Mathematics. — *Proof of the impossibility of a just distribution of an infinite sequence of points over an interval.* By T. VAN AARDENNE-EHRENFEST. (Communicated by Prof. J. G. VAN DER CORPUT.)

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Introduction. About ten years ago Professor J. G. VAN DER CORPUT considered the question whether the distribution of an infinite sequence of points over an interval might be of a nature which he described by the terme *just*, and which is much more regular than that required of what is called a *uniform* distribution. An infinite sequence $\mathfrak{A}$ of points in an interval is said to be *justly distributed* over this interval if there exists a constant C such that for any pair of sub-intervals α, β of equal length and all n

$$|A_n(\alpha) - A_n(\beta)| \leqslant C.$$

Here $A_n(\alpha)$ and $A_n(\beta)$ denote the numbers of points of $\mathfrak{A}$ with indices not exceeding n which belong to α and β respectively.

After some attempts to construct such a sequence, Professor VAN DER CORPUT was led to the conjecture *that a just distribuion does not exist.* In what follows it will be proved that this conjecture is right.

Notations. Greek letters will denote intervals (which may be either closed, or open, or half-open, or consist of one point only). $|\alpha|$ denotes the length of α, $\alpha \subset \beta$ means that α is contained in β.

If $\mathfrak{A}$ is a finite system of (not necessarily mutually different) points, then $A(\lambda)$ denotes the number of points of $\mathfrak{A}$ contained in λ. If $a_1, a_2, \ldots$ is a sequence of points, then $A_n(\lambda)$ will denote the number of points of the initial fragment $a_1, a_2 \ldots a_n$ belonging to λ.

Theorem. To every natural number x there correspond a natural number $N(x)$ and a positive number $u(x)$ with the following property: if ι is an interval of length $|\iota| > 0$, then any finite sequence of $N(x)$ points $a_1, a_2 \ldots a_{N(x)}$ in ι has an initial fragment $a_1, a_2 \ldots a_n$ ($n \leq N(x)$) such that ι contains two intervals σ and τ for which

$$|\tau| = |\sigma| + |\iota|\, u(x) \text{ and } A_n(\sigma) \geqslant A_n(\tau) + x.$$

From this theorem it follows immediately that a just distribution does not exist, for the interval τ mentioned in the theorem contains an interval τ' of length $|\tau'| = |\sigma|$, and we have $A_n(\sigma) \geq A_n(\tau') + x$.

Proof of the theorem. Without loss of generality we suppose $|\iota| = 1$.

$N(1) = 1$ and $u(1) = \frac{1}{2}$ have the required property; for if the sequence consists of one point only, there is a (closed) sub-interval of ι of length 0 containing one point of the sequence, and there is an (open) sub-interval of length $\frac{1}{2}$ containing no point of the sequence.

Suppose that for some particular x the existence of $N = N(x)$ and $u = u(x)$ has been established. We have to show the existence of $N(x+1)$ and $u(x+1)$.

Let T be an even positive integer such that $u \geqq T^{-1}$. We shall prove that the integer M and the positive number w, which are determined by the recursive relations

$$q_{1,0} = 1, \quad f_i = [{}^2\log(5\,x\,q_{i,0})] + 1 \quad (i = 1, \ldots x)$$

$$q_{i,k+1} = (2\,q_{i,k} + 1)\,T \quad (i = 1, \ldots x;\ k = 0, 1, \ldots f_i - 1)$$

$$q_{i+1,0} = q_{i,f_i} + 2 \quad (i = 1, \ldots x)$$

$$M = 2\,q_{x+1,0}(N + 6\,x), \quad d = \frac{1}{q_{x+1,0}}, \quad w = \frac{d}{(x+1)(5x+1)}$$

have the properties required of $N(x+1)$ and $u(x+1)$.

Suppose $\mathfrak{C}$ is a sequence of M points in ι such that each initial fragment $\mathfrak{C}'$ of $\mathfrak{C}$ has the property that

$$C'(\lambda) \leqq C'(\mu) + x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

when $|\mu| = |\lambda| + w$, $\lambda \subset \iota$, $\mu \subset \iota$. We have to show that this is absurd.

We begin by deducing some properties of $\mathfrak{C}$.

Put $M' = 2xq_{x+1,0} + 1$. Let $\mathfrak{A}$ be the system of the first M' points of $\mathfrak{C}$ and let $\mathfrak{B}$ be the sequence which is the remaining fragment of $\mathfrak{C}$.

Property 1. Each sub-interval of ι of length d contains at most $5x$ points of $\mathfrak{A}$ and at least 1 point of $\mathfrak{A}$ and N points of $\mathfrak{B}$.

Proof. $\mathfrak{A}$ is an initial fragment of $\mathfrak{C}$. Hence it follows from (1): if there were a sub-interval of ι of length d containing more than $5x$ points of $\mathfrak{A}$, then each sub-interval of length $d + w$, and a fortiori (since $w < d$) each sub-interval of length $d + d = 2d$, would contain at least $5x + 1 - x = 4x + 1$ points of $\mathfrak{A}$. Since $\frac{1}{2}\,q_{x+1,0}$ is an integer (which follows from the given formulae and from the fact that T is even), the interval ι, being of length $1 = \frac{1}{2}\,q_{x+1,0} \cdot 2d$, would contain at least

$$\tfrac{1}{2}\,q_{x+1,0}(4\,x + 1) = 2\,x\,q_{x+1,0} + \tfrac{1}{2}\,q_{x+1,0}$$

points of $\mathfrak{A}$, which is more than

$$M' = 2\,x\,q_{x+1,0} + 1, \text{ since } \tfrac{1}{2}\,q_{x+1,0} > 1.$$

If, on the other hand, there were a sub-interval of ι of length d containing no point of $\mathfrak{A}$ (or less than N points of $\mathfrak{B}$ and therefore less than $N + 5x$ points of $\mathfrak{A} + \mathfrak{B} = \mathfrak{C}$), it follows from (1), since $\mathfrak{A}$ and $\mathfrak{C}$ are both initial fragments of $\mathfrak{C}$, that each sub-interval of length $d - w$, and a fortiori (since $w < \frac{1}{2}d$) each sub-interval of length $d - \frac{1}{2}d = \frac{1}{2}d$, would contain at most $0 + x = x$ points of $\mathfrak{A}$ (or less than $N + 5x + x = N + 6x$ points of $\mathfrak{C}$). Hence the interval ι, being of length $1 = 2 q_{x+1,0} \cdot \frac{1}{2} d$, would contain at most $2 x q_{x+1,0} = M'—1$ points of $\mathfrak{A}$ (or less than $2 q_{x+1,0} (N + 6x) = M$ points of $\mathfrak{C}$). Either is impossible, and so property 1 is proved.

In what follows we shall denote by ω_r $(r = 0, 1, \dots x)$ a sub-interval of ι of length $q_{x-r+1,0}\, d$ with the property that

$$A(\lambda) \leqq A(\mu) + x - r \quad \dots \dots \dots \quad (2)$$

whenever $|\mu| = |\lambda| + (r + 1) w$, $\lambda \subset \omega_r$, $\mu \subset \omega_r$.

By ϱ_r^k $(r = 0, 1, \dots x—1;\ k = 0, 1, \dots f_{x-r})$ we shall denote an interval of length $q_{x-r,\,k}\, d$ which is contained in an ω_r, and which has a distance not less than d from the end-points of this ω_r.

Property 2. Each ϱ_r^k contains 2^k intervals of length $q_{x-r,0}\, d$ for which

$$A(\lambda) \leqq A(\mu) + x - r - 1$$

whenever $|\mu| = |\lambda| + (r + 2) w$ and λ is contained in one of these 2^k sub-intervals while μ is contained in another of these intervals.

Proof. This is obvious for $k = 0$. Suppose that property 2 has been verified for a particular k $(0 \leq k \leq f_{x-r} — 1)$. We shall verify it for $k + 1$ instead of k.

Consider an interval ϱ which is a ϱ_r^{k+1}. ϱ is contained in an ω_r, to the end-points of which it has a distance not less than d. This ω_r we call ω.

Since $|\varrho| = q_{x-r,\,k+1}\, d > d$, the interval ϱ contains, by property 1, at least N points of $\mathfrak{B}$. Since it was supposed that $N = N(x)$ and $u = u(x)$ satisfy the requirements of the theorem, the sequence $\mathfrak{B}$ has an initial fragment $\mathfrak{B}'$ such that ϱ contains two intervals σ and τ for which

$$B'(\sigma) \geqq B'(\tau) + x \text{ and } |\tau| = |\sigma| + q_{x-r,k+1}\, d\, u.$$

Without loss of generality we assume that σ and τ do not overlap, and that σ lies to the left of τ. Since $u \geq T^{-1}$, there will exist a sub-interval τ' of τ of length

$$|\tau'| = |\sigma| + T^{-1} q_{x-r,k+1}\, d \quad \dots \dots \dots \quad (3)$$

We have

$$B'(\sigma) \geqq B'(\tau') + x. \quad \dots \dots \dots \quad (4)$$

By taking away from τ' on both sides intervals α and β of length

$$|\alpha| = |\beta| = \tfrac{1}{2} \{T^{-1} q_{x-r,k+1}\, d—d\} = q_{x-r,k}\, d$$

we obtain an interval γ of length

$$|\gamma| = |\tau'| - (T^{-1} q_{x-r,k+1}\, d - d) = |\sigma| + d,$$

separating α and β.

First we shall prove that

$$A(\lambda_1) \leqq A(\mu_1) + x - r - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (^*)$$

whenever $|\mu_1| = |\lambda_1| + (r+2)w$ *and either*

$$\lambda_1 \subset \alpha, \mu_1 \subset \beta$$

or

$$\lambda_1 \subset \beta, \mu_1 \subset \alpha.$$

Let ν be the interval which separates λ_1 and μ_1. Put $\lambda_2 = \lambda_1 + \nu$ and $\mu_2 = \mu_1 + \nu$. Then $(^*)$ means the same thing as

$$A(\lambda_2) \leqq A(\mu_2) + x - r - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (^{**})$$

We have

$$|\mu_2| = |\lambda_2| + (r+2)w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Since λ_2 contains ν, and ν contains γ, we have

$$|\lambda_2| \geqq |\gamma| = |\sigma| + d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The interval σ is contained in ϱ; ϱ is contained in ω and has to its end-points a distance not less than d. Therefore σ is contained in ω and has to its end-points a distance not less than d. Since, by property 1, each sub-interval of ι of length d contains at least 1 point of $\mathfrak{A}$, there is a point a of $\mathfrak{A}$ belonging to ω which is situated to the left of σ, and whose distance from σ is smaller than d.

Now consider the *open* interval $\varkappa$ which has its left-hand end-point in a and for which

$$|\varkappa| = |\lambda_2| + (r+1)w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

By (5) we have

$$|\varkappa| = |\mu_2| - w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If $\bar{\varkappa}$ is the *closed* interval with the same end-points as $\varkappa$, we have

$$A(\varkappa) \leqq A(\bar{\varkappa}) - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

since $\varkappa$ is open, and since at least one point of $\mathfrak{A}$ coincides with an end-point of $\bar{\varkappa}$.

The interval σ is contained in $\varkappa$, for, by (6) and (7), we have $|\varkappa| > |\sigma| + d$, and the left-hand end-point a of $\varkappa$, which is situated to the left of σ, has a distance to it which is smaller than d. Therefore we have by (4), observing that μ_2 in contained in τ',

$$B'(\varkappa) \geqq B'(\sigma) \geqq B'(\tau') + x \geqq B'(\mu_2) + x \quad . \quad . \quad . \quad . \quad (10)$$

The left-hand end-point a of $\varkappa$ belongs to ω and is situated to the left of σ and a fortiori to the left of τ', which was supposed to be situated to the right of σ; the length of $\varkappa$ is, by (8), smaller than the length of τ', since μ_2 is contained in τ'; since τ' is contained in ω, it follows that the right-hand end-point of $\bar{\varkappa}$ belongs also to ω. Hence the interval $\bar{\varkappa}$ is contained in ω.

Since $\varkappa$ and λ_2 are contained in ω, which is an ω_r, it follows from (2) and (7) that

$$A(\lambda_2) \leqslant A(\varkappa) + x - r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Since $\bar{\varkappa}$ and μ_2 are contained in ι, and since $\mathfrak{A} + \mathfrak{B}'$ is an initial fragment of $\mathfrak{C}$, it follows from (1) and (8) that

$$A(\bar{\varkappa}) + B'(\bar{\varkappa}) \leqslant A(\mu_2) + B'(\mu_2) + x. \quad . \quad . \quad . \quad . \quad (12)$$

From (9), (10), (11) and (12) follows (**), and so (*) is proved.

Each of the intervals α and β is a ϱ_r^k; for

$$|\alpha| = |\beta| = q_{x-r,k}\, d$$

and α and β are contained in ϱ, and hence they are contained in ω, which is an ω_r, and have to its end-points a distance not smaller than d. Since we supposed that property 2 had been verified for our particular value of k, it follows that α contains 2^k intervals α_j $(j = 1, \ldots 2^k)$ and β contains 2^k intervals β_j $(j = 1, \ldots 2^k)$ such that $|\alpha_j| = |\beta_j| = q_{x-r,0}\, d$, and that

$$A(\lambda) \leqslant A(\mu) + x - r - 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

whenever $|\mu| = |\lambda| + (r+2)w$ and either

$$\lambda \subset \alpha_{j_1},\ \mu \subset \alpha_{j_2},\ j_1 \neq j_2$$

or

$$\lambda \subset \beta_{j_1},\ \mu \subset \beta_{j_2},\ j_1 \neq j_2.$$

From (*) it follows that (13) holds also whenever $|\mu| = |\lambda| + (r+2)w$ and either

$$\lambda \subset \alpha_{j_1},\ \mu \subset \beta_{j_2}$$

or

$$\lambda \subset \beta_{j_1},\ \mu \subset \alpha_{j_2}.$$

Hence the 2^{k+1} sub-intervals $\alpha_1, \alpha_2 \ldots \alpha_{2^k}, \beta_1, \beta_2 \ldots \beta_{2^k}$ of ϱ satisfy the requirements. Thus property 2 is verified for $k + 1$ instead of k.

Property 3. For each of the values $r = 0, 1, \ldots x$ there exists an ω_r.

Proof. This is obvious for $r = 0$, since ι itself is an ω_0. Therefore we suppose that there exists an ω_r for a particular value of r $(0 \leq r \leq x-1)$. This ω_r we call ω. We have to show that there exists an ω_{r+1}.

The interval ϱ which we obtain by taking away from ω on both sides intervals of length d, has the length $q_{x-r+1,0}\, d - 2d = q_{x-r, f_{x-r}}\, d$.

Hence ϱ is a $\varrho_r^{f_{x-r}}$. From property 2 with $k = f_{x-r}$ it follows that ϱ contains $2^{f_{x-r}}$ sub-intervals α_j $(j = 1, \ldots 2^{f_{x-r}})$ of length $|\alpha_j| = q_{x-r,0}\, d$ such that

$$A(\lambda) \leqslant A(\mu) + x - r - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

whenever

$$|\mu| = |\lambda| + (r+2)\, w,\ \lambda \subset \alpha_{j_1},\ \mu \subset \alpha_{j_2},\ j_1 \neq j_2.$$

We shall prove that among these $2^{f_{x-r}}$ intervals α_j, being of length $q_{x-r,0}\, d$, there is at least one interval which is an ω_{r+1}.

If this were not true, each interval α_j would contain two intervals σ_j and τ_j for which $|\tau_j| = |\sigma_j| + (r+2)w$ and $A(\sigma_j) > A(\tau_j) + x - r - 1$. We have $A(\sigma_j) > A(\tau_j)$, since $x - r - 1 \geq 0$. By property 1 each sub-interval of ι of length d contains at most $5x$ points of $\mathfrak{A}$, and therefore each α_j, being of length $q_{x-r,0}\, d$, contains at most $5x \cdot q_{x-r,0}$ points of $\mathfrak{A}$. Therefore $5x\, q_{x-r,0} \geq A(\sigma_j) > A(\tau_j) \geq 0$, and so $A(\sigma_j)$ can take *only* the $5x\, q_{x-r,0}$ values $1, 2, \ldots 5x\, q_{x-r,0}$. On the other hand, it follows from $f_{x-r} = [{}^2\log(5x\, q_{x-r,0})] + 1$ that $2^{f_{x-r}} > 5x\, q_{x-r,0}$ and hence the number of intervals α_j is *greater* than $5x\, q_{x-r,0}$.

Therefore there are two different intervals α_{j1} and α_{j2} $(j_1 \neq j_2)$ such that $A(\sigma_{j1}) = A(\sigma_{j2})$. Without loss of generality we suppose that $|\sigma_{j2}| \geq |\sigma_{j1}|$ and hence $|\tau_{j2}| \geq |\sigma_{j1}| + (r+2)w$ while $A(\sigma_{j1}) = A(\sigma_{j2}) > A(\tau_{j2}) + x - r - 1$. The interval τ_{j2} contains an interval τ'_{j_2} of length

$$|\tau'_{j_2}| = |\sigma_{j_1}| + (r+2)\, w$$

and we have

$$A(\sigma_{j_1}) > A(\tau'_{j_2}) + x - r - 1,\ \sigma_{j_1} \subset \alpha_{j_1},\ \tau'_{j_2} \subset \alpha_{j_2},\ j_1 \neq j_2.$$

This contradicts (14).

It follows that among the intervals α_j there is at least one which is an ω_{r+1}, and thus property 3 is proved.

From property 3 with $r = x$ it follows that there exists a sub-interval ω of ι which is an ω_x. From the definition of ω_x it follows that ω has the length $q_{1,0}\, d = d$ and further that

$$A(\lambda) \leqslant A(\mu) \quad \ldots \ldots \ldots \quad (15)$$

whenever $|\mu| = |\lambda| + (x+1)w$, $\lambda \subset \omega$, $\mu \subset \omega$.

This is absurd since ω, being of length d, contains by property 1 at least 1 and at most $5x$ points of $\mathfrak{A}$; it will therefore contain a (closed) interval λ of length 0 to which belongs at least 1 point of $\mathfrak{A}$, and an (open) interval μ of length $\dfrac{d}{5x+1} = (x+1)w$ containing no point of $\mathfrak{A}$; hence

$$A(\lambda) > A(\mu),\ |\mu| = |\lambda| + (x+1)w,$$

which contradicts (15).

Therefore the sequence $\mathfrak{C}$ does not exist, and so the theorem is proved.