Physics. — *Note on the theory of vector wave fields.* By L. J. F. BROER (ZEEMAN-laboratory of the University of Amsterdam) and A. PAIS (Instituut voor Theoretische Natuurkunde Rijks-Universiteit Utrecht). (Communicated by Prof. J. D. VAN DER WAALS JR.)

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Summary.

It is remarked that FERMI's method of treating the electromagnetic field also can be applied to the classical, but not to the quantum theory of the vector meson field. The rôle of the LORENTZ condition is examined. The connection between the electromagnetic and the (neutral) vector meson field is discussed.

§ 1. *Introduction.* Whereas, from a classical point of view, the electromagnetic field equations may be considered as the equations for a neutral vector meson field with vanishing meson rest mass, it is well known, however, that the quantized MAXWELL equations cannot be obtained by letting

$$\varkappa = \frac{mc}{\hbar} \to 0$$

in the PROCA [1]) equations *). This "lack of continuity" can be traced back to the following two causes:

a. In the customary treatment of the MAXWELL as well as of the PROCA field, the canonically conjugate of the scalar potential, (i.e. the time component of the four-vector potential) vanishes. However, in the latter case the scalar potential can be treated as "derived variable" [2]) i.e., by expressing it in terms of (spatial derivatives of) canonical variables, it can be eliminated from the Hamiltonian of the field; this is impossible in the electromagnetic case.

b. The different rôle of the so-called LORENTZ condition in both cases: This condition is introduced in the MAXWELL case for the sake of convenience, it having no bearing on the field strengths and field energy. In dealing with the vector meson field it is, however, again in the customary treatment, a direct consequence of the field equations themselves.

As to the first point, it has been shown by FERMI [3]) that the canonically conjugate of the scalar potential in the MAXWELL case need not be zero, provided one starts from a suitable Lagrangian of the field. In this note

*) In this note we shall understand by PROCA equations the field equations for a *real* vector meson field.

FERMI's method [4]) will be applied to the PROCA field from which it will be shown (§ 2):

a'. That neither in the MAXWELL nor in the PROCA case the canonically conjugate of the scalar potential vanishes.

b'. That in both cases the LORENTZ condition can be treated as an accessory condition.

Still it will appear that it is even now impossible to treat the quantized MAXWELL equations as a special case of those of PROCA by taking the limit for $\varkappa = 0$. The reason for this will be seen to be the different consequences which the occurrence of the LORENTZ condition has in the *quantum* theory of the corresponding wave fields. In fact, it turns out (§ 4) that the limiting transition $\varkappa \to 0$ is singular in the same way as a LORENTZ transformation for $v/c \to 1$.

The possibility of the treatment of the LORENTZ condition as an accessory one being given, one might ask whether it is necessary to introduce it at all. If one would do without it, however, the field energy appears not to be positive definite. This has already been stated by FIERZ [5]) who has, moreover, pointed out that as a consequence of the LORENTZ condition only spin 1 particles occur upon field quantization, while a theory without this condition would also yield particles with spin zero. As it is, furthermore, just these latter which make the energy not positive definite, it is clear that one cannot do without the LORENTZ condition, whatever its way of occurrence in the theory may be.

The present paper is mainly of a methodical character. Though some points raised here cannot be claimed to be new, the lack of a general survey of the problems discussed here let it seem justified to include them in this note.

§ 2. *Variation principles.* The following notations are used:

$$a_{[\lambda\mu]} \equiv a_{\lambda\mu} - a_{\mu\lambda},\ \partial_\mu = \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t},\ (c = 1).$$

Greek indices run from 1 to 4; summation signs are suppressed. Co- and contravariant tensor components are connected in the usual way.

The MAXWELL equations can be written as:

$$F_{\lambda\mu} = f_{[\lambda\mu]};\ f_{\lambda\mu} \equiv \partial_\lambda A_\mu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

$$\partial_\mu F^{\mu\nu} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

where A_λ is the four-vector potential. Using the same notation for the PROCA field, we have:

$$F_{\lambda\mu} = f_{[\lambda\mu]};\ f_{\lambda\mu} \equiv \partial_\lambda A_\mu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1b)$$

$$\partial_\mu F^{\mu\nu} = \varkappa^2 A^\nu. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2b)$$

Elimination of the field tensors $F_{\mu\nu}$ yields the second order differential equations:

$$\Box A^\nu - \partial^\nu \partial_\mu A^\mu = 0, \quad \Box \equiv \partial_\lambda \partial^\lambda, \text{ MAXWELL} \quad . \quad . \quad . \quad (3a)$$

$$\Box A^\nu - \partial^\nu \partial_\mu A^\mu = \varkappa^2 A^\nu, \text{ PROCA.} \quad . \quad . \quad . \quad . \quad . \quad (3b)$$

If we next introduce, in the MAXWELL case, the accessory condition

$$\partial_\mu A^\mu = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which is known as the LORENTZ condition, (3a) becomes:

$$\Box A^\nu = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5a)$$

In the PROCA case, on the other hand, (4) is a consequence of (2b) on account of the antisymmetry of $F_{\mu\nu}$. Thus (3b) becomes:

$$\Box A^\nu = \varkappa^2 A^\nu. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5b)$$

Considering (1a, b) as defining $F_{\mu\nu}$, the equations (2a, b) can be derived from the variation principle:

$$\delta \int \mathcal{L}\, dv\, dt = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

with

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\varkappa^2}{2} A_\mu A^\mu; \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

in the MAXWELL case $\varkappa = 0$ of course. The quantization of the fields starting from (6) and (7) is well known and needs no recapitulation here. We only remind that the canonical conjugate of A_4 vanishes as $\partial_4 A_4$ does not occur in (7). This leads to some complications in the MAXWELL but not in the PROCA case as A_4 can there be expressed in terms of the canonical conjugates of A_1, A_2 and A_3 with the aid of the fourth equation (2b).

FERMI's method now consists in starting from (6) but taking *)

$$\mathcal{L} = -\tfrac{1}{2} f_{\mu\nu} f^{\mu\nu} - \frac{\varkappa^2}{2} A_\mu A^\mu, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where $f_{\mu\nu}$ is defined by (1a, b). It is easily seen that the field equations directly derived from (8) are (5a) and (5b) respectively. Thus if we start from (8), it is, at this stage, not necessary to use (4) to establish these field equations. In point of fact, (4) cannot now be derived from the field equations and the definition of $f_{\mu\nu}$, because the latter is not antisymmetrical; and this is true whether $\varkappa$ is or is not equal to zero. It may be noted that instead of (4) we now get

$$(\Box - \varkappa^2) S = 0; \quad S \equiv \partial_\nu A^\nu. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4')$$

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*) An objection of principle which may be raised against the use of (8) with $\varkappa = 0$ is that the Lagrangian is not gauge invariant. (If $\varkappa \neq 0$ there is no gauge group of course.)

However, as will appear below, we do need (4) in the MAXWELL as well as in the PROCA case in order to obtain a positive definite field energy. But, if we use (8), (4) should now be considered as an accessory condition in both cases. That, using (8) in stead of (7), the accessory condition is necessary from a physical point of view is connected with the circumstance that $f_{\mu\nu}$ is not antisymmetrical here. This ensues that a choice must be made from the solutions of (5) so as to get the correct value of the energy. This choice will appear to be (4).

§ 3. FOURIER *development; energy of the field.* Assuming the system to be enclosed in a cube of volume 1 with periodicity conditions on its boundary, and introducing the vector notation

$$A_\mu = \vec{A}, -B,$$

$\vec{A}$ and B can be developed in a FOURIER series in the following way:

$$\left.\begin{aligned} \vec{A} &= \sum_{j=0,1,2} \sum_{\vec{k}} \vec{e}_{j\vec{k}} \left[A(j,\vec{k})\, e^{i(\vec{k}\vec{x})} + A^\dagger(j,\vec{k})\, e^{-i(\vec{k}\vec{x})}\right], \\ B &= \sum_{\vec{k}} \left[B(\vec{k})\, e^{i(\vec{k}\vec{x})} + B^\dagger(k)\, e^{-i(\vec{k}\vec{x})}\right], \\ \vec{e}_{0\vec{k}} &= \frac{\vec{k}}{|\vec{k}|}, \quad \vec{e}_{j\vec{k}}\, \vec{e}_{j'\vec{k}} = \delta_{jj'}, \end{aligned}\right\} \quad . \quad . \quad (9)$$

i.e. $j = 0$ denotes longitudinal, $j = 1,2$ transverse waves. Substituting (9) in (8) we get *):

$$\left.\begin{aligned} L = \int \mathcal{L}\, dv = \sum_{j=0,1,2} \sum_{\vec{k}} &\left[\dot{A}^\dagger(j,\vec{k})\, \dot{A}(j,\vec{k}) - \bar{\nu}_k^2\, A^\dagger(j,\vec{k})\, A(j,\vec{k})\right] \\ &- \sum_{\vec{k}} \left[\dot{B}^\dagger(\vec{k})\, B(\vec{k}) - \bar{\nu}_k^2\, B^\dagger(\vec{k})\, B(\vec{k})\right], \\ \bar{\nu}_k^2 = \nu_k^2 + \varkappa^2, \quad \nu_k &= |\vec{k}|. \end{aligned}\right\} \quad . \quad . \quad (10)$$

In the usual way we obtain from (10) the equations of motion for the amplitudes:

$$\ddot{A}(j,\vec{k}) + \bar{\nu}_k^2\, A(j,\vec{k}) = 0, \quad \ddot{B}(\vec{k}) + \bar{\nu}_k^2\, \vec{B}(k) = 0 \quad . \quad . \quad . \quad (11)$$

(11) can also be obtained by substituting (9) in (5a, b); if $\varkappa = 0$, $\bar{\nu}_k = \nu_k$.

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*) In expressions like (10) the right member should be hermitized. For simplicity we write (10) as it stands.

The theory is therefore consistent so far. But difficulties arise because the Hamiltonian derived from (10):

$$H = \sum_j \sum_{\vec{k}} [\dot{A}^\dagger (j, \vec{k})\, \dot{A}\, (j, \vec{k}) + \bar{\nu}_k^2\, A^\dagger (j, \vec{k})\, A\, (j, \vec{k}) - \\ - \sum_k [\dot{B}^\dagger (\vec{k})\, \dot{B}\, (\vec{k}) + \bar{\nu}_k^2\, B^\dagger (\vec{k})\, B\, (\vec{k})], \quad . \quad . \quad (12)$$

is not positive definite. And it is here that (4) plays an essential rôle. In amplitudinal form, (4) can be written as

$$\dot{B}\, (\vec{k}) = - i\, \nu_k\, A\, (0, \vec{k}), \quad \dot{B}^\dagger (\vec{k}) = i\, \nu_k\, A^\dagger (0, \vec{k}).$$

This can with the aid of the equations of motion be brought in the form:

$$\bar{\nu}_k\, B\, (\vec{k}) = \nu_k\, A\, (0, \vec{k}), \quad \bar{\nu}_k\, B^\dagger (\vec{k}) = \nu_k\, A^\dagger (0, \vec{k}). \quad . \quad . \quad . \quad (13)$$

Now there are two ways of introducing an accessory condition in classical theory which we shall discuss successively with regard to their bearing on our problem.

The first method, which is not applicable generally, consists of looking for solutions of the restricted problem (i.e. the problem involving the condition) under those of the general problem (not involving the condition). In our case this means that we have to look for solutions of (5*b*) which satisfy (4). $S \equiv 0$ obviously is a solution of (4') *). It is characterized by the boundary conditions: three dimensional periodicity, S and $\dot{S} = 0$ for $t = t_0$. As these conditions are fulfilled for a class of solutions of (5*b*), the first method is valid here. This result is independent of the value of $\varkappa$. Thus, both if $\varkappa = 0$ and $\varkappa \neq 0$ the LORENTZ condition $S \equiv 0$ can be replaced by $S = \dot{S} = 0$ at a given time, in vacuum as well as with a charge current density.

The second method, closely related to LAGRANGE's parameter method, simply consists in substituting (13) into (12) which yields the positive definite form:

$$H = \sum_{j=1,2} \sum_{\vec{k}} [\dot{A}^\dagger (j, \vec{k})\, \dot{A}\, (j, \vec{k}) + \bar{\nu}_k^2\, A^\dagger (j, \vec{k})\, A\, (j, \vec{k})] + \\ + \sum_{\vec{k}} \frac{\varkappa^2}{\bar{\nu}_k^2} [\dot{A}^\dagger (0, \vec{k})\, \dot{A}\, (0, \vec{k}) + \bar{\nu}_k^2\, A^\dagger (0, \vec{k})\, A\, (0, \vec{k})] \quad . \quad (14)$$

which holds for any $\varkappa$; thus the second method too applies to both cases. An equivalent, but more direct, way to obtain (14) is to start from the Lagrangian (7) instead of (8). If $\varkappa = 0$ we only get a contribution of the transverse waves, as it should be.

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*) It is interesting, though not essential for the present purpose, to note that (4') remains valid when the charge current density is introduced on the right hand side of (5*b*).

§ 4. *Quantization of the field.* According to (10), the canonical conjugate of $A(j, \vec{k})$ is $\dot{A}^\dagger(j, \vec{k})$, that of $B(\vec{k})$ is $-\dot{B}^\dagger(\vec{k})$, etc. In order to quantize the field we have therefore to postulate the relations:

$$\left.\begin{aligned} [\dot{A}^\dagger(j, \vec{k}), A(j', \vec{k}')] &= -i\hbar\,\delta_{jj'}\,\delta_{\vec{k}\,\vec{k}'} \\ [\dot{B}^\dagger(\vec{k}), B(\vec{k}')] &= i\hbar\,\delta_{\vec{k}\,\vec{k}'} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (15)$$

all other pairs commuting. In the usual way we find from (15) that the Hamiltonian takes the form, (the notation will be obvious):

$$H = \sum_{\vec{k}} \hbar\,\nu_k\,[N_A(1, \vec{k}) + N_A(2, \vec{k}) + N_A(0, \vec{k}) - N_B(\vec{k})] \quad . \quad . \quad (16)$$

where the N's are diagonal matrices with the characteristic values 0, 1, 2, ... As was to be expected, H is not positive definite in this case either. We must therefore introduce the LORENTZ condition here too. We thus have to find the quantum mechanical analogues of the two methods of the previous section for an accessory condition $X = 0$ in a quantum mechanical problem characterized by a Hamiltonian H.

First method: Although the most straightforward way would be to interprete $X = 0$ as an operator identity and then to proceed in a similar way as in the corresponding classical treatment, this method is readily seen not to be applicable to the problem on hand, as (13), interpreted in this way, is incompatible with the commutation relations. Thus we shall have to find another treatment. FERMI's idea now is to solve the unrestricted problem

$$H\psi = \frac{i}{\hbar}\,\dot{\psi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and then to look for functionals ψ for which $X\psi = 0$. In general, however, such ψ's do not exist.

Second method: this simply consists in starting from the Lagrangian (7) and thus from the Hamiltonian (14) instead of using (8) and (12) *).

We shall now discuss the applicability of both these methods, first in the MAXWELL, then in the PROCA case.

$\varkappa = 0$. It is well known that here the first method can be used [3]). There appear to exist occupation number functionals ψ, solutions of (17), where H is taken from (12), which vanish when operated on by $B(\vec{k}) - A(0, \vec{k})$ or $B^\dagger(\vec{k}) - A^\dagger(0, \vec{k})$. For these functionals we will have $N_A(0, \vec{k})\,\Psi = N_B(\vec{k})\,\Psi$. The energy (16) reduces therefore to the positive definite form:

$$H = \sum_{\vec{k}} h\,\nu_k\,[N_A(1, \vec{k}) + N_A(2, \vec{k})]. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

*) A less direct way of formulating this method is: start from the *classical* Hamiltonian (12), eliminate with the help of the *classical* condition $X = 0$, (i.e. (13)) one of the variables and then solve the ensuing restricted quantum mechanical problem.

Let us now consider the second method. Thus we have to start from (14), (with $\varkappa = 0$ of course). Quantization of this reduced problem leads at once to (18). This treatment consists essentially in gauging the fields in such a way that the longitudinal and scalar part of the four-vector potential are eliminated before performing the quantization.

$\varkappa \neq 0$. Using the first method, and interpreting (13) accordingly, we would get

$$\bar{\nu}_k^2 N_B(\vec{k}) \Psi = \nu_k^2 N_A(0, \vec{k}) \Psi,$$

which is nonsense, as the N's can only be whole numbers. This means that, in the FERMI-interpretation, the two relations (13) are *mutually* incompatible.

Thus the first method has to be discarded (although it can be seen to give a positive definite H), and we must resort to the second one, i.e. we must start with (14). The canonical conjugates of $A(1, \vec{k})$ etc. are $\dot{A}\dagger(1, \vec{k})$, that of $A(0, \vec{k})$ is $\frac{\varkappa^2}{\bar{\nu}_k^2} \cdot \dot{A}^\dagger(0, \vec{k})$. The commutation relations therefore can be written as:

$$[\dot{A}^\dagger(j, \vec{k}), A(j', \vec{k}')] = -i\hbar \delta_{jj'} \delta_{\vec{k}\vec{k}'}, \quad j = 1, 2 \quad . \quad . \quad (19a)$$

$$[\dot{A}^\dagger(0, \vec{k}), A(0, \vec{k}')] = -i\hbar \nu \delta_{\vec{k}\vec{k}'} \cdot \frac{\bar{\nu}_k^2}{\varkappa^2} \quad . \quad . \quad . \quad . \quad . \quad (19b)$$

This yields:

$$H = \sum_{\vec{k}} h \bar{\nu}_k [N_A(1, \vec{k}) + N_A(2, \vec{k}) + N_A(0, \vec{k})],$$

which is the desired result.

The relations (19) can be put in a more symmetrical form by performing the following transformation to the new variables $C(\vec{k})$ and $A(3, \vec{k})$:

$$\left.\begin{aligned} \varkappa A(0, \vec{k}) &= \bar{\nu}_k C(\vec{k}) + \nu_k A(3, \vec{k}) \\ \varkappa B(\vec{k}) &= \nu_k C(\vec{k}) + \bar{\nu}_k A(3, \vec{k}). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$A(3, \vec{k})$ and $C(\vec{k})$ are the amplitudes of the plane waves representing free particles with spin 1, the amplitude being directed along the direction of momentum, and spin 0 respectively [5]). The LORENTZ condition in the new variables is simply:

$$C(\vec{k}) = 0, \quad C^\dagger(\vec{k}) = 0.$$

The Hamiltonian becomes:

$$H = \sum_{j=1}^{j=3} \sum_{\vec{k}} [\dot{A}^\dagger(j, \vec{k}) \dot{A}(j, \vec{k}) + \bar{\nu}_k^2 A^\dagger(j, \vec{k}) A(j, \vec{k})].$$

The commutation relations are now:

$$[\dot{A}^{\dagger}(j, \vec{k}), A(j', \vec{k'})] = -i\hbar\,\delta_{jj'}\,\delta_{kk'}, \quad j = 1, 2, 3.$$

and the diagonalized Hamiltonian is:

$$H = \sum_{\vec{k}} h\bar{\nu}_k\,[N_A(1, \vec{k}) + N_A(2, \vec{k}) + N_A(3, \vec{k})].$$

Though we apparently can treat the quantization of the MAXWELL and PROCA case in the same manner, viz. by using the second method, yet the former cannot be derived from the latter by letting $\varkappa$ tend to zero. The reason for this is that the relation (19*b*) would become meaningless or, in other words, that the transformation (20), which is essentially a LORENTZ transformation with $\beta = \nu_k/\bar{\nu}_k$, would become singular.

Comparing, finally, the first method with regard to its applicability to the classical PROCA field on the one, to the same quantized field on the other hand, we remark: the condition $C(\vec{k}) = C^{\dagger}(\vec{k}) = 0$ classically means that we have to put equal to zero an infinite number of oscillator amplitudes, viz. those referring to spinless mesons, for all $\vec{k}$. This, of course, is practicable. But such a procedure is impossible in the quantum mechanical case on account of the zero point vibrations; this is another way of expressing why the first method applies the classical, but not to the quantum PROCA case.

In the MAXWELL case, the transformation (20) loses its sense, as explained above. The LORENTZ condition now amounts to the equality of *two* oscillator amplitudes, (for given $\vec{k}$; cf. (13) with $\bar{\nu}_k = \nu_k$), and this is indeed, also quantum mechanically, possible. Thus we see from (20) how it can be that the first method applies to the quantum mechanical MAXWELL but not to the quantum mechanical PROCA case.

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