Applied Mechanics. — *The circular ring under the combined action of compressive and bending loads.* By C. B. BIEZENO and J. J. KOCH.

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1. *Introduction.* In a previous paper [1]), (in these lines to be quoted as "I") it was stated that every equilibrium-system (q, t) of radial and tangential forces, acting on a circular ring can be decomposed into a "compressive" system $A(q, t)$ giving rise only to a normal force N in any cross-section of the ring (so that both the bending moment M and the shearing force D are zero) and a "bending" system $B(q, t)$ characterized by $N = 0$. The first system A, if suitably magnified, leads to elastic instability of the ring. The required factors of magnification λ_1, λ_2, arranged after their order of magnitude have been introduced in "I" as the "characteristic numbers" of the buckling problem connected with the A-system and their numerical computation has been the object of that paper. Here it will be supposed, that the A-system is sufficiently small as to guarantee the elastic stability of the construction. It is obvious, that the deflections (and internal stresses) of the ring, due to the *single* action of the B-system, will be affected in a rather complicated way by the simultaneous action of the A-system, which — alone — would produce no deflections at all.

In this paper it will be shown how the deflection and internal stresses of the ring, under the *combined* action of the A and B-system can be derived from the corresponding quantities occurring with the B-system alone, with the aid of the characteristic numbers λ and the corresponding characteristic functions U (comp. "I") of the A-system.

To prevent lengthy repetitions, the reader is supposed to be fully acquainted with paper "I".

2. *The method.* If the ring is subjected to the k—*th* characteristic (or critical) load $\lambda_k A$, the corresponding characteristic mode of distortion U_k is determined except for a factor of proportionality. To remove this ambiguity we restict our attention to that deflection u_k, which corresponds to the "normalized" characteristic function $U_k = u_k + u_k = -\frac{r^2}{EI} M_k$ (comp. "I", 3, and "I", 4, 36). The same deflection u_k and consequently the same bending moment $M_k = -\frac{EI}{r^2} U_k$ can likewise be sustained by a "bending"

[1]) C. B. BIEZENO and J. J. KOCH. The generalized buckling problem of the circular ring. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 48, 447 (1945).

loadsystem $B_k(q,t)$, which readily can be derived from the equations of equilibrium

$$\left.\begin{aligned} -D' &= qr \\ D &= -tr \\ M' &= -tr^2 \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

(comp. ("I", 2, 1), in which N en N' must be put equal to zero). From this it follows that the *simultaneously* acting loads A and $\varkappa B_k$ ($\varkappa$ designing an arbitrary constant) will produce a bending moment $M = -\frac{EI}{r^2}(u'' + u)$, which for every value of $\varkappa$ is *proportional* to $M_k = -\frac{EI}{r^2} U_k$. If the particular value of $\varkappa$ is required for which the combined loads A and $\varkappa B_k$ produce a bending moment not proportional but *equal* to $M_k = -\frac{EI}{r^2} U_k$ (and consequently produce the distortion u_k) it can be remarked, that the single compressive loadsystem $\lambda_k A_k$ is capable of sustaining the prescribed deflection u_k and consequently to rouse the prescribed moment M_k. The loadsystem A therefore stands up for the bending moment $\frac{1}{\lambda_k} M_k$, and the remaining part $\frac{\lambda - 1}{\lambda} M_k$ of the prescribed bending moment M_k must be supplied by the bending loadsystem $\varkappa B_k$ and consequently $\varkappa = \frac{\lambda_k - 1}{\lambda_k}$.

We learn from these remarks that the effect of the "bending" loadsystem B_k is magnified in the proportion $\frac{\lambda_k}{\lambda_k - 1}$ by the presence of the "compressive" loadsystem A.

The "reduced" moment $-\frac{r^2}{EI} M_B \equiv U_B$ belonging to any arbitrary B-system can be expanded into a series

$$-\frac{r^2}{EI} \cdot M_B \equiv U_B = \sum_{k=1}^{\infty} b_k U_k \quad . \; . \; . \; . \; . \; . \quad (2)$$

of the characteristic functions U_k of the A-system under consideration (Comp. "I", 4).

If the ring is subjected to the simultaneous action of the B- and A-system, the magnifying influence of the latter will exert itself to each component B_k of the B-system, and in particular the resultant bending moment M, respectively the resultant function U, due to the joint systems A and B, is represented by

$$-\frac{r^2}{EI} M = U = \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k - 1} b_k U_k . \quad . \; . \; . \; . \; . \; . \quad (3)$$

The resultant distortion u of the ring is governed by the differential equation

$$u'' + u = U. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Summarizing it can be stated, that the stress distribution and the distortion of a circular ring under the combined action of a A- and B-load can be calculated as soon as the characteristic numbers λ_k of the A-system and the corresponding characteristic functions U_k are known. The function $U_B = -\frac{r^2}{EI} M_B$, which has to be expanded into the series (2), must be computed by integrating the third eq. (1), and the coefficients b_k occurring in this series can easily be found by using the orthogonality of the functions U_k.

3. *Application. The ring compressed by two diametral forces P.* With reference to ("I", 2) we replace the two compressing diametral forces P (whose points of attack may coincide with $\varphi = 0$ and $\varphi = \pi$) by their equivalent Fourier-series q and t:

$$q = a_0 + \sum_1^\infty a_k \cos k\varphi + \sum_1^\infty b_k \sin k\varphi;\ t = c_0 + \sum_1^\infty c_k \cos k\varphi + \sum_0^\infty d_k \sin k\varphi \quad (5)$$

in which evidently all coefficients a_{2n+1}, b, c and d are zero and in which

$$a_0 = -\frac{P}{\pi r} \qquad a_{2n} = -\frac{2P}{\pi r}$$

so that

$$q = -\frac{P}{\pi r} - \frac{2P}{\pi r}\sum_1^\infty \cos 2k\varphi \qquad t = 0. \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

From (6) it follows (comp. "I" 2, 18*a*, *b*) that the A and B-loadsystems are represented by

$$(A)\left\{\begin{aligned} q^* &= -\frac{P}{\pi r} + \frac{2P}{\pi r}\sum_1^\infty \frac{\cos 2k\varphi}{4k^2-1} \\ t^* &= \frac{2P}{\pi r}\sum_1^\infty \frac{2k\sin 2k\varphi}{4k^2-1} \end{aligned}\right. \qquad (B)\left\{\begin{aligned} q^{**} &= -\frac{2P}{\pi r}\sum_1^\infty \frac{4k^2}{4k^2-1}\cos 2k\varphi \\ t^{**} &= -\frac{2P}{\pi r}\sum_1^\infty \frac{2k}{4k^2-1}\sin 2k\varphi \end{aligned}\right. \quad . \ (7)$$

The normal force N_0, acting upon the cross-section (φ) of the ring and caused by the A-system is given by

$$N_0 = -\frac{P}{\pi} + \frac{2P}{\pi}\sum_1^\infty \frac{\cos 2k\varphi}{4k^2-1} = -\frac{P}{\pi}\left[1 + \sum_1^\infty \varepsilon_{2k}\cos 2k\varphi\right];\ \varepsilon_{2k} = -\frac{2}{4k^2-1} \quad (8)$$

and the bending moment M_B belonging to the B-system by

$$M_B = -\frac{2Pr}{\pi}\sum_2^\infty \frac{\cos 2k\varphi}{4k^2-1} = \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

$$-\frac{2Pr}{\pi}[0{,}3333 \cos 2\varphi + 0{,}06667 \cos 4\varphi + 0{,}02857 \cos 6\varphi + 0{,}01587 \cos 8\varphi + \\ + 0{,}0101 \cos 10\varphi + 0{,}00699 \cos 12\varphi]$$

(comp. the third eq. (1) and the second eq. (7).

With reference to ("I" 8), where an example has been treated analogous with the "A"-problem under consideration (viz. $N = \lambda N_0 = \lambda(1 + \varepsilon_2 \cos 2\varphi + \varepsilon_4 \cos 4\varphi))$, we restrict ourselves to the summing up of the following numerical results with respect to the first two characteristic numbers λ_1 and λ_2 and the two corresponding characteristic functions:

$$\lambda_1 = 1.0336 \frac{3EI}{r^2} \frac{\pi}{P}; \qquad \lambda_2 = 5{,}3628 \frac{3EI}{r^2} \frac{\pi}{P}$$

$$\left.\begin{aligned} U_1/r \backsim V_5^*/r = 0{,}8396 \cos 2\varphi - 0{,}0784 \cos 4\varphi - 0{,}0041 \cos 6\varphi - \\ - 0{,}0014 \cos 8\varphi - 0{,}0005 \cos 10\varphi - 0{,}0003 \cos 12\varphi \end{aligned}\right\} \quad (10)$$

$$\left.\begin{aligned} -\frac{\pi}{Pr} N_0 V_5^* = -\frac{\pi}{Pr} W_5^* = +0{,}8123 \cos 2\varphi - 0{,}3796 \cos 4\varphi - \\ - 0{,}0460 \cos 6\varphi - 0{,}0266 \cos 8\varphi - 0{,}0162 \cos 10\varphi - 0{,}0112 \cos 12\varphi \end{aligned}\right\} \quad (11)$$

$$\left.\begin{aligned} U_2/r \backsim \overline{V}_{11}^*/r = 6{,}9010 \cos 2\varphi + 15{,}3749 \cos 4\varphi - 4{,}9193 \cos 6\varphi + \\ + 0{,}0784 \cos 8\varphi - 0{,}0651 \cos 10\varphi - 0{,}0308 \cos 12\varphi \end{aligned}\right\} \quad (12)$$

$$\left.\begin{aligned} -\frac{\pi}{Pr} N_0 \overline{V}_{11}^* = -\frac{\pi}{Pr} \overline{W}_{11}^* = 1{,}2819 \cos 2\varphi + 14{,}320 \cos 4\varphi - \\ - 10{,}7560 \cos 6\varphi + 0{,}3678 \cos 8\varphi - 0{,}4107 \cos 10\varphi - 0{,}2678 \cos 12\varphi \end{aligned}\right\} \quad (13)$$

(in these latter series all terms following that with $\cos 12\varphi$ are neglected).

In the expansion

$$-\frac{M_B r^2}{EI} = b_1 U_1 + b_2 U_2 + \ldots \qquad (14)$$

the coefficients b_1 and b_2 are defined by

$$\left.\begin{aligned} -\int_0^{2\pi} N_0 U_1 \cdot \frac{M_B r^2}{EI} d\varphi = b_1 \int_0^{2\pi} N_0 U_1^2 d\varphi \text{ resp.} \\ -\int_0^{2\pi} N_0 U_2 \frac{M_B r^2}{EI} = b_2 \int_0^{2\pi} N_0 U_2^2 d\varphi \end{aligned}\right\} \ldots (15)$$

If we replace U_1 and U_2 by their approximate values V_5^* and $\overline{V}_{11}^*$ we find:

$$\left.\begin{aligned} b_1 = -\int_0^{2\pi} W_5^* \frac{M_B r^2}{EI} d\varphi : \int_0^{2\pi} V_5^* W_5^* d\varphi; \\ b_2 = -\int_0^{2\pi} \overline{W}_{11}^* \frac{M_B r^2}{EI} d\varphi : \int_0^{2\pi} \overline{V}_{11}^* \overline{W}_{11}^* d\varphi \end{aligned}\right\} \ldots (16)$$

The computation of these coefficients only requires the evaluation of the integrals

$$\int_0^{2\pi} W_5^* \frac{M_B r^2}{EI} d\varphi, \int_0^{2\pi} V_5^* W_5^* d\varphi \text{ and the analogous ones for } b_2, \quad (17)$$

If we write:

$$M_B = -\frac{2Pr}{\pi} \sum_{k=1}^{6} m_{2k} \cos 2k\varphi, \quad V_5^* = r \sum_{k=1}^{6} v_{2k} \cos 2k\varphi,$$

$$W_5^* = -\frac{Pr}{\pi} \sum_1^6 w_{2k} \cos 2k\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

(comp. 9 and 11) we find:

$$\int_0^{2\pi} W_5^* \frac{M_B r^2}{EI} d\varphi = \frac{2P^2 r^4}{\pi^2 EI} \sum_{k=1}^{6} \int_0^{2\pi} w_{2k} m_{2k} \cos^2 2k\varphi \, d\varphi =$$

$$= \frac{2P^2 r^4}{\pi EI} \sum_{k=1}^{6} w_{2k} m_{2k} = 0{,}24348 \frac{2P^2 r^4}{\pi EI}$$

$$\int_0^{2\pi} V_5^* W_5^* d\varphi = -\frac{Pr^2}{\pi} \sum_{k=1}^{6} \int_0^{2\pi} v_{2k} w_{2k} \cos^2 2k\varphi \, d\varphi = -Pr^2 \sum_{k=1}^{6} v_{2k} w_{2k} =$$

$$= -0{,}71021 \, Pr^2$$

whence

$$b_1 = 0{,}34196 \frac{2Pr^2}{\pi EI} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Analogously it is found that

$$b_2 = 0.003822 \frac{2Pr^2}{\pi EI} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The deflections u_1 and u_2 of the ring corresponding with the components $-b_1 \frac{EI}{r^2} U_1, -b_2 \frac{EI}{r^2} U_2 \ldots$ of the bending moment M_B can be calculated from the equations

$$u_1'' + u_1 = b_1 U_1 \qquad u_2'' + u_2 = b_2 U_2 \ldots . \quad . \quad . \quad . \quad (21)$$

resp.:

$$u_1'' + u_1 = 0{,}28715 \cos 2\varphi - 0{,}02681 \cos 4\varphi - 0{,}00140 \cos 6\varphi -$$
$$- 0{,}00048 \cos 8\varphi - 0{,}00017 \cos 10\varphi - 0{,}00010 \cos 12\varphi$$

$$u_2'' + u_2 = 0{,}02638 \cos 2\varphi + 0{,}05876 \cos 4\varphi - 0{,}01880 \cos 6\varphi +$$
$$+ 0{,}00030 \cos 8\varphi - 0{,}00025 \cos 10\varphi - 0{,}00012 \cos 12\varphi$$

(comp. (11) and (19), resp. (12) and (20)), from which it follows:

$$\left.\begin{aligned} u_1 &= [-0{,}09572 \cos 2\varphi + 0{,}00179 \cos 4\varphi + 0{,}00004 \cos 6\varphi + \\ &\qquad + 0{,}00001 \cos 8\varphi + \ldots] \frac{2Pr^3}{EI} \\ u_2 &= [-0{,}00879 \cos 2\varphi - 0{,}00392 \cos 4\varphi + 0{,}00054 \cos 6\varphi + \ldots] \frac{2Pr^3}{\pi EI} \end{aligned}\right\} . \quad (22)$$

The resultant distortion u^* of the ring, caused by the simultaneous action of the A and B loadsystems is given by

$$u^* = \frac{\lambda_1}{\lambda_1 - 1} u_1 + \frac{\lambda_2}{\lambda_2 - 1} u_2 + \ldots = u_1 + u_2 + \ldots + \frac{1}{\lambda_1 - 1} u_1 + \frac{1}{\lambda_2 - 1} u_2 + \ldots \quad (23)$$

The series $u_1 + u_2 +$ ad inf. represents the deflection u of the ring which occurs if no secondary effect is present. This deflection can be calculated by one of the well-known elementary methods (f.i. with the aid of CASTIGLIANO's theorem). The series $\frac{u_1}{\lambda_1 - 1} + \frac{u_2}{\lambda_2 - 2} + \ldots$ can readily be broken off with the third term in consequence of the fact, that u_k decreases and λ_k increases rather rapidly with increasing k. We therefore write:

$$u^* = u + \frac{u_1}{\lambda_1 - 1} + \frac{u_2}{\lambda_2 - 1} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

At $\varphi = 0$ and $\varphi = \frac{\pi}{2}$ the following results are obtained

$$u_0^* = - \frac{2Pr^3}{\pi EI} \left[0{,}11685 + \frac{0{,}09388}{1{,}0336 \frac{3\pi EI}{Pr^2} - 1} + \frac{0{,}01217}{5{,}3628 \frac{3\pi EI}{Pr^2} - 1} \right] . \quad (25)$$

$$u_{\pi/2}^* = + \frac{2Pr^3}{\pi EI} \left[0{,}10730 + \frac{0{,}09748}{1{,}0336 \frac{3\pi EI}{Pr^2} - 1} + \frac{0{,}00433}{5{,}3628 \frac{3\pi EI}{Pr^2} - 1} \right] . \quad (26)$$

The secondary effect to which the "compressive" A-system gives rise in influencing the deflections caused by the "bending" B-system as represented by the second and third terms in these expressions has been expressed in terms of the quotient $\frac{3\pi EI}{Pr^2}$, viz. the quotient of the first critical all-sided pressure q_{cr} of the ring ($q_{cr} = \frac{3EI}{r^2}$) and the "mean normal pressure P/π beloging to the two forces P.