Mathematics. — *On the absolute convergence of Fourier series.* By A. C. ZAANEN. (Communicated by Prof. W. VAN DER WOUDE.)

S. BERNSTEIN [1]) has proved that if the real, periodic function $f(x)$ (period 2π) satisfies a Lipschitz-condition of order a, where $a > \frac{1}{2}$, then the Fourier series of $f(x)$ converges absolutely. For $a = \frac{1}{2}$ this is no longer true. O. SZÁSZ [2]) generalized this theorem by considering the series

$$\sum_{n=1}^{\infty} (|a_n|^\beta + |b_n|^\beta), \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where a_n and b_n are the Fourier coefficients of $f(x)$, and showing that if $f(x)$ satisfies a Lipschitz-condition of order $a (0 < a \leq 1)$, the series (1) converges for every $\beta > 2/(2a + 1)$, but not necessarily for $\beta = 2/(2a + 1)$. Recently L. NEDER [3]) made the following addition to BERNSTEIN's Theorem: If, for $h > 0$,

$$l_1(h) = \log(e + h^{-1}),$$
$$l_2(h) = \log \log(e^e + h^{-1}),$$
$$\text{etc.,}$$

and if, for a certain $\varepsilon > 0$,

$$|f(x + h) - f(x)| \leq \frac{C\, h^{1/2}}{l_1(h)\, l_2(h) \ldots l_k(h)^{1+\varepsilon}},$$

then the Fourier series of $f(x)$ converges absolutely.

We shall prove a similar addition to SZÁSZ's Theorem.

Theorem 1. *If* $0 < a \leq 1$, $\varepsilon > 0$, $h > 0$, *and*

$$|f(x + h) - f(x)| \leq \frac{C\, h^a}{[l_1(h)\, l_2(h) \ldots l_k(h)^{1+\varepsilon}]^{\frac{2a+1}{2}}}, \quad \ldots \quad \ldots \quad (2)$$

then the series (1) *converges also for* $\beta = 2/(2a + 1)$.

Proof. We shall give the proof for $k = 2$. If

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

[1]) S. BERNSTEIN. Sur la convergence absolue des séries trigonométriques, C. R. de l'Acad de Sc. de Paris **158** (1914), 1661—1664.

[2]) O. SZÁSZ, Über den Konvergenzexponent der Fourierschen Reihen. Münchener Sitzungsberichte (1922), 135—150.

[3]) L. NEDER, Ein Satz über die absolute Konvergenz der Fourier-Reihe, Math. Zeitschrift **49** (1944), 644—646.

then

$$f(x+h) - f(x-h) \sim 2 \sum_{n=1}^{\infty} b_n(x) \sin nh,$$

where

$$b_n(x) = b_n \cos nx - a_n \sin nx,$$

so that

$$\frac{1}{\pi} \int_0^{2\pi} [f(x+h) - f(x-h)]^2 \, dx = 4 \sum_{n=1}^{\infty} \varrho_n^2 \sin^2 nh,$$

where

$$\varrho_n^2 = a_n^2 + b_n^2.$$

From (2) it follows now that

$$\frac{1}{\pi} \int_0^{2\pi} [f(x+h) - f(x-h)]^2 \, dx \leqslant \frac{C_1 h^{2\alpha}}{[l_1(2h) \, l_2(2h)^{1+\varepsilon}]^{2\alpha+1}},$$

so that, taking $h = \pi/2N$,

$$\sum_{n=1}^{N} \varrho_n^2 \sin^2 \frac{\pi n}{2N} \leqslant \frac{C_2 N^{-2\alpha}}{\left[l_1\left(\frac{\pi}{N}\right) l_2\left(\frac{\pi}{N}\right)^{1+\varepsilon}\right]^{2\alpha+1}}.$$

Let now $N = 2^\nu$, where ν is an integer, $\geqslant \nu_0 \geqslant (\log 2)^{-2} + 3$. Then

$$l_1\left(\frac{\pi}{N}\right) = \log\left(e + \frac{2^\nu}{\pi}\right) > \log 2^{\nu-2} = (\nu-2) \log 2,$$

$$l_2\left(\frac{\pi}{N}\right) = \log\log\left(e^e + \frac{2^\nu}{\pi}\right) > \log\log 2^{\nu-2} = \log(\nu-2) + \log\log 2 > \tfrac{1}{2}\log(\nu-2),$$

hence

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \varrho_n^2 \leqslant \frac{C_3 \, 2^{-2\nu\alpha}}{[(\nu-2) \log^{1+\varepsilon}(\nu-2)]^{2\alpha+1}},$$

and from this, by HÖLDER's inequality

$$\sum \varrho_n^\beta \leqslant (\sum \varrho_n^2)^{\beta/2} (\sum 1)^{1-\beta/2}$$

with $\beta = 2/(2\alpha+1)$, so that $1 - \beta/2 = 2\alpha/(2\alpha+1)$,

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \varrho_n^\beta \leqslant \frac{C_4 \, 2^{-\frac{2\nu\alpha}{2\alpha+1}}}{(\nu-2) \log^{1+\varepsilon}(\nu-2)} \cdot 2^{\frac{2\nu\alpha}{2\alpha+1}} = \frac{C_4}{(\nu-2) \log^{1+\varepsilon}(\nu-2)}.$$

This shows finally that

$$\sum_{n=2^{\nu_0-1}+1}^{\infty} \varrho_n^\beta = \sum_{\nu=\nu_0}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^\nu} \varrho_n^\beta \leq C_4 \sum_{\nu=\nu_0}^{\infty} \frac{1}{(\nu-2)\log^{1+\varepsilon}(\nu-2)} < \infty.$$

The series $\Sigma \varrho_n^\beta$ converges therefore, and, since $|a_n|^\beta$, and also $|b_n|^\beta$, does not exceed ϱ_n^β, the same is true of the series

$$\sum_{n=1}^{\infty}(|a_n|^\beta + |b_n|^\beta).$$

It may be asked whether there exist functions $f(x)$, satisfying a condition of the form (2), without belonging to a Lipschitz-class of order $a' > a$. We shall show this to be the case.

Theorem 2. *If*

$$0 < a < 1, \quad \varepsilon > 0, \quad \gamma = \frac{2a+1}{2}, \quad \delta = 1 + \gamma(1+\varepsilon),$$

the function

$$f(x) = \sum_{n=1}^{\infty} \frac{e^{i n \log n}}{n^\gamma(\log n)^\delta} e^{inx}$$

satisfies the condition

$$|f(x+h) - f(x)| \leq \frac{C h^a}{[l_1(h)^{1+\varepsilon}]^\gamma},$$

while, for $a' > a$, $f(x)$ does not belong to the Lipschitz-class of order a'. The real and imaginary components of $f(x)$ may therefore serve as illustrations to Theorem 1.

Proof. Writing

$$s_\nu(x) = \sum_{n=1}^{\nu} e^{i n \log n} e^{inx},$$

it may be proved [4] that

$$s_\nu(x) = O(\nu^{1/2}) \quad \text{and} \quad s_\nu'(x) = \frac{d s_\nu(x)}{d x} = O(\nu^{3/2}),$$

uniformly in x. Furthermore

$$\int_N^\infty \frac{dx}{x^{a+1}\log^\delta x} \leq \frac{1}{N^a}\int_N^\infty \frac{dx}{x\log^\delta x} = \frac{1}{N^a}\int_{\log N}^\infty \frac{dy}{y^\delta} = O\left(\frac{N^{-a}}{\log^{\delta-1} N}\right). \quad (3)$$

and

$$\int_2^N \frac{dx}{x^a\log^\delta x} = \int_2^N \frac{x^{1-a}\,dx}{x\log^\delta x} \leq N^{1-a}\int_{\log 2}^{\log N} \frac{dy}{y^\delta} = O\left(\frac{N^{1-a}}{\log^{\delta-1} N}\right). \quad (4)$$

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[4]   See e.g. A. ZYGMUND, Trigonometrical Series, Warsaw (1935), 5. 32.

Introducing the difference $\triangle g(\nu) = g(\nu) - g(\nu + 1)$, we find now by ABEL's transformation

$$f(x+h) - f(x) = \sum_{\nu=1}^{\infty} \{s_\nu(x+h) - s_\nu(x)\} \triangle \nu^{-\gamma} \log^{-\delta} \nu = \sum_{\nu=1}^{N} + \sum_{\nu=N+1}^{\infty} = P + Q,$$

where $N = \left[\dfrac{1}{h}\right]$. Since, by the mean-value theorem,

$$\triangle \nu^{-\gamma} \log^{-\delta} \nu = O(\nu^{-\gamma-1} \log^{-\delta} \nu), \text{ and } s_\nu(x) = O(\nu^{1/2}),$$

the terms in Q are

$$O(\nu^{1/2-\gamma-1} \log^{-\delta} \nu) = O(\nu^{-\alpha-1} \log^{-\delta} \nu),$$

so that, on account of (3),

$$|Q| = |\sum_{N+1}^{\infty}| = O\left(\frac{N^{-\alpha}}{\log^{\delta-1} N}\right) = O\left(\frac{h^\alpha}{l_1(h)^{\delta-1}}\right) = O\left(\frac{h^\alpha}{l_1(h)^{\gamma(1+\varepsilon)}}\right) \leqslant \frac{C_1 h^\alpha}{[l_1(h)^{1+\varepsilon}]^\gamma}.$$

Since $s'_\nu(x) = O(\nu^{3/2})$, we find by the mean-value theorem that $s_\nu(x+h) - s_\nu(x) = O(h\nu^{3/2})$; the terms in P are therefore

$$O(h\nu^{3/2}) \triangle \nu^{-\gamma} \log^{-\delta} \nu = O(h\nu^{3/2-\gamma-1} \log^{-\delta} \nu) = O(h\nu^{-\alpha} \log^{-\delta} \nu);$$

hence, on account of (4),

$$|P| = |\sum_{1}^{N}| = O\left(h\frac{N^{1-\alpha}}{\log^{\delta-1} N}\right) = O\left(\frac{h^\alpha}{l_1(h)^{\delta-1}}\right) \leqslant \frac{C_2 h^\alpha}{[l_1(h)^{1+\varepsilon}]^\gamma}.$$

Finally

$$|f(x+h) - f(x)| \leqslant |P| + |Q| \leqslant \frac{C h^\alpha}{[l_1(h)^{1+\varepsilon}]^\gamma}.$$

If, for any $a' > a$, $f(x)$ would belong to the Lipschitz-class of order a', we should have, by SZÁSZ's Theorem for the real component of $f(x)$, $\sum \varrho_n^p < \infty$ for every p satisfying

$$2/(2a' + 1) < p < 2/(2a + 1) = \gamma^{-1}.$$

However, if $p < \gamma^{-1}$, the series $\sum \varrho_n^p$ diverges, since

$$\sum \left(\frac{1}{n^\gamma (\log n)^\delta}\right)^p$$

diverges.

G. H. HARDY [5] has shown that if $f(x)$ satisfies a Lipschitz-condition of order a $(0 < a \leqslant 1)$, then

$$\sum_{n=1}^{\infty} n^{3-\frac{1}{2}} \varrho_n$$

[5] G. H. HARDY, WEIERSTRASS's non-differentiable function, Tr. of the Am. Math. Soc. **17** (1916), 301—325.

converges for $\beta < a$. We shall prove the following addition:

Theorem 3. *If* $0 < a \leq 1$, $\varepsilon > 0$, $h > 0$, *and*

$$|f(x+h) - f(x)| \leqslant \frac{C\,h^\alpha}{l_1(h)\,l_2(h)\ldots l_k(h)^{1+\varepsilon}}\,. \qquad . \qquad . \qquad (5)$$

then the series $\sum\limits_{n=1}^{\infty} n^{\alpha - \frac{1}{2}}\, \varrho_n$ *converges.*

Proof. We shall again give the proof for $k = 2$. In the same way as in Theorem 1 we find

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \varrho_n^2 \leqslant \frac{C_3\, 2^{-2\nu\alpha}}{[(\nu-2)\log^{1+\varepsilon}(\nu-2)]^2}\,,$$

and from this, by SCHWARZ's inequality,

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \varrho_n \leqslant \left(\Sigma\, \varrho_n^2\right)^{1/2} \cdot 2^{\nu/2} \leqslant \frac{C_4\, 2^{\nu(\frac{1}{2}-\alpha)}}{(\nu-2)\log^{1+\varepsilon}(\nu-2)}\,,$$

so that

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} n^{\alpha-\frac{1}{2}}\, \varrho_n \leqslant \frac{C_5}{(\nu-2)\log^{1+\varepsilon}(\nu-2)}\,,$$

and finally

$$\sum_{n=2^{\nu_0-1}+1}^{\infty} n^{\alpha-\frac{1}{2}}\, \varrho_n \leqslant \sum_{\nu=\nu_0}^{\infty} \frac{C_5}{(\nu-2)\log^{1+\varepsilon}(\nu-2)} < \infty.$$

Remark. It is not difficult to prove that the function

$$f(x) = \sum_{n=1}^{\infty} \frac{e^{i\,n\log n}}{n^\gamma\,(\log n)^{2+\varepsilon}}\, e^{i\,n\,x}$$

where $\gamma = \dfrac{2a+1}{2}$, $\varepsilon > 0$, satisfies a condition of the form (5), without belonging to a Lipschitz-class of order $a' > a$.

Correction. added after reading the proof-sheets. The results of (3) and (4) can be improved, since in both the right sides $\delta-1$ may be replaced by δ. For (3) this is trivial, and for (4) it follows from the splitting up of the integral into two parts, one from 2 to $N^{1/2}$ and the other from $N^{1/2}$ to N. The first integral is then

$$O(N^{\frac{1}{2}(1-\alpha)}) = O(N^{1-\alpha}\log^{-\delta} N)$$

and the second does not exceed

$$\log^{-\delta} N^{1/2}\, O(N^{1-\alpha}) = O(N^{1-\alpha}\log^{-\delta} N).$$

These results enable us to replace, in the announcement of Theorem 2, $\delta = 1 + \gamma(1+\varepsilon)$ by $\delta = \gamma(1+\varepsilon)$. For the same reason we may replace, in the last remark, $2+\varepsilon$ by $1+\varepsilon$.