Mathematics. — *A simple proof of certain inequalities concerning polynomials.* By C. VISSER. (Communicated by Prof. J. G. VAN DER CORPUT.)

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1. *Introduction.* The following theorem is due to TCHEBYCHEF.

Theorem 1. If $P(x)$ is a polynomial of degree n, and the coefficient of x^n is 1, then

$$\underset{-1 \leqq x \leqq 1}{Max} |P(x)| \geqq \frac{1}{2^{n-1}}.$$

There is equality if and only if

$$P(x) = \frac{1}{2^{n-1}} \cos nt, \; x = \cos t.$$

This theorem is well-known. It is extensively dealt with in PÓLYA und SZEGÖ, Aufgaben und Lehrsätze aus der Analysis, 2nd volume, where further literature is indicated.

Another inequality, to which I was led, some years ago, while working at the interesting problem of Mrs. T. VAN AARDENNE–EHRENFEST in the Wiskundige Opgaven 18, Problem No. 1, is

Theorem 2. If $P(x)$ is a polynomial of degree n, and the coefficient of x^n is 1, then

$$\int_{-1}^{1} |P(x)| \, dx \geqq \frac{1}{2^{n-1}}.$$

There is equality if and only if

$$P(x) = \frac{1}{2^n} \frac{\sin (n+1) t}{\sin t}, \; x = \cos t.$$

This inequality seems to be rather unknown. I am indebted to Professor KOKSMA of Amsterdam for calling my attention to the fact that the determination of the minimum of the integral of the absolute value over the interval $(-1, 1)$ for polynomials of degree n and with highest coefficient 1 was put forward as a problem by KORKIN and ZOLOTAREF in the Nouvelles

Annales Mathématiques of 1873, and that a solution was given in recent years by V. BRZECKA, Sur un problème d'extrémum (in Russian), Comm. Inst. Sci. Math. Mécan. Univ. Kharkoff etc., IV S., 16, pp. 33—44. Later I found that a proof of the inequality is implicitly contained in a paper by STIELTJES in 1876, De la représentation approximative d'une fonction par une autre, Oeuvres complètes, 1st volume, pp. 11—20.

In what follows I shall give a simple proof of both Theorem 1 and 2, and some generalizations.

2. *Proof of Theorem 1.* Since

$$P(\cos t) = \cos^n t + \dots = \frac{1}{2^{n-1}}(\cos n t + \dots),$$

Theorem 1 will follow from

Theorem 1a. If

$$C(t) = \cos n t + a_1 \cos (n-1) t + \dots + a_n$$

with arbitrary complex coefficients $a_1, \dots, a_n$, then

$$Max\, | C(t) | \geqq 1.$$

There is equality if and only if $C(t) = \cos nt$.

Proof. Put $\alpha = \frac{\pi}{n}$. Then for any t

$$\sum_{l=0}^{2n-1} (-1)^l e^{i k (t+l\alpha)} = \begin{cases} 0 \text{ for } k = 0, 1, \dots, n-1 \\ 2 n e^{int} \text{ for } k = n. \end{cases} \qquad (1)$$

Hence

$$\sum_{l=0}^{2n-1} (-1)^l \cos k (t+l\alpha) = \begin{cases} 0 \text{ for } k = 0, 1, \dots, n-1 \\ 2 n \cos n t \text{ for } k = n \end{cases}$$

It follows that

$$\sum_{l=0}^{2n-1} (-1)^l C(t+l\alpha) = 2 n \cos n t.$$

Putting $t = 0$, we infer from this that

$$\underset{l=0,1,\dots,2n-1}{Max} | C(l\alpha) | \geqq 1.$$

There is equality if and only if $(-1)^l C(l\alpha)$ is equal to 1 for all l. Then $C(t) - \cos nt$ must vanish identically, being a trigonometric polynomial of order $n-1$, and vanishing $2n$ times in the interval $0 \leqq t < 2\pi$.

3. *Proof of Theorem 2.* Since

$$\int_{-1}^{1} |P(x)|\,dx = \int_{0}^{\pi} |P(\cos t)| \sin t\,dt$$

$$= \frac{1}{2^{n-1}} \int_{0}^{\pi} |\cos n\,t + \ldots.| \sin t\,dt$$

$$= \frac{1}{2^{n-1}} \int_{0}^{\pi} |\cos n\,t \sin t + \ldots| \,dt$$

$$= \frac{1}{2^{n}} \int_{0}^{\pi} |\sin (n+1)\,t + \ldots.| \,dt.$$

Theorem 2 will follow from

Theorem 2a. If

$$S(t) = \sin n\,t + b_1 \sin (n-1)\,t + \ldots. + b_n$$

with arbitrary complex coefficients $b_1, \ldots, b_n$, then

$$\int_{0}^{2\pi} |S(t)|\,dt \geqq 4.$$

There is equality if and only if $S(t) = \sin nt$.

Proof. It follows from (1) that

$$\sum_{l=0}^{2n-1} (-1)^l S(t + l\,a) = 2\,n \sin nt.$$

Integrating over $0 \leqq t \leqq a$, we find

$$\sum_{l=0}^{2n-1} \int_{l\,a}^{(l+1)a} (-1)^l S(t)\,dt = 4.$$

It results that

$$\int_{0}^{2\pi} |S(t)|\,dt \geqq 4$$

and that there is equality if and only if $(-1)^l S(t)$ is non-negative on all intervals $la < t < (l+1)a$. Then $S(t)$ has $2n$ zeros in the points la, so that $S(t) - \sin nt$ must vanish identically, being of order $n-1$, and having $2n$ zeros in the interval $0 \leqq t < 2\pi$.

4. *Analogous inequalities for polynomials of a complex variable.*

Theorem 3. If

$$f(z) = a_0 + a_1 z + \dots a_n z^n$$

is a polynomial of the complex variable z with arbitrary complex coefficients, then

$$\underset{|z| \leqq 1}{Max} \; |f(z)| \geqq |a_0| + |a_n|.$$

There is equality if and only if $f(z) = a_0 + a_n z^n$.

Proof. Put $\omega = e^{i\frac{2\pi}{n}}$ Then

$$\sum_{l=0}^{n-1} f(z\,\omega^l) = n\,a_0 + n\,a_n\,z^n. \quad \dots \dots \quad (2)$$

For some z on $|z| = 1$, say z_0, this becomes $n(|a_0| + |a_n|)e^{i\alpha}$ with real α. It follows that

$$\underset{l=0,1,\dots,n-1}{Max} \; |f(z_0\,\omega^l)| \geqq |a_0| + |a_n|.$$

There is equality if and only if $f(z_0\omega^l) = a_0 + a_n z_0^n$ for all $l = 0, 1, \dots, n-1$. Then, obviously, $f(z) = a_0 + a_n z^n$.

Theorem 4. If

$$f(z) = a_0 + a_1 z + \dots + a_n z^n$$

is a polynomial of the complex variable z, then

$$\int_0^{2\pi} |f(e^{it})|\, dt \geqq 4(|a_0| + |a_n|).$$

There is equality if and only if $f(z) = a_0 + a_n z^n$ with $|a_0| = |a_n|$.

Proof. Putting $z = e^{it}$, we find from (2) that

$$\sum_{l=0}^{n-1} e^{-\frac{int}{2}} f\left(e^{i\left(t + l\frac{2\pi}{n}\right)}\right) = n\,a_0\,e^{-\frac{int}{2}} + n\,a_n\,e^{\frac{int}{2}}$$

Integrating over a suitably chosen interval $t_0 \leqq t \leqq t_0 + \dfrac{2\pi}{n}$, we have

$$\sum_{l=0}^{n-1} \int_{t_0 + l\frac{2\pi}{n}}^{t_0 + (l+1)\frac{2\pi}{n}} (-1)^l\, e^{-\frac{int}{2}} f(e^{it})\, dt = 4(|a_0| + |a_n|)\, e^{i\beta}$$

with real β. It follows that

$$\int_0^{2\pi} |f(e^{it})|\, d\,t \geqq 4(|a_0| + |a_1|).$$

It is necessary for equality that

$$(-1)^l e^{-\frac{int}{2}} f(e^{it})$$

has the same argument β on all intervals $t_0 + l\frac{2\pi}{n} < t < t_0 + (l+1)\frac{2\pi}{n}$

This involves the vanishing of $f(e^{it})$ in all points $t_0 + l\frac{2\pi}{n}$, and so $f(z) = a_n(z^n - e^{int_0})$, i.e. $f(z) = a_0 + a_n z^n$ with $|a_0| = |a_n|$. That, conversely, in this case there is equality is easily recognized.

5. *General trigonometric polynomials.*

Theorem 5. If

$$F(t) = \sum_{k=-n}^{n} A_k e^{ikt}$$

is a trigonometric polynomial with arbitrary coefficients, then

$$Max\, |F(t)| \geqq |A_{-n}| + A_n|,$$

with equality if and only if

$$F(t) = A_{-n} e^{-int} + A_n e^{int},$$

and

$$\int_0^{2\pi} |F(t)|\, dt \geqq 4(|A_{-n}| + |A_n|),$$

with equality if and only if

$$F(t) = A_{-n} e^{-int} + A_n e^{int}$$

with $|A_{-n}| = |A_n|$.

Proof. Put

$$f(z) = z^n \sum_{k=-n}^{n} A_k z^k.$$

Then $f(z)$ is a polynomial of the complex variable z. Application of Theorems 3 and 4 yields Theorem 5.

Remark. For a trigonometric polynomial

$$F(t) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos k\,t + b_k \sin kt)$$

with real coefficients, the inequalities become

$$Max\,|\,F(t)\,| \geqq \sqrt{a_n^2 + b_n^2},$$

with equality if and only if $F(t) = a_n \cos nt + b_n \sin nt$, and

$$\int_0^{2\pi} |\,F(t)\,|\,dt \geqq 4\sqrt{a_n^2 + b_n^2}\,,$$

also with equality if and only if $F(t) = a_n \cos nt + b_n \sin nt$. The first of these is well-known; see e.g. PÓLYA und SZEGÖ, Aufgaben und Lehrsätze, 2nd volume.