Mathematics. — *On the G-function.* II. By C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT.)

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§ 4. **The fundamental systems of the differential equation satisfied by $G_{p,q}^{m,n}(z)$**

The function $y = G_{p,q}^{m,n}(z)$ defined by (2) satisfies the homogeneous linear differential equation of the q-th order

$$\left\{(-1)^{p-m-n} z \prod_{j=1}^{p} (\theta - a_j + 1) - \prod_{j=1}^{q} (\theta - b_j)\right\} y = 0, \quad . \quad . \quad (34)$$

where θ denotes the operator $z \dfrac{d}{dz}$.

This may be established in a very simple way. For the particular case with $m = 1$, $n = p = q = 2$ and $b_1 = 0$, that is if the function $G_{p,q}^{m,n}(z)$ reduces to the ordinary hypergeometric function ${}_2F_1$, it has been proved by BARNES [21]). The proof of the general case is almost similar to that of BARNES' special case, so that it may be omitted here.

Replacing z by $ze^{(p-m-n+1)\pi i}$ we find that the q functions [22])

$$\left.\begin{aligned} & e^{(m+n-p-1)\pi i\, b_h}\, G_{p,q}^{1,p}\left(z\, e^{(p-m-n+1)\pi i}\, \middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_h, b_1, \ldots, b_{h-1}, b_{h+1}, \ldots, b_q \end{matrix}\right) \\ & = \frac{\prod\limits_{j=1}^{p} \Gamma(1+b_h-a_j)}{\prod\limits_{\substack{j=1 \\ j \neq h}}^{q} \Gamma(1+b_h-b_j)}\, z^{b_h}\, {}_pF_{q-1}\left(\begin{matrix} 1+b_h-a_1, \ldots, 1+b_h-a_p; \\ 1+b_h-b_1, \ldots{}^{*} \ldots, 1+b_h-b_q; (-1)^{p-m-n} z \end{matrix}\right), \end{aligned}\right\} (35)$$

where $h = 1, \ldots, q$ satisfy too the differential equation (34). When

$$b_h - b_j \neq 0, \pm 1, \pm 2, \ldots \qquad (h = 1, \ldots, q; j = 1, \ldots, q; h \neq j),$$

the functions (35) are obviously linearly independent and form therefore a fundamental system of solutions valid in the vicinity of the origin [23]). Hence the function $G_{p,q}^{m,n}(z)$ must be linearly expressible in terms of the functions (35). The actual expression is given by (7).

We now proceed to determine a fundamental system of solutions of (34) valid near infinity; for this purpose we distinguish two cases:

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21) BARNES, [5], 145.

22) Equation (35) follows from (7).

23) If one or more of the differences $b_h - b_j$ $(h \neq j)$ is equal to 0, $\pm$ 1, $\pm$ 2, ..., some of the fundamental solutions (35) must be replaced by expressions involving logarithmic terms.

First case: $q > p$. To every value of $\arg z$ it is possible to determine an integer λ such that

$$-(\tfrac{1}{2}q-\tfrac{1}{2}p+1)\pi < \arg z+(q-m-n-2\lambda+1)\pi < (\tfrac{1}{2}q-\tfrac{1}{2}p+1)\pi\ . \quad (36)$$

It is also possible to determine an integer ω such that [24]

$$-(q-p+\varepsilon)\pi < \arg z+(q-m-n-2\psi)\pi < (q-p+\varepsilon)\pi. \quad (37)$$

for $\psi=\omega,\ \omega+1,\dots,\omega+q-p-1$.

We now suppose

$$a_j-a_h \neq 0,\ \pm 1,\ \pm 2,\dots (j=1,\dots,p;\ h=1,\dots,p;\ j\neq h)\ . \quad (38)$$

and consider the functions

$$G^{q,1}_{p,q}(z\,e^{(q-m-n-2\lambda+1)\pi i}\,\|\,a_t) \qquad (t=1,\dots,p). \quad (39)$$

They obviously satisfy the differential equation (34). On account of (36) and (13) it follows from theorem A that these functions tend algebraically to zero or to infinity for $|z|\to\infty$; it is further clear in virtue of (38) and (13) that they have a mutually different algebraic behaviour for large values of $|z|$. Hence they form a system of p linearly independent solutions of the differential equation (34). The fundamental system is therefore not yet complete.

In order to determine the failing solutions we consider the $q-p$ functions

$$G^{q,0}_{p,q}(z\,e^{(q-m-n-2\psi)\pi i}) \qquad (\psi=\omega,\ \omega+1,\dots,\omega+q-p-1)\ . \quad (40)$$

It is easily seen that these functions satisfy too the differential equation (34). Because of (37) and (25) it follows from (26) that they tend exponentially to zero or to infinity as $|z|\to\infty$; besides it appears from (26) and (25) that they are mutually linearly independent. Hence they supply the $q-p$ failing fundamental solutions of (34).

We have therefore proved: *When $q > p$ and the conditions* (36), (37) *and* (38) *are satisfied, a fundamental system of* (34) *valid in the vicinity of $z=\infty$ is formed by the p functions* (39) *and the $q-p$ functions* (40).

Second case: $q = p$. We shall assume that the numbers $a_1 \dots, a_p$ satisfy the condition (38). Besides we suppose

$$\arg z+(p-m-n)\pi \neq 0,\ \pm 2\pi,\ \pm 4\pi,\dots \quad (41)$$

Then it is possible to determine an integer λ such that

$$-\pi < \arg z+(p-m-n-2\lambda+1)\pi < \pi. \quad (42)$$

We now consider the functions

$$G^{p,1}_{p,p}(z\,e^{(p-m-n-2\lambda+1)\pi i}\,\|\,a_t) \qquad (t=1,\dots,p)\ . \quad (43)$$

[24] The number ε is defined by (24).

These functions satisfy the differential equation (34) with $q = p$. Because of (42), (31) and (38) it follows from theorem F that they have outside the unit-circle a mutually different behaviour for $|z|$ tending to infinity. The functions (43) form therefore a system of p linearly independent solutions of the differential equation (34) with $q = p$.

Hence, if $q = p$ and the conditions (41), (42) and (38) are satisfied, the p functions (43) form a fundamental system adapted for the region outside the circle $|z| = 1$.

Remark. Sometimes there occur in a formula not the p functions (39), respect. (43) but only the first n of these functions, viz. the functions

$$G_{p,q}^{q,1}(z\,e^{(q-m-n-2\lambda+1)\pi i}\,||\,a_t) \qquad (t = 1, \ldots, n),$$

respect.

$$G_{p,p}^{p,1}(z\,e^{(p-m-n-2\lambda+1)\pi i}\,||\,a_t) \qquad (t = 1, \ldots, n).$$

In order that these functions are linearly independent, it is not necessary that the condition (38) is satisfied. We may replace (38) by the less stringent condition (20).

If we have to do with less than $q - p$ functions $G_{p,q}^{q,0}$, namely with the ν ($\nu < q - p$) functions

$$G_{p,q}^{q,0}(z\,e^{(q-m-n-2\psi)\pi i}) \qquad (\psi = \omega, \omega + 1, \ldots, \omega + \nu - 1),$$

then it is evident that the condition (37) needs not to be satisfied for $\psi = \omega, \omega + 1, \ldots, \omega + q - p - 1$, but only for $\psi = \omega, \omega + 1, \ldots, \omega + \nu - 1$.

§ 5. Contents of the paper.

The function $G_{p,q}^{m,n}(z)$ can by means of (7) be linearly expressed in terms of the functions of the fundamental system of (34) valid near $z = 0$. But now that we have constructed a system of fundamental solutions of (34) valid in the neighbourhood of $z = \infty$, it must also be possible to express the function $G_{p,q}^{m,n}(z)$ as a linear combination of these last fundamental solutions. That is to say: There must exist relations of the form

$$G_{p,q}^{m,n}(z) = \sum_{t=1}^{p} C_t\, G_{p,q}^{q,1}(z\,e^{(q-m-n-2\lambda+1)\pi i}\,||\,a_t) + \sum_{h=0}^{q-p-1} D_h\, G_{p,q}^{q,0}(z\,e^{(q-m-n-2h-2\omega)\pi i}), \quad (44)$$

where

$$G_{p,q}^{q,1}(z\,e^{(q-m-n-2\lambda+1)\pi i}\,||\,a_t) \text{ and } G_{p,q}^{q,0}(z\,e^{(q-m-n-2h-2\omega)\pi i})$$

are the in § 4 defined fundamental solutions in the neighbourhood of $z = \infty$ and where the coefficients C_t and D_h do not depend on z.

On account of the generality of the function $G_{p,q}^{m,n}(z)$ and the great number of important functions which are particular cases of it [25], it will

[25]) See the lists in [18]—[25]; these lists are not complete.

be interesting to investigate this subject more fully. In the present paper I will determine the coefficients C_t and D_h for all values of m, n, p and q, for which the function $G_{p,q}^{m,n}(z)$ has been defined and for all values of $|z|$ and $\arg z$ [26]). It appears necessary to distinguish several cases. In most cases the coefficients C_t and D_h are much more difficult to determine and much more complicated than the coefficients in formula (7).

§§ 6, 7, 8 and 13 supply some definitions and a great number of lemmas. In §§ 9, 10, 11, 12 and 14 I prove four expansion formulae (theorems 1, 2, 3 and 5) which render it possible to write the function $G_{p,q}^{m,n}$ in a special way as a linear combination of functions $G_{p,q}^{k,l}$ and $G_{p,q}^{k,l-1}$.

These expansion formulae are the most powerful instruments of the present paper. Their most important particular case appears when we put $l = 1$ and $k = q$. The in this manner specialized expansion formulae are written out in § 16. In §§ 17 and 19 I will show that the desired relations of the type (44) are particular cases of the specialized expansion formulae of § 16. § 17 is devoted to the case with $q > p$; the case with $q = p$ will be examined in § 19.

In § 2 I have mentioned BARNES' asymptotic expansion of the function $G_{p,q}^{m,n}(z)$ $(q > p)$. His results bear upon the two following cases only:

A. $1 \leqq n \leqq p < q$, $1 \leqq m \leqq q$, $m+n > \frac{1}{2}p + \frac{1}{2}q$, $|\arg z| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$.
B. $n = 0$, $m = q$, $|\arg z| < (q-p+\varepsilon)\pi$.

But now that it is possible to express the function $G_{p,q}^{m,n}(z)$ for all values of m, n, p, q and $\arg z$ in terms of functions $G_{p,q}^{q,1}(\zeta \| a_t)$ and $G_{p,q}^{q,0}(\zeta)$ of which the asymptotic expansions for large values of $|\zeta|$ can be written down by means of BARNES' formulae (18) and (26), it is also possible to deduce asymptotic expansions for the function $G_{p,q}^{m,n}(z)$ $(q > p)$ for all cases which do not come under those of BARNES. Hence we are now able, for all values of m, n, p, q and $\arg z$, to determine asymptotic expansions of the function $G_{p,q}^{m,n}(z)$ $(q > p)$ as $|z| \to \infty$. These expansions are given in § 18; I have stated my results in the form of six theorems.

The function $G_{p,p}^{m,n}(z)$ has been defined in § 1 for $|z| < 1$ and any value of $\arg z$. In § 3 (theorem E) I have given a formula for the analytic continuation of this function outside the circle $|z| = 1$ in the case

I. $m + n \geqq p + 1$, $|\arg z| < (m + n - p)\pi$.

Hence there still fail formulae for the analytic continuation of $G_{p,p}^{m,n}(z)$ in the two following cases:

II. $m + n \leqq p$, all values of $\arg z$.
III. $m + n \geqq p + 1$, $|\arg z| > (m + n - p)\pi$.

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[26]) If $q = p$, I suppose $|z| > 1$; the values of z on the cross-cut are excluded.

As I have already stated in § 3, I will make in these cases a cross-cut along the real axis from $(-1)^{m+n-p}$ to $\infty \cdot (-1)^{m+n-p}$. Now it is possible to express the function $G_{p,p}^{m,n}(z)$ for all values of m, n, p and $\arg z$ (except values of z on the cross-cut) in terms of the fundamental solutions near $z = \infty$ of the differential equation satisfied by $G_{p,p}^{m,n}(z)$. Since the continuations for $|z| > 1$ of these fundamental solutions can be immediately written down by means of theorem F, it is also possible to determine in all cases (except when z lies on the cross-cut) the analytic continuation of $G_{p,p}^{m,n}(z)$ for $|z| > 1$. The formula in question is given in § 19.

A simple and important special case of the function $G_{p,q+1}^{m,n}(z)$ is the generalized hypergeometric function ${}_pF_q\ (\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_p; z)$. The behaviour of this function for large values of $|z|$ was a subject of investigation of several writers. Most of these investigations are fairly difficult. In § 20 I will shew that the asymptotic expansions of the function ${}_pF_q\ (z)$ can with slight labour be deduced from the analogous expansions of $G_{p,q+1}^{1,p}(z)$.

Now one would be apt to think, on account of (7), that it must on the contrary also be possible to deduce the asymptotic expansions of $G_{p,q}^{m,n}(z)$ from the known expansions of the function ${}_pF_{q-1}(z)$. In many cases however we then find for $G_{p,q}^{m,n}(z)$ an expansion wherein the coefficients all vanish. Now it is clear that a such expansion is worthless, so that in these cases it is not possible to deduce a suitable asymptotic expansion for $G_{p,q}^{m,n}(z)$ from the corresponding expansion of ${}_pF_{q-1}(z)$. For instance the asymptotic expansions of $G_{p,q}^{m,n}(z)$ which are exponentially small cannot be deduced from the asymptotic expansions of ${}_pF_{q-1}(z)$.

In § 15 I will examine some relations of the form

$$M(z) = \sum_{h=1}^{k} r_h\, M(z\, e^{(\gamma-2h)\pi i}).$$

Finally, in § 21 I will determine for all values of $\arg z$ the asymptotic expansion of WHITTAKER's function $W_{k,m}(z)$ as $|z| \to \infty$.

§ 6. Definitions.

Definition 4. *If the numbers a_s are defined for $s = 0, \pm 1, \pm 2, \ldots$ and h is a positive integer or zero, the sums $\sum_{s=0}^{h} a_s$ and $\sum_{s=0}^{-1} a_s$ have the usual meaning*

$$\sum_{s=0}^{h} a_s = a_0 + a_1 + \ldots + a_h \text{ and } \sum_{s=0}^{-1} a_s = 0.$$

If k is an integer $\geqq 2$, the sum $\sum_{s=0}^{-k} a_s$ is defined as follows:

$$\sum_{s=0}^{-k} a_s = -a_{-1} - a_{-2} - \ldots - a_{-k+1}.$$

Definition 5. *Suppose that m, n and q are integers with*

$$0 \leqq m \leqq q \text{ and } n \geqq 0.$$

Then the coefficients $A^{m,n}_{\ q}$ *and* $\bar{A}^{m,n}_{\ q}$ *are defined as follows:*

$$A^{m,n}_{\ q} = (-2\pi i)^{m+n-q}\, e^{(a_1+\ldots+a_n-b_{m+1}-\ldots-b_q)\pi i}, \quad \ldots \quad (45)$$

$$\bar{A}^{m,n}_{\ q} = (2\pi i)^{m+n-q}\, e^{-(a_1+\ldots+a_n-b_{m+1}-\ldots-b_q)\pi i}. \quad \ldots \quad (46)$$

Definition 6. *Suppose that m, n and p are integers with*

$$0 \leqq n \leqq p \text{ and } m \geqq 0.$$

Then the coefficients $B^{m,n}_p$ *and* $\bar{B}^{m,n}_p$ *are defined as follows:*

$$B^{m,n}_p = (-2\pi i)^{m+n-p}\, e^{(b_1+\ldots+b_m-a_{n+1}-\ldots-a_p)\pi i} \quad \ldots \quad (47)$$

$$\bar{B}^{m,n}_p = (2\pi i)^{m+n-p}\, e^{-(b_1+\ldots+b_m-a_{n+1}-\ldots-a_p)\pi i}. \quad \ldots \quad (48)$$

Definition 7. *Suppose that m, n and q are integers with*

$$0 \leqq m \leqq q \text{ and } n \geqq 0;$$

suppose further that t is an arbitrary integer.

Then the coefficients $\Omega^{m,n}_{\ q}(t)$ *and* $\bar{\Omega}^{m,n}_{\ q}(t)$ *are defined as follows:* $\Omega^{m,n}_{\ q}(t)$, *respect.* $\bar{\Omega}^{m,n}_{\ q}(t)$ *is the coefficient of* x^t *in the expansion of*

$$\frac{\prod_{j=m+1}^{q}(1-xe^{2\pi i b_j})}{\prod_{j=1}^{n}(1-xe^{2\pi i a_j})}, \text{ respect. } \frac{\prod_{j=m+1}^{q}(1-xe^{-2\pi i b_j})}{\prod_{j=1}^{n}(1-xe^{-2\pi i a_j})}$$

in ascending powers of x.

Definition 8. *Suppose that m, n and p are integers with*

$$0 \leqq n \leqq p \text{ and } m \geqq 0;$$

suppose further that λ *is an arbitrary integer.*

Then the coefficients $\Gamma^{m,n}_p(\lambda)$ *and* $\bar{\Gamma}^{m,n}_p(\lambda)$ *are defined as follows:* $\Gamma^{m,n}_p(\lambda)$, *respect.* $\bar{\Gamma}^{m,n}_p(\lambda)$ *is the coefficient of* x^λ *in the expansion of*

$$\frac{\prod_{j=n+1}^{p}(1-xe^{2\pi i a_j})}{\prod_{j=1}^{m}(1-xe^{2\pi i b_j})}, \text{ respect. } \frac{\prod_{j=n+1}^{p}(1-xe^{-2\pi i a_j})}{\prod_{j=1}^{m}(1-xe^{-2\pi i b_j})}$$

in ascending powers of x.

Definition 9. *Suppose that h, m, n, p and q are integers with*

$$0 \leqq n \leqq p,\ 0 \leqq m \leqq q \text{ and } h \geqq 1;$$

suppose further that λ *is an arbitrary integer.*

Then the coefficients $\Phi^{m,n}_{p,q}(h;\lambda)$ *and* $\bar{\Phi}^{m,n}_{p,q}(h;\lambda)$ *are defined as follows:*

$\Phi^{m,n}_{p,q}(h;\lambda)$, respect. $\overline{\Phi}^{m,n}_{p,q}(h;\lambda)$ is the coefficient of $x^{h+\lambda-1}$ in the expansion of

$$\left\{\sum_{t=0}^{h-1}\Omega^{0,p}_{q}(t)\,x^t\right\}\left\{\frac{\prod_{j=n+1}^{p}(1-xe^{2\pi ia_j})}{\prod_{j=1}^{m}(1-xe^{2\pi ib_j})}\right\},$$

respect.

$$\left\{\sum_{t=0}^{h-1}\overline{\Omega}^{0,p}_{q}(t)\,x^t\right\}\left\{\frac{\prod_{j=n+1}^{p}(1-xe^{-2\pi ia_j})}{\prod_{j=1}^{m}(1-xe^{-2\pi ib_j})}\right\}$$

in ascending powers of x.

Definition 10. *Suppose that h, m, n, p and q are integers with*

$$0\leqq n\leqq p,\ 0\leqq m\leqq q \text{ and } h\geqq 1;$$

suppose further that λ is an arbitrary integer.

Then the coefficients $\Psi^{m,n}_{p,q}(h;\lambda)$ and $\overline{\Psi}^{m,n}_{p,q}(h;\lambda)$ are defined as follows: $\Psi^{m,n}_{p,q}(h;\lambda)$, respect. $\overline{\Psi}^{m,n}_{p,q}(h;\lambda)$ is the coefficient of x^λ in the expansion of

$$\left\{\sum_{t=1}^{h}\overline{\Omega}^{o,p}_{q}(h-t)\,x^t\right\}\left\{\frac{\prod_{j=n+1}^{p}(1-xe^{2\pi ia_j})}{\prod_{j=1}^{m}(1-xe^{2\pi ib_j})}\right\},$$

respect.

$$\left\{\sum_{t=1}^{h}\Omega^{o,p}_{q}(h-t)\,x^t\right\}\left\{\frac{\prod_{j=n+1}^{p}(1-xe^{-2\pi ia_j})}{\prod_{j=1}^{m}(1-xe^{-2\pi ib_j})}\right\}$$

in ascending powers of x.

Definition 11. *Suppose that l, m, n and p are integers with*

$$1\leqq l\leqq p,\ 0\leqq n\leqq p \text{ and } m\geqq 0;$$

suppose further that r is an arbitrary integer.

Then the coefficients $\Theta^{m,n}_{p}(l;r)$ and $\overline{\Theta}^{m,n}_{p}(l;r)$ are defined as follows: $\Theta^{m,n}_{p}(l;r)$, respect. $\overline{\Theta}^{m,n}_{p}(l;r)$ is the coefficient of x^r in the expansion of

$$\frac{\prod_{j=n+1}^{p}(1-xe^{2\pi ia_j})}{(1-xe^{2\pi ia_l})\prod_{j=1}^{m}(1-xe^{2\pi ib_j})},\ \text{respect.}\ \frac{\prod_{j=n+1}^{p}(1-xe^{-2\pi ia_j})}{(1-xe^{-2\pi ia_l})\prod_{j=1}^{m}(1-xe^{-2\pi ib_j})}$$

in ascending powers of x.

Remark. From the above definitions it follows

$$\Omega^{m,n}_{q}(t)=\overline{\Omega}^{m,n}_{q}(t)=0 \text{ for } t=-1,-2,-3,\ldots, \quad . \quad . \quad (49)$$

$$\Omega^{m,n}_{q}(0)=\overline{\Omega}^{m,n}_{q}(0)=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

$$\Gamma^{m,n}_{p}(\lambda)=\overline{\Gamma}^{m,n}_{p}(\lambda)=0 \text{ for } \lambda=-1,-2,-3,\ldots, \quad . \quad . \quad (51)$$

$$\Gamma^{m,n}_{p}(0)=\overline{\Gamma}^{m,n}_{p}(0)=1,$$

$$\Phi^{m,n}_{p,q}(h;\lambda)=\overline{\Phi}^{m,n}_{p,q}(h;\lambda)=0 \text{ for } \lambda=-h,-h-1,-h-2,\ldots, \quad (52)$$

$$\Psi^{m,n}_{p,q}(h;\lambda)=\overline{\Psi}^{m,n}_{p,q}(h;\lambda)=0 \text{ for } \lambda=0,-1,-2,\ldots, \quad . \quad . \quad (53)$$

$$\Theta^{m,n}_{p}(l;r)=\overline{\Theta}^{m,n}_{p}(l;r)=0 \text{ for } r=-1,-2,-3,\ldots \quad . \quad . \quad (54)$$

and

$$\Theta^{m,n}_{p}(l;0)=\overline{\Theta}^{m,n}_{p}(l;0)=1.$$

§ 7. **Lemmas.**

Lemma 1.

$$G^{m,n}_{p,q}(z)=\frac{1}{2\pi i}\left\{e^{\pi i b_{m+1}}\,G^{m+1,n}_{p,q}(ze^{-\pi i})-e^{-\pi i b_{m+1}}\,G^{m+1,n}_{p,q}(ze^{\pi i})\right\}. \quad . \quad (55)$$

$$G^{m,n}_{p,q}(z)=\frac{1}{2\pi i}\left\{e^{\pi i a_{n+1}}\,G^{m,n+1}_{p,q}(ze^{-\pi i})-e^{-\pi i a_{n+1}}\,G^{m,n+1}_{p,q}(ze^{\pi i})\right\}. \quad . \quad (56)$$

Proof. From (2) it follows

$$\frac{1}{2\pi i}\left\{e^{\pi i b_{m+1}}\,G^{m+1,n}_{p,q}(ze^{-\pi i})-e^{-\pi i b_{m+1}}\,G^{m+1,n}_{p,q}(ze^{\pi i})\right\}$$

$$=\frac{1}{2\pi i}\int_C \frac{e^{\pi i(b_{m+1}-s)}-e^{-\pi i(b_{m+1}-s)}}{2\pi i}\;\frac{\prod_{j=1}^{m+1}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+2}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\,z^s\,ds$$

$$=\frac{1}{2\pi i}\int_C \frac{\sin(b_{m+1}-s)\pi}{\pi}\;\frac{\prod_{j=1}^{m+1}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+2}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\,z^s\,ds$$

$$=\frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\,z^s\,ds=G^{m,n}_{p,q}(z),$$

so that formula (55) has been established.

The proof of (56) is similar to that of (55).

Lemma 2. *If k, l, p, q, s and t are integers with*

$$1 \leqq l \leqq s \leqq p \leqq q, \ 1 \leqq t \leqq s-l+1 \text{ and } 0 \leqq k \leqq q,$$

then

$$G_{p,q}^{k,l,s}(z \| a_t) = e^{-2\pi i a_t} G_{p,q}^{k,l,s}(z e^{2\pi i} \| a_t) + 2\pi i e^{-\pi i a_t} G_{p,q}^{k,l-1,s}(z e^{\pi i}) \quad (57)$$

and

$$G_{p,q}^{k,l,s}(z \| a_t) = e^{2\pi i a_t} G_{p,q}^{k,l,s}(z e^{-2\pi i} \| a_t) - 2\pi i e^{\pi i a_t} G_{p,q}^{k,l-1,s}(z e^{-\pi i}). \quad (58)$$

Proof. Since $G_{p,q}^{k,\lambda}(z)$ is a symmetric function of $a_1, \ldots, a_\lambda$ and also of $a_{\lambda+1}, \ldots, a_p$, we have by (56) and the definitions of $G_{p,q}^{k,l,s}$ and $G_{p,q}^{k,l-1,s}$ (see the end of § 1)

$$2\pi i \, G_{p,q}^{k,l-1,s}(z) = e^{\pi i a_t} G_{p,q}^{k,l,s}(z e^{-\pi i} \| a_t) - e^{-\pi i a_t} G_{p,q}^{k,l,s}(z e^{\pi i} \| a_t).$$

From this formula follow (57) and (58) without difficulty.

The most important lemma is

Lemma 3. *Suppose that m, n and q are integers with* $0 \leqq m \leqq q$ *and* $n \geqq 0$; *further that r is an arbitrary integer; finally that the numbers* $a_1, \ldots, a_n$ *satisfy the condition*

$$a_j - a_t \neq 0, \pm 1, \pm 2, \ldots (j=1, \ldots, n; \ t=1, \ldots, n; \ j \neq t). \quad (20)$$

Then the following formula holds:

$$\left. \begin{aligned} &\sum_{t=1}^{n} e^{(m+n-q+2r)\pi i a_t} \triangle^{m,n}_{q}(t) \\ &= -\frac{1}{2\pi i} \{ A^{m,n}_{q} \Omega^{m,n}_{q}(r) - \bar{A}^{m,n}_{q} \bar{\Omega}^{m,n}_{q}(q-m-n-r) \}. \end{aligned} \right\} \quad (59)$$

Proof. Let us suppose that the numbers $\alpha_1, \ldots, \alpha_n$ satisfy the condition

$$\alpha_j^2 \neq \alpha_t^2 \qquad (j=1, \ldots, n; \ t=1, \ldots, n; \ j \neq t).$$

By U_r we denote the coefficient of x^r in the expansion of

$$\frac{\prod_{j=1}^{s} (1-\beta_j^2 x)}{\prod_{j=1}^{n} (1-\alpha_j^2 x)} \qquad (s \geqq 0; \ n \geqq 0)$$

in ascending powers of x. By V_r we denote the coefficient of x^r in the expansion of

$$\frac{\prod_{j=1}^{s} \left(1-\frac{x}{\beta_j^2}\right)}{\prod_{j=1}^{n} \left(1-\frac{x}{\alpha_j^2}\right)}$$

in ascending powers of x.

Finally we write

$$W_t = \frac{\prod_{j=1}^{s}\left(\frac{\beta_j}{\alpha_t} - \frac{\alpha_t}{\beta_j}\right)}{\prod_{\substack{j=1\\ j\neq t}}^{n}\left(\frac{\alpha_j}{\alpha_t} - \frac{\alpha_t}{\alpha_j}\right)}.$$

From the definition of the coefficient U_r it follows that for small values of $|x|$ the following expansion holds

$$\frac{\prod_{j=1}^{s}\left(\beta_j x - \frac{1}{\beta_j}\right)}{\prod_{j=1}^{n}\left(\alpha_j x - \frac{1}{\alpha_j}\right)} = (-1)^{n-s} \frac{\prod_{j=1}^{n} \alpha_j}{\prod_{j=1}^{s} \beta_j} \sum_{r=0}^{\infty} U_r x^r; \quad . \quad . \quad . \quad (60)$$

for large values of $|x|$ however we have according to the definition of the coefficient V_r

$$\frac{\prod_{j=1}^{s}\left(\beta_j x - \frac{1}{\beta_j}\right)}{\prod_{j=1}^{n}\left(\alpha_j x - \frac{1}{\alpha_j}\right)} = \frac{\prod_{j=1}^{s} \beta_j}{\prod_{j=1}^{n} \alpha_j} \sum_{h=0}^{\infty} V_h x^{s-n-h}. \quad . \quad . \quad . \quad . \quad (61)$$

The coefficients U_r and V_r satisfy further the relations

$$U_r = V_r = 0 \text{ for } r = -1, -2, -3, \ldots.$$

We now consider the expansion of the left-hand side of (61) in partial fractions. This expansion runs as follows [27])

$$\frac{\prod_{j=1}^{s}\left(\beta_j x - \frac{1}{\beta_j}\right)}{\prod_{j=1}^{n}\left(\alpha_j x - \frac{1}{\alpha_j}\right)} - \frac{\prod_{j=1}^{s} \beta_j}{\prod_{j=1}^{n} \alpha_j} \sum_{h=0}^{s-n} V_h x^{s-n-h} = \sum_{t=1}^{n} \frac{\alpha_t^{n-s-1} W_t}{\alpha_t x - \frac{1}{\alpha_t}}. \quad . \quad . \quad (62)$$

Now the coefficient of x^r in the expansion of the right-hand side of (62) in ascending powers of x is $-\sum_{t=1}^{n} \alpha_t^{n-s+2r} W_t$.

If $r \geqq 0$, we find therefore on account of (62) and (60)

$$\sum_{t=1}^{n} \alpha_t^{n-s+2r} W_t = (-1)^{n-s+1} \frac{\prod_{j=1}^{n} \alpha_j}{\prod_{j=1}^{s} \beta_j} U_r + \frac{\prod_{j=1}^{s} \beta_j}{\prod_{j=1}^{n} \alpha_j} V_{s-n-r}, \quad . \quad (63)$$

this relation still being true when $r > s - n$, since $V_h = 0$ for $h = -1, -2, -3, \ldots$.

We may yet show that (63) also holds when $r < 0$.

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[27]) Since $V_h = 0$ for $h < 0$, formula (62) also holds when $n > s$ (comp. definition 4).

For we have by (61)

$$\frac{\prod_{j=1}^{s}\left(\beta_j x-\frac{1}{\beta_j}\right)}{\prod_{j=1}^{n}\left(\alpha_j x-\frac{1}{\alpha_j}\right)}-\frac{\prod_{j=1}^{s}\beta_j}{\prod_{j=1}^{n}\alpha_j}\sum_{h=0}^{s-n}V_h x^{s-n-h}=\frac{\prod_{j=1}^{s}\beta_j}{\prod_{j=1}^{n}\alpha_j}\sum_{h=s-n+1}^{\infty}V_h x^{s-n-h}.$$

From this formula and (62) it follows

$$\sum_{t=1}^{n}\frac{\alpha_t^{n-s-1}W_t}{\alpha_t x-\frac{1}{\alpha_t}}=\frac{\prod_{j=1}^{s}\beta_j}{\prod_{j=1}^{n}\alpha_j}\sum_{h=s-n+1}^{\infty}V_h x^{s-n-h}.$$

Since the coefficient of $x^{-\mu}$ in the expansion of the left-hand side of this relation is equal to $\sum_{t=1}^{n}\alpha_t^{n-s-2\mu}W_t$, we get for $\mu>0$

$$\sum_{t=1}^{n}\alpha_t^{n-s-2\mu}W_t=\frac{\prod_{j=1}^{s}\beta_j}{\prod_{j=1}^{n}\alpha_j}V_{s-n+\mu}.$$

If herein μ is replaced by $-r$, we find (63) (since $U_r=0$ for $r=-1, -2, -3, \ldots$) so that (63) is true for any integer.

We now put $s=q-m$, $\alpha_1=e^{\pi i a_1}, \ldots, \alpha_n=e^{\pi i a_n}$, $\beta_1=e^{\pi i b_{m+1}}, \ldots, \beta_s=e^{\pi i b_q}$. Formula (63) reduces then to (59), which proves the lemma.

Lemma 4. *Suppose that m, n, q and λ are integers with*

$$0\leqq m\leqq q,\ n\geqq 1 \qquad (64)$$

and

$$0\leqq\lambda\leqq m+n-q-1; \qquad (65)$$

suppose further that the numbers $a_1, \ldots, a_{n+1}$ satisfy the condition

$$a_j-a_t\neq 0, \pm 1, \pm 2, \ldots\ (j=1, \ldots, n+1;\ t=1, \ldots, n;\ j\neq t).$$

Then the following formula holds:

$$\left.\begin{aligned}&\sum_{t=1}^{n}e^{(m+n-q-2\lambda-1)\pi i a_t}\Delta_q^{m,n+1}(t)\\&=-\pi^{m+n-q}e^{(m+n-q-2\lambda-1)\pi i a_{n+1}}\frac{\prod_{j=m+1}^{q}\sin(b_j-a_{n+1})\pi}{\prod_{j=1}^{n}\sin(a_j-a_{n+1})\pi}.\end{aligned}\right\} \qquad (66)$$

P r o o f. If λ satisfies (65), it follows from (59) with $r=-\lambda-1$ and $n+1$ instead of n, on account of (49)

$$\sum_{t=1}^{n+1}e^{(m+n-q-2\lambda-1)\pi i a_t}\Delta_q^{m,n+1}(t)=0.$$

Hence we have in view of (16)

$$\sum_{t=1}^{n} e^{(m+n-q-2\lambda-1)\pi i a_t} \triangle^{m,\,n+1}_{q}(t) = -e^{(m+n-q-2\lambda-1)\pi i a_{n+1}} \triangle^{m,\,n+1}_{q}(n+1)$$

$$= -\pi^{m+n-q} e^{(m+n-q-2\lambda-1)\pi i a_{n+1}} \frac{\prod_{j=m+1}^{q} \sin(b_j - a_{n+1})\pi}{\prod_{j=1}^{n} \sin(a_j - a_{n+1})\pi},$$

so that the lemma has been proved.

Lemma 5. *Suppose that m, n, q and λ are integers which satisfy the conditions (64) and (65); suppose further that the numbers $a_1, \ldots, a_n$ fulfil the condition (20) and satisfy besides the inequality*

$$a_t - \beta \neq 0, \pm 1, \pm 2, \ldots \quad (t = 1, \ldots, n).$$

Then the following formula holds:

$$\left.\begin{aligned} &\sum_{t=1}^{n} e^{(m+n-q-2\lambda-1)\pi i a_t} \frac{\triangle^{m,\,n}_{q}(t)}{\sin(\beta - a_t)\pi} \\ &= -\pi^{m+n-q-1} e^{(m+n-q-2\lambda-1)\pi i \beta} \frac{\prod_{j=m+1}^{q} \sin(b_j - \beta)\pi}{\prod_{j=1}^{n} \sin(a_j - \beta)\pi} \end{aligned}\right\} \quad . \quad . \quad (67)$$

P r o o f. From the definition of the coefficient $\triangle$ it follows that $\triangle^{m,\,n+1}_{q}(t)$ with $1 \leqq t \leqq n$ and $a_{n+1} = \beta$ is equal to

$$\frac{\pi \triangle^{m,\,n}_{q}(t)}{\sin(\beta - a_t)\pi}.$$

Formula (67) is therefore a particular case of (66).

Lemma 6. *Suppose that h, m, n, q and λ are integers with*

$$1 \leqq m + 1 \leqq h \leqq q \text{ and } 0 \leqq \lambda \leqq m + n - q - 1;$$

suppose further that the numbers $a_1, \ldots, a_n$ fulfil the condition (20) and satisfy besides the inequality

$$a_t - b_h \neq 0, \pm 1, \pm 2, \ldots \quad (t = 1, \ldots, n;\ m + 1 \leqq h \leqq q).$$

Then the following formula holds:

$$\sum_{t=1}^{n} e^{(m+n-q-2\lambda-1)\pi i a_t} \frac{\triangle^{m,\,n}_{q}(t)}{\sin(b_h - a_t)\pi} = 0. \quad . \quad . \quad . \quad . \quad (68)$$

P r o o f. Formula (68) is a particular case of (67), since the right-hand side of (67) vanishes for $\beta = b_h$ $(m + 1 \leqq h \leqq q)$.

Lemma 7. *Suppose that m, n and p are integers with $0 \leqq n \leqq p$ and $m \geqq 0$; further that λ is an arbitrary integer; finally that the numbers $b_1, \ldots, b_m$ satisfy the condition*

$$b_j - b_s \neq 0, \pm 1, \pm 2, \ldots \quad (j = 1, \ldots, m;\ s = 1, \ldots, m;\ j \neq s).$$

Then the following formula holds:

$$\left.\begin{aligned} & B_p^{m,n}\,\Gamma_p^{m,n}(\lambda)-\overline{B}_p^{m,n}\,\overline{\Gamma}_p^{m,n}(p-m-n-\lambda) \\ & =-2i\pi^{m+n-p}\sum_{s=1}^{m} e^{(m+n-p+2\lambda)\pi i b_s}\,\frac{\prod\limits_{j=n+1}^{p}\sin(a_j-b_s)\pi}{\prod\limits_{\substack{j=1\\ j\neq s}}^{m}\sin(b_j-b_s)\pi} \end{aligned}\right\} \quad . \quad . \qquad (69)$$

P r o o f. If we replace m, n, q, r, a_j and b_j successively by n, m, p, λ, b_j and a_j, the coefficients $A^{m,n}_{q}$, $\overline{A}^{m,n}_{q}$, $\Omega^{m,n}_{q}(r)$ and $\overline{\Omega}^{m,n}_{q}(r)$ transform successively into $B_p^{m,n}$, $\overline{B}_p^{m,n}$, $\Gamma_p^{m,n}(\lambda)$ and $\overline{\Gamma}_p^{m,n}(\lambda)$; on account of (16) formula (59) reduces then to (69).