($a_i = 0$ or 2) will consist of all points of the form

$$\sum_{i=1}^{n} \frac{a_i}{3^i} + \sum_{i=n+1}^{\infty} \frac{b_i}{3^i} \qquad (b_i = 0 \text{ or } 2).$$

For arbitrary but fixed n the 2^n sets $I_{a_1 a_2 \ldots a_n}$ apparently determine a covering, the so-called nth covering of D. Every monotonicly decreasing sequence of sets

$$I_{a_1} \supset I_{a_1 a_2} \supset I_{a_1 a_2 a_3} \supset \ldots \tag{1}$$

apparently determines exactly one point $d_{a_1 a_2 a_3 \ldots}$ of D and, conversely, for every point $d \subset D$ there exists exactly one defining sequence (1). A number of sequences (1) defines exactly the points $a \subset A$. If in an I lies at least one point of A we also denote this I by I. Now consider the first covering of D, consisting of the two sets I_0 and I_2. If both lump-sets I_0 and I_2 are at the same time sets $\hat{I}_0$ and $\hat{I}_2$, nothing happens. If, however, one of these lump-sets, e.g. I_2, is not an $\hat{I}$-set, so that $I_2 \cdot A = 0$, we map the set $I_2 \cdot N$, if it is not vacuous, onto one fixed but arbitrarily chosen point of $\hat{I}_0 \cdot A$. Then we strike out the set I_2 and also all sets $I_{2\, b_2 b_3 \dots b_n}$ ($b_i = 0$ or 2; $n = 2, 3, \dots$) in the subsequent coverings (a set of this shape can never be an $\hat{I}$-set and all (possible) points $I \cdot N$ have already been mapped for every I of that kind.

Now consider the $(n-1)$th covering of D consisting of the 2^{n-1} sets $I_{a_1 a_2 \dots a_{n-1}}$ ($a_i = 0$ or 2). A number of these I-sets is also an $\hat{I}$-set. The remaining I-sets (possibly none at all) have been struck out according to the supposed induction, while the points of N which they may contain have already, at one of the first $(n-1)$ steps, been mapped onto points of A. We now construct the mapping at the nth step as follows. Consider the nth covering of D by the sets $I_{a_1 a_2 \dots a_{n-1} a_n}$. Every $\hat{I}_{a_1 a_2 \dots a_{n-1}}$ is divided into two sets $I_{a_1 a_2 \dots a_{n-1} 0}$ and $I_{a_1 a_2 \dots a_{n-1} 2}$. If both of these are $\hat{I}$-sets nothing happens. If they are not, then at least one, for instance $I_{a_1 a_2 \dots a_{n-1} 0} = \hat{I}_{a_1 a_2 \dots a_{n-1} 0}$, is an $\hat{I}$-set. We map all possible points of $N \cdot I_{a_1 a_2 \dots a_{n-1} 2}$ on a fixed but arbitrarily chosen point of $A \cdot \hat{I}_{a_1 a_2 \dots a_{n-1} 0}$. This process takes place for the considered n for every $\hat{I}_{a_1 a_2 \dots a_{n-1}}$. After this all sets $I_{a_1 a_2 \dots a_n}$ which are not $\hat{I}$-sets, are struck out.

This process we think infinitely continued in the same way and we shall prove that the thus constructed mapping f, if we moreover put $f(a) = a$ for $a \subset A$, is a retraction of N in A.

First every point $n \subset N$ is indeed mapped onto one point of A. For, by definition, $f(a) = a$ for the points a of A; a point $n \subset N - A$ lies at a certain distance ε from A, as A is closed in N, so that there may be found a jth covering of A by $\hat{I}$-sets for sufficiently great j such that the point n does not belong to any of the $\hat{I}$-sets of this covering, in other words is mapped on a point of A.

The mapping f further is continuous. Let a sequence of points n_i ($i = 1, 2, \dots$) of N be converging to a point $n \subset N - A$. The point n is, e.g. at the kth step, mapped on a point a of A as it is belonging to an I-set which is struck out (which in other words is not an $\hat{I}$-set). But this I-set is a neighbourhood of n so that also nearly all points n_i are mapped on the point a; the mapping f therefore is continuous for such a sequence. — There remains the possibility that the sequence of points n_i converges to a point $a \subset A$. a is determined by the intersection of the lump-sets of a sequence

$$D \supset \hat{I}_{a_1} \supset \hat{I}_{a_1 a_2} \supset \hat{I}_{a_1 a_2 a_3} \supset \dots$$

For every natural number j almost all points n_i are lying in $\hat{I}_{a_1 a_2 \dots a_j}$. Then these points n_i have not yet been mapped by previous steps, since they cannot be lying in a struck-out set I. At the $(j+1)$th and subsequent steps the points n_i are mapped on points within $\hat{I}_{a_1 a_2 \dots a_j}$, so that also nearly all points $f(n_i)$ $(i = 1, 2, \dots)$ are lying in $\hat{I}_{a_1 a_2 \dots a_j}$, i.e., f is continuous.

Conversely, we still have to prove that a retract A of N is closed in N. This is, however, a matter of course.

3. We now determine all continuous invariants of the family F_c of all compact 0-dimensional sets. The countability is a not-trivial continuous invariant; further, it has been shown in [3] that the (compact) countable (and so naturally 0-dimensional) sets possess exactly the not-trivial continuous invariants $[<\lambda, l]$. These are already all (not-trivial) continuous invariants of F_c. For, be N a compact, not-countable, 0-dimensional set of F_c; then, because of the non-countability, N has a not-vacuous kernel, dense in itself. This kernel is, according to a theorem of BROUWER [2], homoeomorphic with the discontinuum D. Further, if A is an arbitrary compact 0-dimensional set, then according to the already mentioned theorem of SIERPINSKI [6] A is homoeomorphic with a closed subset of D, and therefore also with a closed subset of N; according to theorem I A may be considered as a retraction of N. So *every not-countable compact 0-dimensional set N may be mapped continuously on every compact 0-dimensional set A*, by which we have shown that there exist no other continuous invariants in F_c than the already mentioned ones.

Theorem II [1]. *The family of the compact 0-dimensional sets has, as only (not-trivial) continuous invariants, the properties of the "countability" and $[<\lambda, l]$.*

4. We shall now proceed to determine all continuous invariants of the family F_m of all microcompact 0-dimensional sets. All sets mentioned in this number will be supposed to be microcompact and 0-dimensional. We know already the existence of the following (not-trivial) continuous invariants in F_m: the properties $[<\lambda, l]$, the property $\varkappa \cdot \alpha$, i.e. "both compact and countable", the property α (countability), the property $\varkappa$ (compactness). Also the property $\varkappa + \alpha$ (a set has the property $\varkappa + \alpha$ if the set is the sum of two (disjunct) sets one of which is compact while the other is countable; one of these two sets may be vacuous) is an apparently continuous invariant. The logical dependency of these axioms is illustrated by fig. 1. We shall prove that these are the only continuous invariants of F_m.

Theorem III. *The family of microcompact 0-dimensional sets has, as*

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[1]) Mentioned without proof in [4], theorem IV.

only not-trivial continuous invariants, the properties: $[<\lambda, l]$ [2], $\varkappa\, \alpha$, α, $\varkappa$ and $\varkappa + \alpha$.

By this theorem especially all (e.g. in an n-dimensional Euclidian space) *closed 0-dimensional sets have been classified continuously* [3].

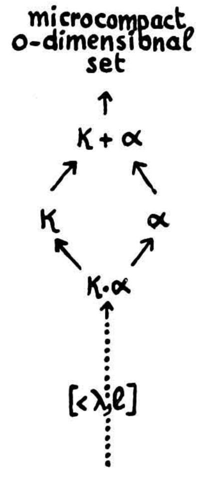

fig.1

Proof. The compact sets have already been investigated (theorem II); the investigation of the countable sets has been described in [3] (theorem V), where we even proved that *every* (not only microcompact) not-compact countable set may be mapped continuously on *every* countable set.

Therefore we now first must investigate a set with property $\varkappa + \alpha$, which does not possess the property $\varkappa$ or (and) α (for those have already been investigated); in other words, we start from a set $N = C + A$ where C is compact and not-countable, while A is countable and not-compact. We have to prove that N may be mapped continuously on every set $N' = C' + A'$ (C' compact, A' countable; C' or A' may be vacuous) with property $\varkappa + \alpha$.

Before proceeding to the construction of the continuous mapping in question we insert a necessary remark. Every micro-compact 0-dimensional set N may, as known, be compactified to a compact 0-dimensional (separable) space $\overline{N} = N + P$ by one point P. Putting this for $N = C + A$, so $\overline{N} = C + A + P$, then apparently there may be found a neighbourhood $U = U(C/N)$ of C in N, which, like C, is compact and not-countable, so that $N - U$ is countable and not-compact, while both U and $N - U$ are closed in N. U is again denoted by C and $N - U$ by A; then $N = C + A$ has therefore been divided into two disjunct closed sets, of which C is compact and not-countable, and A is countable but not-compact.

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[2]) There are apparently $\aleph_1$ continuous invariants of this kind.

[3]) Since every such closed set is microcompact.

We now construct the continuous mapping on $N' = C' + A'$. If N' is compact, then N' is topologically equivalent with a closed subset N'' of C, since C has a discontinuum as a subset; a retraction of N in N'' (theorem I) together with the topological mapping of N'' on N' gives the required continuous mapping of N on N'.

If N' is not compact, so if A' is not compact, and moreover if C' is not vacuous, then we map C continuously on C' (which apparently is possible according to the above-written) and A on A'. Both mappings together give an apparently continuous mapping of N on N'.

If, finally, N' is not compact but C' is vacuous, then we map A on A' and all C on one arbitrary point of A'. The required continuous mapping is therefore always possible.

Secondly we have to consider an 0-dimensional microcompact set N which does not possess the property $\varkappa + \alpha$, and we must prove that every such N may be mapped on every other (0-dimensional micro-compact) set N'. Suppose N' is not compact (if N' is compact the following proof may be simplified). We compactify the micro-compact sets N and N' each by one point, P and P' respectively, to the compact 0-dimensional sets $\overline{N} = N + P$ and $\overline{N}' = N' + P'$. We determine two monotonicly decreasing systems of neighbourhoods U_i and U'_i $(i = 0, 1, 2, \ldots)$ of P and P' in $\overline{N}$ and $\overline{N}'$, consisting of lump-sets, while $U_0 = N$ and $U'_0 = N'$. Since N does not possess the property $\varkappa + \alpha$ there are infinitely many values i (e.g. j_k $(k = 0, 1, 2, \ldots)$) so that the lump-set $V_{j_k} = U_{j_k} - U_{j_{k+1}}$ has a discontinuum as a subset, in other words, every V_{j_k} may be mapped continuously on the lump-set $U'_k - U'_{k+1}$ $(k = 0, 1, 2, \ldots)$. The possibly remaining sets $U_i - U_{i+1}$ $(i \neq j_k)$ we all map on one point of N'. Thus an apparently continuous mapping of N on N' is constructed.

It follows immediately from the above that there exist no other continuous invariants than the already mentioned, by which the proof has been completed.

5. *Final remark*. The above used method is not suited to the determination of the continuous invariants of the family of all 0-dimensional sets nor to the determination of the topological invariants of the family of all compact 0-dimensional sets.

On the contrary, in these cases things turn out to be very complicated. This is already shown by the following example. The 0-dimensional sets N, having a discontinuum D as a subset, play an exceptional part, since every such N may be mapped continuously on every compact 0-dimensional N' (N' is topologically equivalent with a closed subset N'' of D and N'' is a retraction of N). At the same time there exist, however, not-countable 0-dimensional sets which do not contain a discontinuum as a subset (according to a theorem of BERNSTEIN, comp. [5], p. 176); sets of this kind are called *totally imperfect*.

We now think a discontinuum D divided into two totally imperfect subsets S_1 and S_2 (this is possible) and define S'_1 and S'_2 as pointsets on the real axis, homoeomorphic with S_1 and S_2, and isolated in respect to each other. The sum $S' = S'_1 + S'_2$ therefore is also totally imperfect. Yet it is apparently possible *to map this totally imperfect set S' one-to-one and continuously on the discontinuum D,* namely by mapping the corresponding points of S_1 and S'_1, resp. S_2 and S'_2, on each other.

But to what extent is it possible to map an arbitrary totally imperfect set continuously on a given (not-countable) 0-dimensional set?

LITERATURE

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[6] SIERPINSKI, W. Sur les ensembles connexes et non-connexes. Fund. Math. **2**, 81—95 (1921).

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[4]) I take this opportunity to correct a disturbing error in this paper:
p. 238, second line: read $\aleph_1$ instead of $\aleph_0$.
p. 243, nr. 1. 10, 11th line: read $\aleph_1$ instead of $\aleph_0$.
p. 243, nr. 1. 10, 12th line: cross out the word "even".