Mathematics. — *An elementary inequality.* By C. VISSER. (Communicated by Prof. J. G. VAN DER CORPUT.)

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§ 1. Suppose that we have a certain number of rows of n elements

$$a_1, a_2, \ldots, a_n \qquad (n > 1).$$

The elements may be arbitrary objects. Suppose that the number of mutually different rows is s. Two rows are called different when they differ in at least one element occurring at the same place.

Let us omit from each of the given rows the i-th element, thus obtaining a number of rows of $n-1$ elements. Let s_i denote the number of different rows among these. We shall prove that

$$s_1\, s_2 \ldots s_n \geqq s^{n-1}, \qquad (1)$$

with equality if and only if the given rows can be obtained from n sets of elements by taking all rows $a_1, a_2, \ldots, a_n$, where a_1 belongs to the first set, a_2 to the second set, $\ldots, a_n$ to the n-th set.

§ 2. We shall prove the inequality (1) by induction. It is clear that it is true for $n=2$. Suppose that $n > 2$, and that the inequality is true when n is replaced by $n-1$.

Let

$$a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(k)}$$

denote all mutually different elements a_n occurring at the last place in the given rows. Let $E^{(p)}$ denote the set of all rows which have $a_n^{(p)}$ as their last element. We shall indicate the numbers s, s_i for the set $E^{(p)}$ by $\sigma^{(p)}, \sigma_i^{(p)}$.

We have

$$s_i = \sigma_i^{(1)} + \sigma_i^{(2)} + \ldots + \sigma_i^{(k)} \text{ for } i = 1, 2, \ldots, n-1\,; \qquad (2)$$

$$s_n \geqq \sigma_n^{(p)} = \sigma^{(p)} \text{ for } p = 1, 2, \ldots, k\,; \qquad (3)$$

$$s = \sigma^{(1)} + \sigma^{(2)} + \ldots + \sigma^{(k)}\,; \qquad (4)$$

and by the inductive hypothesis

$$\sigma_1^{(p)}\, \sigma_2^{(p)} \ldots \sigma_{n-1}^{(p)} \geqq \sigma^{(p)^{n-2}} \qquad (5)$$

Now, by HÖLDER's inequality,

$$\prod_{i=1}^{n-1} (\sigma_i^{(1)} + \sigma_i^{(2)} + \ldots + \sigma_i^{(k)})^{\frac{1}{n-1}} \geqq \sum_{p=1}^{k} (\sigma_1^{(p)}\, \sigma_2^{(p)} \ldots \sigma_{n-1}^{(p)})^{\frac{1}{n-1}}.$$

Hence, by (2) and (5),

$$(s_1\, s_2 \dots s_{n-1})^{\frac{1}{n-1}} \geqq \sigma^{(1)\frac{n-2}{n-1}} + \sigma^{(2)\frac{n-2}{n-1}} + \dots + \sigma^{(k)\frac{n-2}{n-1}}$$

and so, by (3) and (4),

$$(s_1\, s_2 \dots s_{n-1}\, s_n)^{\frac{1}{n-1}} \geqq \sigma^{(1)\frac{n-2}{n-1}}\, \sigma^{(1)\frac{1}{n-1}} + \sigma^{(2)\frac{n-2}{n-1}}\, \sigma^{(2)\frac{1}{n-1}} + \dots + \sigma^{(k)\frac{n-2}{n-1}}\, \sigma^{(k)\frac{1}{n-1}}$$

$$= \sigma^{(1)} + \sigma^{(2)} + \dots + \sigma^{(k)}$$

$$= s.$$

This proves (1). If there is equality, then we must have *first*

$$\sigma_n^{(p)} = s_n$$

for all $p = 1, 2, \dots, k$. This implies that the rows obtained from the set $E^{(p)}$ by omitting the n-th element $a_n^{(p)}$ from its rows are the same set for each p. *Secondly* there must be equality (with $n-1$ instead of n) for this set. It follows, sind the statement concerning the sign of equality is correct for $n = 2$, that the given set of rows is of the nature described at the end of the preceding section. On the other hand, it is clear that for any such set of rows there is equality.

§ 3. There is a more general inequality of which (1) is a particular case. Let $i_1, i_2, \dots, i_r$ be r of the indices $1, 2, \dots, n$ $(1 \leqq r < n)$. Let us omit from each of the given rows the elements having these indices, thus obtaining a number of rows of $n-r$ elements. Denote the number of different rows among these by $s_{i_1\, i_2 \dots i_r}$. Then

$$\Pi\, s_{i_1\, i_2 \dots i_r} \geqq s^{\frac{(n-1)(n-2)\dots(n-r)}{1.2.\dots r}}, \quad \dots\dots\dots \quad (6)$$

where the product must be extended over all possible combinations $i_1\, i_2 \dots i_r$. The case of equality is the same as for (1). It is not difficult to see that (6) can be proved by successive application of (1).

§ 4. We next state a geometric application of (1). Let $x_1, x_2, \dots, x_n$ be rectangular coordinates in an n-dimensional space. Suppose we have a set of s different points in the space. We project these points on the $(n-1)$-dimensional coordinate spaces $x_i = 0$. Let there be s_i projections on the space $x_i = 0$. It is understood that each point which is a projection is counted only once. Then we have

$$s_1\, s_2 \dots s_n \geqq s^{n-1}.$$

There is equality if and only if the given set consists of all points $(a_1, a_2, \dots, a_n)$ obtained from n finite sets of real numbers by taking a_1 in the first set, a_2 in the second set, $\dots$, a_n in the n-th set.

There is, of course, an analogous geometric application of (6).

§ 5. Finally we discuss an extension of the inequality (1) which concerns the measures of a set of points E in an n-dimensional space with rectangular coordinates $x_1, x_1, \ldots, x_n$ and its projections E_i on the (n-1)-dimensional coordinate spaces $x_i = 0$. In order to avoid the possible occurrence of non-measurable sets, we shall assume that E is either a closed or an open set. Further we suppose that E is bounded. It will be clear, however, from what follows that it is possible to make much milder restrictions. Let e denote the measure of E, and e_i the $(n-1)$-dimensional measure of E_i. Then

$$e_1 e_2 \ldots e_n \geqq e^{n-1}. \qquad (7)$$

It is a simple matter to derive this from the result of the preceding section, but it will be difficult to find in this way when there is equality. Therefore we shall give a direct proof.

It is clear that (7) is true for $n = 2$. Suppose that $n > 2$, and that (7) is true when n is replaced by $n-1$.

Consider the set of all points of E for which $x_n = \lambda$. Let $\varepsilon(\lambda)$ denote its $(n-1)$-dimensional measure and let $\varepsilon_i(\lambda)$ denote the $(n-2)$-dimensional measure of its projection on the space $x_i = 0$ $(i = 1, 2, \ldots, n-1)$. We have

$$e_i = \int \varepsilon_i(\lambda)\, d\lambda \text{ for } i = 1, 2, \ldots, n-1; \qquad (8)$$

$$e_n \geqq \varepsilon(\lambda) \text{ for all } \lambda; \qquad (9)$$

$$e = \int \varepsilon(\lambda)\, d\lambda; \qquad (10)$$

and by the inductive hypothesis

$$\varepsilon_1(\lambda)\, \varepsilon_2(\lambda) \ldots \varepsilon_{n-1}(\lambda) \geqq \varepsilon(\lambda)^{n-2} \qquad (11)$$

Now, by HÖLDER's inequality for integrals,

$$\prod_{i=1}^{n-1} \left(\int \varepsilon_i(\lambda)\, d\lambda \right)^{\frac{1}{n-1}} \geqq \int [\varepsilon_1(\lambda)\, \varepsilon_2(\lambda) \ldots \varepsilon_{n-1}(\lambda)]^{\frac{1}{n-1}}\, d\lambda.$$

Hence, by (8) and (11),

$$(e_1 e_2 \ldots e_{n-1})^{\frac{1}{n-1}} \geqq \int \varepsilon(\lambda)^{\frac{n-2}{n-1}} . \, d\lambda.$$

From this it follows, by (9) and (10), that

$$\begin{aligned}(e_1 e_2 \ldots e_{n-1} e_n)^{\frac{1}{n-1}} &\geqq \int \varepsilon(\lambda)^{\frac{n-2}{n-1}} \varepsilon(\lambda)^{\frac{1}{n-1}}\, d\lambda \\ &= \int \varepsilon(\lambda)\, d\lambda \\ &= e.\end{aligned}$$

This proves (7). A reasoning analogous to that at the end of section 2 shows that there is equality if and only if E is, a set of n-dimensional measure zero being neglected, the product of n one-dimensional sets on the n coordinate axes.

It is hardly necessary to mention that there is an analogous continuous extension of (6) concerning the $(n-r)$-dimensional measures of the projections of E on the coordinate spaces $x_{i_1} = x_{i_2} = \ldots = x_{i_r} = 0$.

Remark. For $n = 3$ it was proved by MINKOWSKI that for a *convex* set E

$$e_1 + e_2 + e_3 \geqq 3\, e^{\frac{2}{3}},$$

with equality if and only if E is a cube with its edges parallel to the coordinate axes[1]). The corresponding inequality for arbitrary n is

$$e_1 + e_2 + \ldots + e_n \geqq n\, e^{\frac{n-1}{n}} \quad \ldots \ldots \quad (12)$$

It is true for an arbitrary set E. Indeed, by the inequality of the arithmetic and geometric means, (12) is a consequence of (7). There is equality if and only if E is, a set of measure zero being neglected, the product of n sets of equal one-dimensional measure on the coordinate axes.

It is possible, also, to derive (7) from (12). Consider the set E^1 consisting of all points $(e_1 x_1, e_2 x_2, \ldots, e_n x_n)$, where $(x_1, x_2, \ldots, x_n)$ is a point in E. The measure of E^1 is $e.\, e_1 e_2 \ldots e_n$, and the $(n-1)$-dimensional measures of its projections on the $(n-1)$-dimensional coordinate spaces are all equal to $e_1 e_2 \ldots e_n$, and so, by (12),

$$n \,.\, e_1 e_2 \ldots e_n \geqq n\, e^{\frac{n-1}{n}},$$

and therefore

$$e_1 e_2 \ldots e_n \geqq e^{n-1},$$

with equality when E^1 satisfies the condition for equality in (12), which, as is seen at once, means that E is, but for a set of measure zero, the product of n sets on the coordinate axes.

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1) H. MINKOWSKI. Volumen und Oberfläche, Mathematische Annalen **57** (1903), pp. 447—495.