Mathematics. — *Topological classification of all closed countable and continuous classification of all countable pointsets.* By J. DE GROOT. (Communicated by Prof. J. G. VAN DER CORPUT).

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Dedicated to the late Prof. Dr. G. SCHAAKE.

I.

1. 1. The problem of determining the existence or not-existence of a homoeomorphism between two spaces (pointsets) is often called the chief problem of topology. In other words one has to enumerate, if possible, all classes of not-homoeomorphic sets. In this generality the problem is by no means solved. Only for special families of sets [1]) a solution is known. We are giving some instances. The closed surfaces (closed 2-dimensional manifolds) have been completely classified topologically (compare for instance SEIFERT–THRELFALL, Lehrbuch der Topologie, Chpt. VI), for the closed 3-dimensional manifolds however the problem is as yet unsolved. For the perfect (i.e. compact, dense in themselves) 0-dimensianal sets the problem has been solved, for every two sets of that kind are homoeomorphic as both are homoeomorphic with the discontinuum D of CANTOR (Theorem of BROUWER); for the countable sets the problem of homoeomorphism has not been solved, and so of course not for the 0-dimensional sets in general. Only a theorem of SIERPIŃSKI is known, saying that every two countable sets, dense in themselves, are homoeomorphic.

We shall first solve the problem of homoeomorphism for compact countable sets (1.2.—1.6.). After that a generalisation will prove possible for the more general class of microcompact countable sets and therefore especially for the closed countable sets.

1. 2. We consider an arbitrary compact countable set A as a subset of an n-dimensional Euclidian space R_n (or as a subset of the Hilbert space R_ω). Moreover it is possible to consider A as a subset of the discontinuum D, as every countable set is homoeomorphic with a subset of D (Theorem of SIERPIŃSKI). Among all possible subsets of D which together form a family with potency $2^{\aleph}$ [2]), there are $\aleph$ (different) compact sets. So D too contains at most $\aleph$ compact countable subsets. The family of sets containing one point of D already gives $\aleph$ different compact countable sets. So D has exactly $\aleph$ compact countable subsets.

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1) A set of sets is called a family or a system.

2) $\aleph_0$ is the potency of the set of natural numbers, $\aleph$ that of the set of real numbers.

We now have to decide how many and which of these subsets are really *topologically different*. We shall find $\aleph_0$ topologically different compact countable sets and so of course just as many topological invariants.

1\. 3. Let A be an arbitrary compact countable set. By taking the set I_0 of all isolated points out of A we have the set of all limit-points of A. the first derived set A_1, left. By taking out of A_1 the set of isolated points I_1 one has the second derived set A_2 left. This process is continued. It appears that all derived sets are compact. It is not possible that a certain (compact) derived set A_i is dense in itself; for every compact set, dense in itself, is homoeomorphic with the discontinuum D, contradictory to the countability of A. Only the following two possibilities may offer themselves: 1^0. after a finite number of steps one finds a vacuous derived set A_n, or 2^0. not one of the derived sets A_i (i being an arbitrary natural number) is vacuous. In the second case we determine the intersection of the compact sets $A = A_0, A_1, A_2, A_3, \ldots$ and thus achieve an apparently compact set $A_\omega = \prod_{i=0}^{\infty} A_i$, the ω-th derived set. Of A_ω we consider the derived set $A_{\omega+1}$ and this process is continued in the same way along the transfinite sequence of ordinal numbers. Always $A_\alpha = I_\alpha + A_{\alpha+1}$ for any ordinal number α, where I_α is the set of isolated points of A_α. This process will end after an at most countable number of steps (as A is countable), in other words, there is first ordinal number μ so that $A_\mu = 0$. Here we have a special case of the well-known theorem of CANTOR–BENDIXSON. As μ belongs to the first or second class of numbers, i.e. to the (well-ordered) set of all ordinal numbers fixing a countable potency, we may, supposing that ω_1 is the first ordinal number of the third class of numbers (therefore the first ordinal number of potency $\aleph_1$), denote the well-ordered family of derived sets by

$$A = A_0 \supset A_1 \supset A_2 \supset \ldots A_\omega \supset A_{\omega+1} \ldots A_{\omega.2} \supset \ldots A_\alpha \supset \ldots A_\mu = 0 \quad |\, \omega_1 \qquad (1)$$

A set A_β therefore is formed in one of the following ways: 1^0. by removing the isolated points from $A_{\beta-1}$, 2^0. if β has no immediately preceding ordinal number — if β is limes-number — by taking the intersection $\prod_{\gamma<\beta} A_\gamma$. We further remark that for every considered ordinal number α: $\bar{I}_\alpha = A_\alpha$, in other words the set of limit-points of I_α is exactly $A_{\alpha+1}$.

We may sharpen the theorem of CANTOR–BENDIXSON a little, as we are considering compact sets. Should μ be limes-number it would be possible to find a sequence of ordinal numbers $\gamma_1, \gamma_2, \gamma_3, \ldots$ with $\lim_{i \to \infty} \gamma_i = \mu$. This implies

$$\prod_i A_{\gamma_i} = A_\mu = 0.$$

On the other hand this intersection can not be vacuous as the A_{γ_i} are compact sets. Then μ is not a limes-number and has a predecessor. So there is a last not-vacuous derived set A_λ $(\lambda + 1 = \mu)$ and we may write (1) as

$$A = A_0 \supset A_1 \supset A_2 \ldots A_\omega \supset \ldots A_\alpha \supset \ldots A_\lambda \supset A_{\lambda+1} = 0 \quad |\omega_1. \quad (1')$$

A_λ then must be finite and consists of, say l points. So we have the following result: for every compact countable set A a "last" set A_λ may be found, consisting of l points. Thus to each A belongs a pair of numbers (λ, l), where λ is an ordinal number $< \omega_1$ and l a natural number. (λ, l) we call the *degree* of A. Conversely it is clear (with the assistance of a construction by means of transfinite induction) that for every pair of numbers (λ, l) of that kind there may be found a compact countable A, having (λ, l) for degree.

We now contend that those ordinal numbers (λ, l) give exactly all possible topologically different compact countable sets.

Theorem I. *Two compact countable sets A and B may be mapped topologically on each other only if the degree (λ, l) of A is the same as that of B. To each degree (λ, l) where λ is an arbitrary ordinal number of the first or second class of numbers and l is an arbitrary natural number, there belongs (topologically spoken) just one not-vacuous compact countable set A. Thus all topological invariants of the (family of) compact countable sets are known* [3]).

In 1. 4., 1. 5. and 1. 6. this theorem will be proved.

1. 4. It is desirable to give some lemmas and notions.

The degree (λ, l) is greater (by definition) than the degree (μ, m) if $\lambda > \mu$ or $\lambda = \mu$ and $l > m$. In the same way we define $=$ and $<$.

By the *degree of an* arbitrary *point* p in A we mean the highest ordinal number α for which p still belongs to A_α. Apparently there is indeed such a highest ordinal number α. It is evident that all points with degree α in A together form exactly the set I_α. If p has degree α and q has degree β the degree of p is greater than, equal to or smaller than that of q if $\alpha > \beta$, $\alpha = \beta$, $\alpha < \beta$ respectively. Apparently, if A has degree (λ, l) and p has degree α in A, then $\alpha \leqq \lambda$. Further the following simple lemmas hold true, which we shall mention without proofs.

Lemma I. The degree of a compact subset $D \subset A$ is smaller than or equal to that of A.

Lemma II. If a point $p \subset A$ has degree α there may be found a (compact) neighbourhood $U = U\,(p/A)$ of p in A so that the set U has degree $(\alpha, 1)$.

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[3]) The ordinal numbers (λ, l) are topological invariants. The property "to be of degree $(1, 1)$ or of degree $(2, 1)$" also is a topological invariant (denoted by $(1, 1) + (2, 1)$). This topological invariant however follows trivially from the given system of invariants. It is easy to prove that indeed every topological invariant of the family in question is equivalent with a "sum of well-chosen numbers of degree".

Lemma III. If D is a *part-set* of A (i.e. an in A both open and closed set) then a point of D has the same degree in D as in A.

Lemma IV. If a point p has degree α in A, then in every neighbourhood of p there are points with degree $< \alpha$ but for the rest arbitrary. Further there may be found a sufficiently small neighbourhood $U(p)$ of p so that *all* points of that neighbourhood have a degree $\leqq \alpha$.

1. 5. In this number we shall prove that, if the compact countable sets A and B both have the degree (λ, l), there exists a topological mapping of A on B.

This is proved by means of transfinite induction to (λ, l) where we suppose all possible values of (λ, l) arranged according to size in a well-ordered sequence.

If A and B both have the degree $(0, l)$, in other words if A and B both consist of l points, then every one-to-one mapping of A on B is a topological mapping. Now let us consider the case where A and B have the degree $(\lambda, 1)$, while the theorem is proved for all compact countable sets with degree $< (\lambda, 1)$. Both in A and in B one may find one point, say a and b, with degree λ. Choose a system of neighbourhoods $U_1, U_2, \ldots$ of a consisting of part-sets with $U_i \supset U_{i+1}$. We put $R_1 = A - U_1$. The degree (ϱ_1, r_1) of R_1 is less than $(\lambda, 1)$. Using the lemmas mentioned above one easily sees that there may be found a part-set $V_1 = V_1(b)$ of b in B so that $S_1 = B - V_1$ has a degree $(\sigma_1, s_1) > (\varrho_1, r_1)$. Further we may certainly find in U_1 s_1 points with degree σ_1 and therefore also (sufficiently small) part-neighbourhoods of those s_1 points, each having the degree $(\sigma_1, 1)$. The sum of R_1 and of a well-chosen number of these s_1 part-neighbourhoods together form a part-set R'_1 with degree (σ_1, s_1) [4]. According to the supposed induction R'_1 may be mapped topologically on S_1.

We now proceed to the second step. Evidently there exists a sufficiently great value i so that U_i has no point in common with R'_1. Put

$$R_2 = A - U_i - R'_1.$$

Let the degree of this part-set R_2 be (ϱ_2, r_2). It is possible to find a part-neighbourhood $V_2 = V_2(b)$ within V_1 so that the degree of $S_2 = V_1 - V_2$ has a value $(\sigma_2, s_2) > (\varrho_2, r_2)$. Within U_i we may find at least s_2 points each with degree σ_2 and therefore just as many part-neighbourhoods of those points with degree $(\sigma_2, 1)$. R_2 and the sum of a number of those s_2 part-neighbourhoods form a set R'_2 with degree (σ_2, s_2). We map R'_2 topologically on S_2.

We now proceed to the third step. It is possible to find a sufficiently great $k > i$ so that U_k has no point in common with R'_2. Put

$$R_3 = U_i - U_k - U_i \cdot R'_2.$$

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[4]) If $\sigma_1 > \varrho_1$ we take all s_1 part-neighbourhoods. If $\sigma_1 = \varrho_1$ we take only $s_1 - r_1$ arbitrary part-neighbourhoods.

Construct a part-neighbourhood $V_3 = V_3(b)$ within V_2 so that $S_3 = V_2 - V_3$ has a degree $(\sigma_3, s_3) > (\varrho_3, r_3)$, where (ϱ_3, r_3) is the degree of R_3. In the above mentioned way we construct an R'_3 which is mapped topologically on S_3.

This process is continued ad infinitum, while we take care that the part-neighbourhoods $V_1, V_2, V_3, \ldots$ have b as intersection, in other words form a system of neighbourhoods of b. Then, when we map a on b this mapping together with the constructed mappings of R'_i on S_i gives a mapping of A on B which is evidently topological.

Finally let us suppose the case that A and B have the degree (λ, l) with $l > 1$ while the theorem has already been proved for the degree $(\lambda, 1)$. In A are exactly l points with degree λ. Determine for $l - 1$ of those points disjunct part-neighbourhoods $W_1, W_2, \ldots, W_{l-1}$ which moreover don't contain the last of the l points. To the l-th point we add the part-neighbourhood $A - \sum_{i=1}^{l-1} W_i$. Construct in the same way for B the part-sets $Z_1, Z_2, \ldots, Z_{l-1}, Z_l$. The part-sets W_i and Z_i apparently all have the degree $(\lambda, 1)$ and we may therefore map each W_i topologically on Z_i. These mappings together give a topological mapping of A on B [5]).

1. 6. Theorem I will be proved completely if we can show that a compact countable set A with degree (λ, l) may not be mapped topologically on a (compact countable) set B of different degree.

Let $t(A) = B$ be a topological mapping of A on B. Then the isolated points i_0 of A will necessarily be mapped on the isolated points k_0 of B: $t(I_0) = K_0$, so $t(A_1) = B_1$. In general $t(A_\alpha) = B_\alpha$ for a derived set A_α of A holds true. For, supposing this has already been proved for all ordinal numbers β with $\beta < \alpha$, we may prove it for α as follows. If α is not a limes-number then

$$t(A_{\alpha-1}) = B_{\alpha-1} \text{ and } t(I_{\alpha-1}) = K_{\alpha-1}, \text{ so } t(A_\alpha) = B_\alpha.$$

If α is a limes-number then

$$A_\alpha = \prod_{\beta<\alpha} A_\beta$$

and

$$t(A_\alpha) = t(\prod_{\beta<\alpha} A_\beta) = \prod_{\beta<\alpha} t(A_\beta) = \prod_{\beta<\alpha} B_\beta = B_\alpha.$$

From $t(A_\alpha) = B_\alpha$ follows in particular: $t(A_\lambda) = B_\lambda$, in other words B_λ is not vacuous and consists of exactly l points. In other words B has degree (λ, l), q.e.d.

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[5]) Thanks to the kind criticism of HANS FREUDENTHAL I was able to shorten the proof of 1. 5., which was at first far too long, considerably. He moreover sent me a somewhat shorter proof. By making use of some simple properties of the ordinal numbers one may prove (also by transfinite induction) that *every A of degree (λ, l) is homoeomorphic with the well-ordered set $\omega^\lambda . l + 1$.*

Remark. With a topological mapping of $\bar{A}$ on B the degree of a point is invariant.

1. 7. In this number let A be a not necessarily compact but only a microcompact countable set. We now consider in the transfinite sequence of the derived sets of the microcompact set A the first ordinal number μ for which A_μ is compact; A_μ is allowed to be vacuous. Let (λ, l) be the degree of A_μ. We define (μ, λ, l) as the degree of A. For a compact $A = A_0$ evidently $\mu = 0$ so that the above defined (λ, l) now is denoted by $(0, \lambda, l)$. If each not-vacuous A_α is not compact the degree of A is evidently $(\mu, 0, 0)$, while the degree of the vacuous set is $(0, 0, 0)$.

Now the following theorem holds true:

Theorem II. *The degrees* (μ, λ, l) $(0 \leqq \mu + \lambda < \omega_1;\ l = 0, 1, 2, \ldots;$ *if* $l = 0$ *then* $\lambda = 0)$ *determine precisely all topological invariants of the family of the microcompact countable sets, in other words a microcompact countable set may be mapped topologically only on a set having the same degree. In this way in particular all closed countable subsets of an n-dimensional Euclidian space have been classified.*

The Theorem will be proved in 1. 8.—1. 9.

1. 8. If the microcompact set A is not compact A may be compactified by one point P, in other words it is possible to find a set, homoeomorphic with A, which we also denote by A, and a point P so that $A + P$ is a compact countable set. Further this new space $A + P$ is completely determined by A in topological respect, in other words if we first compactify A by a point P to $A + P$ and afterwards a set A' homoeomorphic with A by a point P' to $A' + P'$, then $A + P$ may be mapped topologically on $A' + P'$, where P is mapped on P'. This follows from a simple theorem concerning topological extension (comp. J. DE GROOT, Proc. Kon. Ned. Ak. v. Wet. **44** (1941), p. 933): every topological mapping of a separable space A (i.e. a normal space with countable base [6])) on a separable space A' may be extended to a topological mapping of $A + P$ on $A' + P'$, if $\bar{A} = A + P$ and $\bar{A}' = A' + P'$ are compact separable spaces. Every (not compact) microcompact countable set A therefore completely determines the compact countable set $A + P$.

Concerning the degree (μ, λ, l) of A the following principally different possibilities may offer themselves:

1°. $l = 0$. The degree of A then is $(\mu, 0, 0)$ and that of $A + P$ necessarily $(0, \mu, 1)$.

2°. $l \neq 0$. The degree of A then is (μ, λ, l) $(l \neq 0)$ and that of $A + P$ necessarily $(0, \mu + \lambda, l)$ if $\lambda \neq 0$ or $(0, \mu + \lambda, l + 1)$ if $\lambda = 0$.

It is easy to see that for each degree (μ, λ, l) there is a microcompact A with (μ, λ, l) as degree.

[6]) The theorem holds also true for more general, not necessarily metrical spaces.

1\. 9. If A and B both have degree (μ, λ, l) then A may be mapped topologically on B. For, in case 1^0 $\overline{A} = A + P$ may be mapped topologically on the compactified $\overline{B} = B + Q$ as both have degree $(0, \mu, 1)$. According to the remark in 1. 6. P is mapped on Q, which implies that A is mapped topologically on B. If 2^0 is the case we divide $\overline{A}$ into the two disjunct part-sets $\overline{A}_1$ and $\overline{A}_2$ so that $\overline{A}_1$ contains point P and has degree $(0, \mu, 1)$ (the degree of P in $A + P$ is always μ), while $\overline{A}_2$ has degree $(0, \mu + \lambda, l)$. In the same way we may divide $\overline{B}$ into $\overline{B}_1$ and $\overline{B}_2$ where $\overline{B}_1$ will contain point Q. According to Theorem I we may map $\overline{A}_1$ topologically on $\overline{B}_1$ where P is mapped on Q and $\overline{A}_2$ (topologically) on $\overline{B}_2$. Then therefore A is mapped topologically on B.

Finally we have to prove that the topological image B of A has the same degree (μ, λ, l) as A. Let $t(A) = B$. We construct $\overline{A} = A + P$ and $\overline{B} = B + Q$. When we map P on Q this mapping together with t gives a topological mapping of $\overline{A}$ on $\overline{B}$. According to the remark in 1. 6. P and Q in A and B have the same degree while $\overline{A}$ and $\overline{B}$ have the same degree according to Theorem I. This immediately shows what we had to prove.

1\. 10. The method developed above cannot be used in this shape to determine the topological classification for more general classes of countable sets, but by making a few modifications it is possible to obtain some still more general results. I did not, however, achieve much in this way and shall therefore refrain from mentioning the results. In this number we shall only show that the family of all countable sets, and even the family of the countable sets having a vacuous nucleus, dense in itself, contain $\aleph$ *topologically different* sets (and just as many so-called independent topological invariants). In an R_n (or in D) there apparently exist precisely $\aleph^{\aleph_0} = \aleph$ different countable subsets. Among these are, as we saw above $\aleph_0$ *topologically different microcompact* sets, but as we will show even $\aleph$ topologically different (not microcompact) countable sets.

Theorem III. *It is possible to construct $\aleph$ topologically different countable sets (all with a vacuous nucleus dense in itself). There exist exactly $\aleph$ topologically different countable sets.*

Proof. In 1. 3. we constructed the transfinite sequence of the derived sets of a compact countable set A. If A is not compact it is also possible to construct the sequence of the derived sets (every derived set being closed in A). In doing so however we may meet the complication that a certain derived set A_α (and even $A = A_0$ itself) is dense in itself, in other words does no longer contain any isolated points. Then the process is stopped. A_α then is called the nucleus, dense in itself, of A. If A does not contain a nucleus which is dense in itself we eventually find again a first vacuous derived set A^d (but μ now needs not necessarily have a predecessor).

Let us consider $A = I_0 + A_1$ while A has a vacuous nucleus. In a point

$a_1 \subset A_1$ the following two principally different possibilities may offer themselves: 1^0. a_1 has a (sufficiently small) compact neighbourhood $U(a_1/A)$: a_1 is microcompact in A; 2^0. a_1 is not microcompact in A, in other words within every neighbourhood $U(a_1/A)$ we may find a sequence of points of A without a limit-point in A. But then there even exists a sequence of points of I_0 without a limit-point (in A) within every mentioned $U(a_1/A)$ (for, the nucleus of A being vacuous, $\overline{I_0} = A$). In case 1^0. we denote a_1 by a_1^+, in case 2^0. by a_1^-. All points of I_0 are plus-points. The points of A_1 may contain both plus- and minus-points, where the set of minus-points is closed in A as is easily proved. We therefore may divide A_1 into $A_1^+ + A_1^-$. Let us now, without entering further into this general case, consider the following special case.

$I_1 = A_1 - A_2$ will consist exclusively of plus-points in $A_0 = A$. I_2 will, *with regard to the set* A_1, consist exclusively of minus-points, I_3 will, *with regard to the set* A_2, consist exclusively of plus-points. A_4 will be vacuous. Such a set we therefore may denote by $+-+$. In the same way we may construct sets $+--$, $-+-$, etc., where we may find all possible combinations. Let us now consider, however, a set where no A_i (i finite) is vacuous; then we may in the same way find all possible sets $\delta_1 \delta_2 \delta_3 \ldots\ldots$ ($\delta_i = +$ or $-$), for instance by constructing $\delta_1 \delta_2 \delta_3 \ldots\ldots$ as the sum of a countable number of sets δ_1, $\delta_1 \delta_2$, $\delta_1 \delta_2 \delta_3 \ldots\ldots$ etc. which are isolated in regard to each other. The potency of all possible different sets $\delta_1 \delta_2 \delta_3 \ldots$ is exactly $2^{\aleph_0} = \aleph$, therefore that of the continuum. What remains to be proved is that a set $A \equiv \delta_1 \delta_2 \delta_3 \ldots\ldots$ may not be mapped topologically on a (different) set $B \equiv \varepsilon_1 \varepsilon_2 \varepsilon_3 \ldots\ldots$. Suppose there existed a topological mapping $t(A) = B$. Then I_0 is necessarily mapped on K_0 (i.e. the set of isolated points of B), and so A_1 on B_1. Thus in general $t(A_\alpha) = B_\alpha$ (comp. 1. 6.; the proof given there holds also true for arbitrary countable sets) and so also $t(I_j) = K_j$. As $\delta_1 \delta_2 \delta_3 \ldots\ldots$ and $\varepsilon_1 \varepsilon_2 \varepsilon_3 \ldots\ldots$ are unequal, there exists a first index j with $\delta_j \neq \varepsilon_j$. Let I_j consist of plus-points and K_j of minus-points with regard to A_{j-1} and B_{j-1}. As $t(A_{j-1}) = B_{j-1}$, a compact neighbourhood $U = U(i/A_{j-1})$ of a point $i \subset I_j$ in A_{j-1} is mapped on a compact neighbourhood $t(U) = V$ of $k = t(i)$ in B_{j-1} which contradicts the fact that k was a minus-point. Therefore there exist indeed $\aleph$ topologically different countable sets with vacuous nucleus.

II.

2. 1. In the following we shall determine all possible continuous invariants of the family of all countable sets. In other words we shall give a *continuous classification of the countable sets.* By a *continuous invariant* (with regard to a certain family F) is naturally meant a property which, if valid for a set A of F, also holds true for all continuous images $f(A)$ belonging to F. Trivial continuous invariants, i.e. properties valid for all

sets of F, will of course be left out of account. The property "to be countable" therefore is not a continuous invariant with regard to the family F of the countable sets, but the property "compactness" on the other hand is a continuous invariant in F. No continuous invariants for instance are the properties "to consist of infinitely many points", "to contain an isolated point", "to be dense in itself", etc. For a certain continuously invariant property α there are (in the family of the countable sets F) a number of sets with property α. We denote the family of those sets, which is a subset of F, by $E(\alpha)$. Therefore, if a set belongs to $E(\alpha)$, all its continuous images also belong to $E(\alpha)$. If $E(\beta) \subset E(\alpha)$ (i.e. every set of $E(\beta)$ is a set of $E(\alpha)$), property β evidently always implies property α. Notation $\beta \to \alpha$.

We consider in the first place an arbitrary not compact countable set and we shall show that this set may be mapped continuously on every countable set.

Theorem IV. *Every not compact countable set A may be mapped continuously on every countable set B.*

Proof. We determine a sequence belonging to A of different points $a_1, a_2, a_3, \ldots\ldots$, all isolated in respect to each other and having no limit-point belonging to A. As A is not compact it is evidently possible to construct such a sequence. Then we construct for every point a_i a neighbourhood $U_i = U(a_i)$ of a_i in A such that not a single pair of neighbourhoods has a point in common, while the diameter $[U_i]$ of U_i be less than $\frac{1}{2}i$. Moreover we see to it that every U_i is a part-set

B is countable and consists of the points $b_1, b_2, \ldots, b_k, \ldots$

We define

$$f(U_i) = b_i \qquad f(A - \sum_{i=1}^{\infty} U_i) = b_1.$$

If B is finite and consists of k points we put $b_k = b_{k+1} = \ldots\ldots$ It is easy to see that f is continuous.

2. 2. We denote the property of compactness by $\varkappa$ and show that for an arbitrary continuous invariant $\alpha : \alpha \to \varkappa$. According to 2. 1. we have to prove $E(\alpha) \subset E(\varkappa)$. Suppose there existed a not compact set A having property α. According to Theorem IV then every countable set would have property α and then α would be a trivial continuous invariant, which was excluded. We have therefore achieved the required contradiction. We ascertain that if there exist continuous invariants this implies the property of compactness. From now on we therefore have to consider only the compact countable sets. We shall prove that these have only the following continuous invariants: "the degree is less than (λ, l)" (this implies compactness, as only for a compact countable set a degree (λ, l) has been defined). We denote this property by $[<\lambda, l]$. The continuous invariant "to be a finite set" for instance is determined by $[<1, 1]$. So we get the following continuous classification of the countable sets.

Theorem V. *In the topology of the countable sets there exist as the only continuous invariants the infinitely many properties of compactness* $\varkappa$ *and* $[<\lambda, l]$. *Between these invariants there apparently is the following connection: from* $[<\lambda, l]$ *always follows* $[<\mu, m]$ *and* $\varkappa$ *of* $(\mu, m) \leqq (\lambda, l)$.
The proof will be given in 2. 3. and 2. 4.

2. 3. In this number we prove

Theorem V'. *The compact countable set* A *may be mapped continuously on the (compact) countable set* B *only if the degree of* B *is less than or equal to that of* A.

Proof. Let the degree (α, m) of A be greater than or equal to the degree (β, n) of B. We have to map A continuously on B. This is done by means of transfinite induction to β. For $\beta = 0$, B consists of n points. As A consists of more than n points it will certainly be possible to divide A into n disjunct closed subsets $A_1, A_2, \ldots, A_n$. The points of each A_i are mapped on the i-th point of B. This is the required continuous mapping of A on B. Now let it be possible to construct a continuous mapping of A on B for all ordinal numbers $<\beta$. In B there are precisely n points $b_1, b_2, \ldots, b_n$ with degree β. The degree of all other points of B is $<\beta$. For each point $b_i (i = 1, 2, \ldots, n)$ we choose a sequence of (compact) part-neighbourhoods $U_i^j = U^j(b_i)$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots)$ of b_i in B so that:

$$U_i^j \supset U_i^{j+1}, \quad \prod_j U_i^j = b_i, \quad U_r^j \cdot U_s^j = 0 \qquad (r \neq s).$$

Further we put

$$S_0^0 = B - \sum_{i=1}^{n} U_i^1, \quad S_i^j = U_i^j - U_i^{j+1} \quad (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots).$$

The sets S are evidently compact and have a degree $<(\beta, 1)$. We denote the degree of S_i^j $(i = 0, 1, \ldots, n;\ j = 0, 1, 2, \ldots)$ by (σ_i^j, s_i^j). As

$$(\alpha, m) > (\beta, n)$$

A certainly contains n points $a_1, a_2, \ldots, a_n$ of degree β. Using the lemmas of 1. 4. we may easily prove that it is possible to construct a (compact) part-set R_0^0 which does not contain a single point $a_i (i = 1, 2, \ldots, n)$ and the degree of which is $(\varrho_0^0, r_0^0) > (\sigma_0^0, s_0^0)$. The complement-set $A - R_0^0$ may be divided into n disjunct closed sets $V_i^1 (i = 1, 2, \ldots, n)$, where V_i^1 contains a_i precisely. For every V_i^1 we may construct a part-set $R_i^1 \subset V_i^1$ not containing a_i, while the degree (ϱ_i^1, r_i^1) of R_i^1 is greater than or equal to the degree (σ_i^1, s_i^1). After that we put $V_i^2 = V_i^1 - R_i^1$ and define in the same way an R_i^2 within V_i^2, etc. Because of the supposed induction it is now possible to map every $R_i^j (i = 0, 1, \ldots, n;\ j = 1, 2, \ldots)$ continuously on the corresponding S_i^j. Further, if we map a_i on b_i it is evident that in this way we have constructed a continuous mapping of A on B.

There remains to be proved: if the degree (α, m) of A is less than the degree (β, n) of B, then it is impossible to map A continuously on B. For $\alpha = 0$ A consists of m points and B of more than m points. Therefore a continuous mapping of A on B is impossible. Put $\alpha > 0$. Suppose there exists a continuous mapping f with $f(A) = B$. Consider the set

$$I_0 = A_0 - A_1$$

of the isolated points of A. The image $f(I_0)$ contains $K_0 = B_0 - B_1$, for suppose there existed an isolated point p of B not belonging to $f(I_0)$, then p also did not belong to $\overline{f(I_0)}$, but $f(A) = f(\bar{I}_0) = \overline{f(I_0)} = B$, so p would not belong to B which is incorrect. Now consider a point $b_1 \subset B_1$. It is possible to find a sequence of points in K_0 converging to b_1. These points, as we have just shown, possess all inverse images belonging to I_0. Those inverse images have a limit-point $a_1 \subset A_1$. Now a_1 has to be mapped on b_1 by f. In this way each point of B_1 has an inverse image belonging to A_1, in other words $f(A_1) \supset B_1$. So there is a closed and therefore compact subset A_1^1 of A_1, mapped on $B_1 : f(A_1^1) = B_1$. Should A_1^1 be vacuous we should have reached a contradiction. If A_1^1 is not vacuous we continue as follows. Consider the set $A_1^1 - A_2^1$ (A_2^1 is the derived set of A_1^1) and apply to this and to B_1 the same method as we applied to $A_0 - A_1$ and B. Thus we find a compact set A_2^2 with $A_2^2 \subset A_2^1 \subset A_2$ and $f(A_2^2) = B_2$. This process is continued and thus the following sequence is formed:

$$A \supset A_1^1 \supset A_2^2 \supset \ldots\ldots.$$

The sets of this sequence are mapped by f on the sets of the sequence

$$B \supset B_1 \supset B_2 \supset \ldots\ldots.$$

If one of the sets A_i^i is vacuous we have already reached a contradiction. If none of the A_i^i is vacuous the intersection $\prod\limits_i A_i^i = A_\omega^\omega$ is compact and not vacuous and apparently $f(A_\omega^\omega) = B_\omega$, while $A_\omega^\omega \subset A_\omega$. With A_ω^ω and B_ω, as originally with A and B, the process is continued transfinitely. If we find a vacuous A_γ^γ with $\gamma < \alpha$ a contradiction will have been reached. If not one of the A_γ^γ is vacuous we find eventually (after a countable number of steps) a set A_α^α which is mapped on B_α (B_α is not vacuous because $\alpha \leqq \beta$). A_α^α as a subset of A_α consists of at most m points. B_α on the other hand consists of more than m points as $(\alpha, m) < (\beta, n)$, in contradiction to $f(A_\alpha^\alpha) = B_\alpha$.

2. 4. Suppose there existed besides the continuous invariants mentioned in Theorem V yet another continuous invariant β. We already proved $\beta \to \varkappa$ and so may confine ourselves again to the compact countable sets. Consider $E(\beta)$; if this contains a set with degree (λ, l) it also contains all sets with degree $\leqq (\lambda, l)$, according to Theorem V'. We order the

degree-numbers (λ, l) according to size in a well-ordered set X. Now two possibilities may offer themselves: 1^0. in X there is a first element (μ, m) so that the sets with degree (μ, m) don't belong to $E(\beta)$; 2^0. such an element (μ, m) may not be found in X, in other words $E(\beta)$ contains exactly all compact countable sets (for every compact countable set has a certain degree), or $\beta \equiv \varkappa$. In the first case all sets with degree $\geqq (\mu, m)$ necessarily possess property $-\beta$ (i.e. β does not hold true), for suppose there existed a set with degree $(\nu, n) \geqq (\mu, m)$ having property β, then every set with degree (μ, m) also would possess property β. Thus it appears that a compact countable set has property β only if its degree is $< (\mu, m)$, but that means: $\beta \equiv [<\mu, m]$.

As the properties $[<\lambda, l]$ and "compactness" according to Theorem V' are continuous invariants and according to the above there exist no other continuous invariants, Theorem V has thus been proved.

The Hague, April 1945.